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On monotone mappings in modular function spaces

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Abstract

We prove the existence of fixed points of monotone ρ-nonexpansive mappings in ρ-uniformly convex modular function spaces. This is the modular version of Browder and Göhde fixed point theorems for monotone mappings. We also discuss the validity of this result in modular function spaces where the modular is uniformly convex in every direction. This property has never been considered in the context of modular spaces.

Keywords: Fixed point, Krasnoselskii iteration, modular function spaces, monotone mapping, nonexpansive mapping, partially ordered, uniformly convex, uniformly convex in every direction.
2010 MSC: 46B20, 45D05, 47E10, 34A12.

1. Introduction

Fixed point theory is a powerful tool in different fields such as differential equations, economics, game theory, dynamical systems, optimal control, and artificial intelligence. The extension of the Banach contraction principle [3] by Ran and Reurings [25] to partially ordered metric spaces has seen some excitement among the mathematicians working in fixed point theory. Ran and Reurings extension was carried while investigating the solutions to some special matrix equations. The study of these matrix equations is motivated by the fact that they often arise in the analysis of ladder networks, dynamic programming, control theory, stochastic filtering, statistics and many other applications [7]. Nieto and Rodríguez-López [24] improved Ran and Reurings fixed point theorem and used such arguments to find periodic solutions to some differential equations. In this paper, we investigate the existence of fixed points of monotone nonexpansive

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mappings in modular function spaces. In particular, we prove the analogue to Browder [5] and Göhde [10] fixed point theorem for monotone mappings in modular function spaces. Moreover, we introduce for the first time the uniform convexity in every direction in the modular sense and prove a fixed point theorem for monotone nonexpansive mappings.

In terms of content, this paper overlaps in places with the following popular books on fixed point theory by Goebel and Kirk [9], by Khamsi and Kirk [14], by Khamsi and Kozlowski [16] and by Kozlowski [19].

2. Preliminaries

Since the basic definitions of modular function spaces are widely found in the literature, we omit them here and refer to the paper [4] and the book [16].

Let Σ be a nontrivial σ-algebra of subsets of a nonempty set Ω. Denote by $\mathcal{M}_\infty$ the set of all extended measurable functions. Let $\rho : \mathcal{M}_\infty \to [0,\infty]$ be a regular function modular. Define

$$\mathcal{M} = \{h \in \mathcal{M}_\infty : |h(\omega)| < \infty \ \rho\text{-a.e.}\}.$$

The modular function space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, is defined as

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

Throughout this paper, we will assume function modulars are convex. The Luxemburg norm in L_ρ is defined as:

$$\|f\|_\rho = \inf\left\{t > 0 : \rho\left(\frac{1}{t} f\right) \le 1\right\}.$$

The set of function modulars will be denoted by $\Re$. In the following theorem we recall some of the needed properties of modular spaces.

Theorem 2.1 ([16, 19])**.** *Let $\rho \in \Re$. The following properties hold:*

1. *if $\lim\limits_{n\to+\infty} \rho(\lambda f_n) = 0$, for some $\lambda > 0$, then there exists a subsequence $\{f_{\phi(n)}\}$ such that $\{f_{\phi(n)}\}$ converges ρ-a.e. to 0;*
2. *(Fatout property) if $\{f_n\}$ converges ρ-a.e. to f, then we have*

$$\rho(f) \le \liminf_{n\to+\infty} \rho(f_n).$$

We say that ρ satisfies the Δ_2-type condition if and only if there exists $K > 0$ such that $\rho(2f) \le K\, \rho(f)$, for any $f \in L_\rho$. This property is crucial when studying modular function spaces.

Definition 2.2. Let $\rho \in \Re$.

(a) $\{f_n\}$ is said to be ρ-convergent to f if $\lim\limits_{n\to+\infty} \rho(f_n - f) = 0$.

(b) $\{f_n\}$ is said to be ρ-Cauchy if $\lim\limits_{n,m\to+\infty} \rho(f_n - f_m) = 0$.

(c) $B \subset L_\rho$ is said to be ρ-closed if for any sequence $\{f_n\}$ in B which ρ-converges to f, we have $f \in B$.

(d) $B \subset L_\rho$ is said to be ρ-bounded if

$$diam_\rho(B) = \sup\{\rho(f - g) : f \in B, g \in B\} < \infty.$$

Since ρ fails the triangle inequality, then the ρ-convergence may not imply the ρ-Cauchy behavior. As a consequence to Theorem 2.1, we have:

Proposition 2.3. *Let $\rho \in \Re$. Then L_ρ is ρ-complete, i.e., any ρ-Cauchy sequence is ρ-convergent. Moreover, the ρ-balls*

$$B_\rho(f, r) = \{g \in L_\rho :\ \rho(f - g) \le r\},$$

are ρ-closed, for any $f \in L_\rho$ and $r \ge 0$.

The next theorem is crucial throughout our work. Its proof follows easily from Theorem 2.1.

Theorem 2.4. *Let $\rho \in \Re$.*

(i) *Let $\{f_n\}$ be a monotone increasing sequence, i.e., $f_n \leq f_{n+1}$ ρ-a.e., for any $n \in \mathbb{N}$. If $\{f_n\}$ ρ-converges to f, then $f_n \leq f$ ρ-a.e., for any $n \in \mathbb{N}$.*

(ii) *Let $\{f_n\}$ be a monotone decreasing sequence, i.e., $f_{n+1} \leq f_n$ ρ-a.e., for any $n \in \mathbb{N}$. If $\{f_n\}$ ρ-converges to f, then $f \leq f_n$ ρ-a.e., for any $n \in \mathbb{N}$.*

(iii) *Order intervals in L_ρ are ρ-closed and convex. Recall that an order interval is any of the subsets $[f, \rightarrow) = \{g \in L_\rho : \ f \leq g \ \rho\text{-a.e.}\}$, $(\leftarrow, f] = \{g \in L_\rho : \ g \leq f \ \rho\text{-a.e.}\}$ and $[f, g] = \{h \in L_\rho : \ f \leq h \leq g \ \rho\text{-a.e.}\}$, for any $f, g \in L_\rho$.*

Next we define the type of mappings for which the fixed point problem will be considered.

Definition 2.5. Let $\rho \in \Re$ and K be a nonempty subset of L_ρ. A mapping $T : K \to K$ is monotone if $T(f) \leq T(g)$ ρ-a.e. whenever $f \leq g$ ρ-a.e., for any $f, g \in K$. Moreover, T is called monotone ρ-nonexpansive if T is monotone such that

$$\rho(T(f) - T(g)) \leq \rho(f - g),$$

whenever $f, g \in C$ and $f \leq g$ ρ-a.e. A point $f \in C$ is called a fixed point of T if and only if $T(f) = f$.

Remark 2.6. Recall that $T : C \to C$ is said to be ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f. It is not true that monotone ρ-nonexpansiveness implies ρ-continuity since this result is not true in general when ρ is a norm.

The concept of type functions have been widely used in the proofs of many fixed point results.

Definition 2.7. Let K be a nonempty subset of L_ρ.

(1) A function $\tau : K \to [0, \infty]$ is called a type if there exists a ρ-bounded sequence $\{f_n\}$ in L_ρ such that

$$\tau(f) = \limsup_{n \to +\infty} \rho(f_n - f)$$

for any $f \in K$.

(2) Any sequence $\{g_n\}$ in K which satisfies

$$\lim_{n \to \infty} \tau(g_n) = \inf\{\tau(f) : f \in K\},$$

will be called a minimizing sequence of τ in K.

Note that any type function τ is convex since ρ is convex. Next we give the definition of the modular uniform convexity introduced in [15].

Definition 2.8. Let $\rho \in \Re$.

(i) We say that ρ is uniformly convex (UC) if

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\, \rho\Big(\frac{f+g}{2}\Big) : \ \rho(f) \leq r, \rho(g) \leq r, \rho(f-g) \geq \varepsilon r\right\} > 0$$

for every $r > 0$ and $\varepsilon > 0$.

(ii) We say that ρ satisfies (UUC) if there exists $\eta(s, \varepsilon) > 0$, for every $s \geq 0$, and $\varepsilon > 0$ such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0, \quad for\ r > s.$$

Remark 2.9. It is known that for a wide class of modular function spaces with the Δ_2 property, the uniform convexity of the Luxemburg norm is equivalent to the uniform convexity of the modular ρ. For example, in

Orlicz spaces this result can be traced to early papers by Luxemburg [22], Milnes [23], Akimovic [1], and Kaminska [13]. It is also known that, under suitable assumptions, the uniform convexity of the modular in Orlicz spaces is equivalent to the very convexity of the Orlicz function [6, 17]. Typical examples of Orlicz functions that do not satisfy the Δ_2 condition but are uniformly convex (and hence very convex) are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$, [21, 23]. See also [11] for the discussion of some geometrical properties of Calderon-Lozanovskii and Orlicz-Lorentz spaces.

Definition 2.10 ([17])**.** Let $\rho \in \Re$. L_ρ is said to have the property (R) if and only if for every decreasing sequence $\{K_n\}_{n\geq 1}$ of ρ-bounded, ρ-closed and convex nonempty subsets, we have $\bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$.

We have the following amazing result.

Theorem 2.11 ([15])**.** *Let $\rho \in \Re$ be (UUC), then L_ρ has property (R).*

The following lemma plays a crucial role in the proof of many fixed point results in modular function spaces.

Lemma 2.12 ([15])**.** *Let $\rho \in \Re$. Assume that ρ is (UUC). Let K be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Suppose that τ is a type defined on K and $\{g_n\}$ is a minimizing sequence of τ, i.e.,*

$$\lim_{n\to+\infty} \tau(g_n) = \inf_{f\in K} \tau(f).$$

Then $\{g_n\}$ is ρ-convergent and its ρ-limit is independent of the sequence $\{g_n\}$.

3. Main results

In this section, we establish Browder and Göhde's fixed point theorem [5, 10] for monotone ρ-nonexpansive mappings. Recall that the beginning of the fixed point theory in modular function spaces finds its root in the paper [18].

Let $\rho \in \Re$ and $T : C \to C$ be a monotone ρ-nonexpansive mapping where $C \subset L_\rho$ is nonempty and convex. Let $f_0 \in C$ and $\lambda \in (0, 1)$. The Krasnoselskii-Ishikawa [12, 20] iteration sequence $\{f_n\}$ in C is defined by

$$f_{n+1} = (1-\lambda)f_n + \lambda T(f_n), \quad n \geq 0. \tag{3.1}$$

Assume that f_0 and $T(f_0)$ are comparable. Assume that $f_0 \leq T(f_0)$ ρ-a.e. Since order intervals are convex, we have $f_0 \leq f_1 \leq T(f_0)$ ρ-a.e. Hence $T(f_0) \leq T(f_1)$ ρ-a.e. since T is monotone. By the induction, we will prove that

$$f_n \leq f_{n+1} \leq T(f_n) \leq T(f_{n+1}) \ \ \rho\text{-a.e.}$$

for any $n \in \mathbb{N}$. Note that if $T(f_0) \leq f_0$ ρ-a.e., then we will have

$$T(f_{n+1}) \leq T(f_n) \leq f_{n+1} \leq f_n \ \ \rho\text{-a.e.}$$

for any $n \in \mathbb{N}$. By using the monotone ρ-nonexpansiveness of T, we get

$$\rho(T(f_{n+1}) - T(f_n)) \leq \rho(f_{n+1} - f_n)$$

for any $n \in \mathbb{N}$.

The modular version of Browder and Göhde fixed point theorem for monotone mappings is given below.

Theorem 3.1. *Let $\rho \in \Re$ be (UUC) and C be a nonempty convex ρ-closed ρ-bounded subset of L_ρ not reduced to one point. Let $T : C \to C$ be a monotone ρ-nonexpansive mapping and ρ-continuous. Assume there exists $f_0 \in C$ such that f_0 and $T(f_0)$ are comparable. Then T has a fixed point.*

Proof. Assume that $f_0 \leq T(f_0)$ ρ-a.e. The case $T(f_0) \leq f_0$ ρ-a.e. follows the same ideas. Consider the Krasnoselskii-Ishikawa sequence $\{f_n\}$ generated by (3.1) starting at f_0 with $\lambda \in (0,1)$. Since ρ is (UUC), then L_ρ satisfies the property (R). By using the properties of $\{f_n\}$, we know that

$$C_\infty = \bigcap_{n\geq 0}[f_n, \to) \cap C = \bigcap_{n\geq 0}\{g \in C : \; f_n \leq g \; \rho\text{-a.e.}\} \neq \emptyset.$$

Let $g \in C_\infty$, then $f_n \leq g$ ρ-a.e. and since T is monotone, we get $f_n \leq T(f_n) \leq T(g)$ ρ-a.e., for any $n \geq 0$, i.e., $T(C_\infty) \subset C_\infty$. Consider the type function $\tau : C_\infty \to [0,+\infty)$ generated by $\{f_n\}$, i.e., $\tau(g) = \limsup\limits_{n\to+\infty} \; \rho(f_n - g)$. Let $\{g_n\}$ be a minimizing sequence of τ in C_∞. Lemma 2.12 implies that $\{g_n\}$ is ρ-convergent to some $g \in C_\infty$ and any minimizing sequence of τ in C_∞ will also ρ-converge to g. Since T is monotone ρ-nonexpansive and $g_m \in C_\infty$, for any $m \in \mathbb{N}$, we get

$$\begin{aligned}
\tau(T(g_m)) &= \limsup_{n\to+\infty} \; \rho(f_{n+1} - T(g_m)) \\
&\leq \limsup_{n\to+\infty} \; (1-\lambda)\rho(f_n - T(g_m)) + \lambda \; \rho(T(f_n) - T(g_m)) \\
&\leq \limsup_{n\to+\infty} \; (1-\lambda)\rho(f_n - T(g_m)) + \lambda \; \rho(f_n - g_m) \\
&\leq (1-\lambda)\limsup_{n\to+\infty} \rho(f_n - T(g_m)) + \lambda \; \limsup_{n\to+\infty} \; \rho(f_n - g_m) \\
&\leq (1-\lambda) \; \tau(T(g_m)) + \lambda \; \tau(g_m),
\end{aligned}$$

which implies $\tau(T(g_m)) \leq \tau(g_m)$, since $\lambda \in (0,1)$. Therefore, the sequence $\{T(g_m)\}$ is also a minimizing sequence of τ in C_∞ since $T(C_\infty) \subset C_\infty$. Therefore, $\{T(g_m)\}$ ρ-converges to g as well. Since T is ρ-continuous, $\{T(g_m)\}$ also ρ-converges to $T(g)$. The uniqueness of the ρ-limit implies that $T(g) = g$, i.e., g is a fixed point of T. □

Krasnoselskiĭ and Rutickiĭ [21] studied the fixed points of the Hammerstein and Uryson operators which are not defined in any of the classical L^p spaces. In fact, they showed that the proper setting for such operators is the class of Orlicz spaces. By using the structure of the modular function spaces, we can do more. The main reason is that the norm in Orlicz spaces is defined in a very complicated way while the module is an integral and easy to use. In the next example, we illustrate these ideas and show how to apply our Theorem 3.1. This example was inspired from the one used in [2]. The fundamental difference resides in the fact that most uniformly convex spaces, like L^p, fail to satisfy the Opial property as a key assumption in the paper [2].

Example 3.2. Let $\varphi : [0,+\infty) \to [0,+\infty)$ be an increasing convex function (i.e., φ to some extent behaves like the power functions $f(t) = t^p$, $p \geq 1$). We will assume that φ satisfies the Δ_2-type condition, i.e., there exists $K \geq 1$ such that $\varphi(2t) \leq K\varphi(t)$, for any $t \in [0,+\infty)$. The Orlicz-Birnbaum space L^φ is defined by

$$L^\varphi = \{x : [0,1] \to \mathbb{R}; \; \rho_\varphi(x) = \int_{\mathbb{I}} \varphi(|x(t)|)dt < +\infty\},$$

where $\mathbb{I} = [0,1]$. Next we consider the Uryson integral equation

$$x(t) = g(t) + \int_{\mathbb{I}} F(t,s,x(s))ds \tag{3.2}$$

for $t \in \mathbb{I}$, where g is in L^φ, and $F(t,s,x)$ is measurable in both variables s and t for every x. We shall assume that F satisfies the inequality

$$|F(t,s,x)| \leq h(t,s) + M(t) \; |x(s)|,$$

where $t, s \in \mathbb{I}$, and $x \in L^\varphi$. Assume that $M(t) \in [0,1]$ and

$$\int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(h(t,s))dtds < +\infty \quad \text{and} \quad M_0 = \int_{\mathbb{I}} M(t)dt < +\infty.$$

Moreover, we will assume that F enjoys the following monotonicity condition

$$0 \leq F(t,s,x) - F(t,s,y) \leq x - y, \tag{3.3}$$

where $t, s \in \mathbb{I}$, and $x, y \in L^\varphi$ such that $y \leq x$, i.e., $y(t) \leq x(t)$ for almost every $t \in \mathbb{I}$. Let

$$B = \Big\{ y \in L^\varphi : \quad \text{such that} \quad \rho_\varphi(x) \leq R \Big\},$$

i.e., B is the ρ_φ-closed ball of L^φ centered at 0 with radius R. Consider the operator defined by

$$\widetilde{F}(t)(x)(s) = F(t,s,x(s)),$$

and define the operator $J : L^\varphi \to L^\varphi$ by

$$(Jx)(t) = g(t) + \int_{\mathbb{I}} \widetilde{F}(t)(x)(s)ds.$$

Since φ is convex and satisfies the Δ_2-type condition, there exists $K \geq 1$ such that $\varphi(2t) \leq K\varphi(t)$, for $t \geq 0$. Then we have

$$\varphi(a+b) \leq \frac{K}{2}\Big(\varphi(a) + \varphi(b)\Big)$$

for any $a, b \in [0, +\infty)$. If M_0 is small enough and R is sufficiently large, then we will have $J(B) \subset B$. Indeed, by using the Jensen's inequality we have:

$$\begin{aligned}
\rho_\varphi(Jx) &= \int_{\mathbb{I}} \varphi(Jx(t))dt \\
&= \int_{\mathbb{I}} \varphi\Big(g(t) + \int_{\mathbb{I}} \widetilde{F}(t)(x)(s)ds\Big)dt \\
&\leq K/2 \int_{\mathbb{I}} \varphi(|g(t)|)dt + K/2 \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi\Big(|\widetilde{F}(t)(x)(s)|\Big)dsdt \\
&\leq K/2 \int_{\mathbb{I}} \varphi(|g(t)|)dt + K/2 \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi\Big(h(t,s) + M(t)|x(s)|\Big)dsdt \\
&\leq K/2 \int_{\mathbb{I}} \varphi(|g(t)|)dt + K^2/4 \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(h(t,s))dsdt \\
&\quad + K^2/4 \int_{\mathbb{I}}\int_{\mathbb{I}} M(t)\ \varphi(|x(s)|)dsdt \\
&\leq K/2 \int_{\mathbb{I}} \varphi(|g(t)|)dt + K^2/4 \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(h(t,s))dsdt + M_0K^2/4\ \rho_\varphi(x) \\
&\leq K/2 \int_{\mathbb{I}} \varphi(|g(t)|)dt + K^2/4 \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(h(t,s))dsdt + RM_0K^2/4
\end{aligned}$$

for any $x \in B$. If $M_0K^2 < 4$, choose R such that

$$R \geq \frac{2K}{4 - M_0K^2} \int_{\mathbb{I}} \varphi(|g(t)|)dt + \frac{K^2}{4 - M_0K^2} \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(h(t,s))dsdt.$$

Then we will get $J(x) \in B$ as claimed. Next we prove that J is monotone ρ_φ-nonexpansive. First, from the condition (3.3), J is obviously monotone. Let $x, y \in L^\varphi$ such that $y \leq x$. Since φ is convex, we have:

$$\begin{aligned}
\rho_\varphi(Jx - Jy) &= \int_{\mathbb{I}} \varphi(|Jx(t) - Jy(t)|)dt \\
&= \int_{\mathbb{I}} \varphi\Big(\Big|\int_{\mathbb{I}} (\widetilde{F}(t)(x)(s) - \widetilde{F}(t)(y)(s))ds\Big|\Big)dt \\
&\leq \int_{\mathbb{I}}\int_{\mathbb{I}} \varphi(|x(s) - y(s)|)dsdt \\
&= \rho_\varphi(x - y),
\end{aligned}$$

which implies that J is a monotone ρ_φ-nonexpansive operator as claimed. If φ is very convex, then L^φ is ρ_φ-uniformly convex [17]. Under the above assumptions and by using Theorem 3.1, we obtain the following:

(1) if $g(t) + \int_{\mathbb{I}} F(t,s,0)ds \geq 0$, for almost all $t \in [0,1]$, then the integral equation (3.2) has a positive solution in L^φ;
(2) if $g(t) + \int_{\mathbb{I}} F(t,s,0)ds \leq 0$, for almost all $t \in [0,1]$, then the integral equation (3.2) has a negative solution in L^φ.

In order to weaken the assumptions of Theorem 3.1, we will need the concept of uniform continuity of a modular function.

Definition 3.3. Let $\rho \in \Re$. The function modular ρ is said to be uniformly continuous if for any $\varepsilon > 0$ and $R > 0$ there exists $\eta > 0$ such that
$$|\rho(g) - \rho(f+g)| \leq \varepsilon,$$
whenever $\rho(f) \leq \eta$ and $\rho(g) \leq R$.

As an example of a modular which is uniformly continuous, one may consider Orlicz modulars [6, 13].

Next, we introduce a weakening of (UUC) as it was done in Banach spaces by Garkavi [8] and Zizler [27].

Definition 3.4. Let $\rho \in \Re$. We say that ρ is uniformly convex in every direction (UCED) if for any $r > 0$ and $h \in L_\rho$ such that $h \neq 0$, we have
$$\delta(r,h) = \inf \left\{ 1 - \frac{1}{r}\ \rho\Big(f + \frac{h}{2}\Big) : \rho(f) \leq r,\ \rho(f+h) \leq r \right\} > 0.$$
We say that ρ is $(UUCED)$ if there exists $\eta(s,h) > 0$, for every $s \geq 0$, and $h \neq 0$ such that
$$\delta(r,h) > \eta(s,h), \quad for\ r > s.$$

It is quite easy to show that if ρ is (UC) (resp. (UUC)), then it is $(UCED)$ (resp. $(UUCED)$). The following lemma will be crucial in the proof of our next result and it is seen as an improvement to Lemma 2.12.

Lemma 3.5. *Assume that $\rho \in \Re$ is $(UUCED)$ and is uniformly continuous. Assume that L_ρ satisfies the property (R). Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Let K be ρ-closed convex nonempty subset of C. Let $\{f_n\}$ be in C. Consider the type $\tau : K \to [0,+\infty]$ defined by $\tau(f) = \limsup_{n\to+\infty} \rho(f - f_n)$. Then τ has a unique minimum point in K.*

Proof. First, note that $\tau(f) \leq diam_\rho(C) < +\infty$. Hence
$$\tau_0 = \inf_{f \in K} \tau(f) \leq diam_\rho(C) < +\infty.$$

Next consider a sequence $\{g_m\}$ in K which ρ-converges to $g \in K$. Let us prove that $\tau(g) \leq \liminf_{m\to+\infty} \tau(g_m)$, i.e., τ is ρ-lower semi-continuous. Fix $\varepsilon > 0$ and take $R = diam_\rho(C)$ is the definition of uniform continuity of ρ. Then there exists $\eta > 0$ such that $|\rho(f) - \rho(f+h)| \leq \varepsilon$, whenever $\rho(f) \leq \eta$ and $\rho(h) \leq R$. Since $\{g_m\}$ ρ-converges to g, there exists $m_0 \geq 1$ such that for any $m \geq m_0$ we have $\rho(g_m - g) < \eta$. Since $\rho(g - f_n) \leq diam_\rho(C)$, we conclude that
$$\Big|\rho(g - f_n) - \rho(g_m - g + g - f_n)\Big| = \Big|\rho(g - f_n) - \rho(g_m - f_n)\Big| \leq \varepsilon$$

for any $n \in \mathbb{N}$. In particular, we have $\rho(g - f_n) \leq \rho(g_m - f_n) + \varepsilon$, for any $m \geq m_0$ and $n \in \mathbb{N}$, which implies $\tau(g) \leq \tau(g_m) + \varepsilon$, for any $m \geq m_0$. Hence

$$\tau(g) \leq \liminf_{m \to +\infty} \tau(g_m) + \varepsilon$$

for any $\varepsilon > 0$. Therefore, we must have $\tau(g) \leq \liminf_{m \to +\infty} \tau(g_m)$. By using this result, it is obvious that $K_r = \{f \in K : \tau(f) \leq \inf_{g \in K} \tau(g) + r\}$ is ρ-closed convex and nonempty subset of K, for any $r > 0$, since ρ is convex. The property (R) is satisfied by L_ρ then it will imply $K_\infty = \bigcap_{n \geq 1} K_{1/n} \neq \emptyset$. Let $f \in K_\infty$. Then we have $\tau(f) \leq \inf_{g \in K} \tau(g) + \frac{1}{n}$, for any $n \geq 1$. Hence $\tau(f) \leq \inf_{g \in K} \tau(g)$, which implies $\tau(f) = \inf_{g \in K} \tau(g)$. Therefore any point in K_∞ is a minimum point of τ in K. Next we prove that K_∞ is in fact reduced to one point. Assume not. Then there exist $h_1, h_2 \in K_\infty$ such that $h_1 \neq h_2$. Set $h = h_1 - h_2$, then $h \neq 0$. Since $\tau(h_1) = \tau(h_2) = \tau_0 = \inf_{g \in K} \tau(g)$, then we can assume without loss of any generality that $\tau_0 > 0$. Otherwise if $\tau_0 = 0$, the sequence $\{f_n\}$ will ρ-converge to both h_1 and h_2 which will contradict the uniqueness of the ρ-limit. By using the definition of τ, there exists $n_0 \geq 1$ such that

$$\rho(f_n - h_1) \leq \tau_0 + \varepsilon \quad \text{and} \quad \rho(f_n - h_2) \leq \tau_0 + \varepsilon$$

for any $n \geq n_0$. Since ρ is $(UUCED)$, there exists $\eta(\tau_0, h) > 0$ such that

$$\rho\left(f_n - h_1 + \frac{h_1 - h_2}{2}\right) \leq (\tau_0 + \varepsilon)(1 - \delta(\tau_0 + \varepsilon, h)) \leq (\tau_0 + \varepsilon)(1 - \eta(\tau_0, h)).$$

Since $\rho\left(f_n - h_1 + \frac{h_1 - h_2}{2}\right) = \rho\left(f_n - \frac{h_1 + h_2}{2}\right)$, we get

$$\rho\left(f_n - \frac{h_1 + h_2}{2}\right) \leq (\tau_0 + \varepsilon)(1 - \eta(\tau_0, h))$$

for any $n \geq n_0$. Hence

$$\tau_0 = \tau\left(\frac{h_1 + h_2}{2}\right) \leq (\tau_0 + \varepsilon)(1 - \eta(\tau_0, h))$$

for any $\varepsilon > 0$. Therefore, we must have $\tau_0 \leq \tau_0(1 - \eta(\tau_0, h))$. This contradiction forces K_∞ to be reduced to one point, i.e., τ has a unique minimum point. □

Now we are ready to prove an improved version of Theorem 3.1.

Theorem 3.6. *Assume that $\rho \in \Re$ is $(UUCED)$ and is uniformly continuous. Assume that L_ρ satisfies the property (R). Let C be a nonempty convex ρ-closed ρ-bounded subset of L_ρ not reduced to one point. Let $T : C \to C$ be a monotone ρ-nonexpansive mapping. Assume there exists f_0 such that f_0 and $T(f_0)$ are comparable. Then T has a fixed point.*

Proof. Without loss of generality, assume that $f_0 \leq T(f_0)$ ρ-a.e.. Consider the Krasnoselskii-Ishikawa sequence $\{f_n\}$ generated by (3.1) starting at f_0 with $\lambda \in (0, 1)$. Since L_ρ satisfies the property (R) and by using the properties of $\{f_n\}$, we have

$$C_\infty = \bigcap_{n \geq 0} [f_n, \to) \cap C = \bigcap_{n \geq 0} \{g \in C;\ f_n \leq g \ \rho\text{-a.e.}\} \neq \emptyset.$$

Let $g \in C_\infty$, then $f_n \leq g$ ρ-a.e. and since T is monotone, we get $f_n \leq T(f_n) \leq T(g)$ ρ-a.e., for any $n \geq 0$, i.e., $T(C_\infty) \subset C_\infty$. Consider the type function $\tau : C_\infty \to [0, +\infty)$ generated by $\{f_n\}$, i.e.,

$\tau(g) = \limsup_{n\to+\infty} \rho(f_n - g)$. By using Lemma 3.5, we know that τ has a unique minimum point $g \in C_\infty$. Since T is monotone ρ-nonexpansive, we have

$$\begin{aligned}
\tau(T(g)) &= \limsup_{n\to+\infty} \rho(f_{n+1} - T(g)) \\
&\le \limsup_{n\to+\infty} (1-\lambda)\rho(f_n - T(g)) + \lambda\, \rho(T(f_n) - T(g)) \\
&\le \limsup_{n\to+\infty} (1-\lambda)\rho(f_n - T(g)) + \lambda\, \rho(f_n - g) \\
&\le (1-\lambda) \limsup_{n\to+\infty} \rho(f_n - T(g)) + \lambda \limsup_{n\to+\infty} \rho(f_n - g) \\
&\le (1-\lambda)\, \tau(T(g)) + \lambda\, \tau(g),
\end{aligned}$$

which implies $\tau(T(g)) \le \tau(g)$, since $\lambda \in (0,1)$. Therefore, $T(g)$ is also a minimum point of τ in C_∞ since $T(C_\infty) \subset C_\infty$. The uniqueness of the minimum point of τ implies that $T(g) = g$, i.e., g is a fixed point of T. □

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