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FIXED POINTS OF MONOTONE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

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Abstract. Let C be a nonempty, ρ-bounded, ρ-closed, and convex subset of a modular function space L_ρ and $T : C \to C$ be a monotone asymptotically ρ-nonexpansive mapping. In this paper, we investigate the existence of fixed points of T. In particular, we establish a modular monotone analogue to the original Goebel and Kirk's fixed point theorem for asymptotically nonexpansive mappings. We will also investigate the behavior of the modified Mann iteration process defined by

$$f_{n+1} = \alpha\, T^n(f_n) + (1-\alpha) f_n,$$

for $n \in \mathbb{N}$ and establish the analogue to Schu's fundamental results in the setting of modular function spaces.

1. Introduction

Modular function spaces (MFS) find their roots in the study of the classical function spaces $L^p(\Omega)$ and their extensions by many like Orlicz spaces for example. For more details on MFS, we recommend the book by Kozlowski [11]. Another interesting use of the modular structure, for whoever is looking for more applications, is the excellent book by Diening et al. [4] about Lebesgue and Sobolev spaces with variable exponents. Fixed point theory in MFS was initiated in 1990 in the original paper [9]. Since then this theory has seen an explosion which culminated in the publication of the recent book by Khamsi and Kozlowski [8]. In this work, we continue investigating the fixed point problem in MFS. To be precise, we investigate the case of monotone mappings. This area of metric fixed point theory is new and attracted some attention after the publication of Ran and Reuring's paper [15]. An interesting reference with many applications of the fixed point theory of monotone mappings is the excellent book by Carl and Heikkilä [3].

Since this work deal with the metric fixed point theory, we recommend the book by Khamsi and Kirk [6].

2. Preliminaries

For the basic definitions and properties of MFS, we refer the readers to the books [8, 11].

Throughout this work, Δ stands for a nonempty set, Σ a nontrivial σ-algebra of

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subsets of Δ, $\mathcal{P}$ a δ-ring of subsets of Δ such that $P \cap S \in \mathcal{P}$ for any $P \in \mathcal{P}$ and $S \in \Sigma$. We will assume that there exists an increasing sequence $\{\Delta_n\} \subset \mathcal{P}$ such that $\Delta = \bigcup \Delta_n$. $\mathcal{M}_\infty$ will stand for the space of all extended measurable functions $f : \Delta \to [-\infty, \infty]$ for which there exists $\{g_n\} \subset \mathcal{E}$, with $|g_n| \leq |f|$ and $g_n(t) \to f(t)$, for all $t \in \Delta$, where $\mathcal{E}$ stands for the vector space of simple functions whose supports is in $\mathcal{P}$.

Definition 2.1 ([8, 11])**.** A convex and even function $\rho : \mathcal{M}_\infty \to [0, \infty]$ is called a regular modular if:

(i) $\rho(f) = 0$ implies $f = 0$ $\rho - a.e.$;
(ii) $|f(t)| \leq |g(t)|$ for all $t \in \Delta$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$ (we will say that ρ is monotone);
(iii) $|f_n(t)| \uparrow |f(t)|$ for all $t \in \Delta$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$ (ρ has the Fatou property).

Recall that a subset $A \in \Sigma$ is said to be ρ-null if $\rho(g 1_A) = 0$, for any $g \in \mathcal{E}$, and a property holds ρ-almost everywhere (ρ-a.e.) if the exceptional set is ρ-null. The notation 1_A denotes the characteristic function of the set A. Consider the set

$$\mathcal{M} = \{f \in \mathcal{M}_\infty; |f(t)| < \infty \ \rho - a.e\}.$$

The MFS L_ρ is defined as:

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

In the following theorem we recall some of the properties of modular spaces that will be used throughout this work.

Theorem 2.2 ([8, 11])**.** *Let ρ be a convex regular modular.*

(1) *If $\rho(\beta f_n) \to 0$, for some $\beta > 0$, then there exists a subsequence $\{f_{\psi(n)}\}$ such that $f_{\psi(n)} \to 0$ $\rho - a.e.$*
(2) *If $f_n \to f$ $\rho - a.e.$, then $\rho(g) \leq \liminf_{n\to\infty} \rho(g_n)$.*
(3) *ρ satisfies the Δ_2-type condition if*

$$\omega(\alpha) = \sup \left\{ \frac{\rho(\alpha g)}{\rho(g)}, \ 0 < \rho(g) < \infty \right\} < \infty$$

for any $\alpha \in [0, +\infty)$. If ρ satisfies the Δ_2-type condition, then we have $\rho(\alpha f_n) \to 0$ if and only if $\rho(\alpha f_n) \to 0$, for any $\alpha > 0$.

The following definition is needed since it connects the metric properties with its modular version.

Definition 2.3 ([8, 11])**.** Let ρ be a convex regular modular.

(1) $\{g_n\}$ is said to ρ-converge to g if $\lim_{n\to\infty} \rho(g_n - g) = 0$.
(2) A sequence $\{g_n\}$ is called ρ-Cauchy if $\lim_{n,m\to\infty} \rho(g_n - g_m) = 0$.
(3) A subset C of L_ρ is said to be ρ-closed if for any sequence $\{g_n\}$ in C ρ-convergent to g implies that $g \in C$.
(4) A subset C of L_ρ is called ρ-bounded if its ρ-diameter $\sup\{\rho(g - h); \ g, \ h \in C\} < \infty$.

Note that despite the fact that ρ does not satisfy the triangle inequality in general, the ρ limit is unique and ρ-convergence may not imply ρ-Cauchy behavior. But it is interesting to know that ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \leq r\}$ are ρ-closed, and any ρ-Cauchy sequence in L_ρ is ρ-convergent, i.e. L_ρ is ρ-complete [8, 11].

Using Theorem 2.2, we get the following result:

Theorem 2.4. *Let ρ be a convex regular modular. Let $\{g_n\} \subset L_\rho$ be a sequence which ρ-converges to g. The following hold:*

(i) *if $\{g_n\}$ is monotone increasing, i.e., $g_n \leq g_{n+1}$ ρ-a.e., for any $n \geq 1$, then $g_n \leq g$ ρ-a.e., for any $n \geq 1$.;*

(ii) *if $\{g_n\}$ is monotone decreasing, i.e., $g_{n+1} \leq g_n$ ρ-a.e., for any $n \geq 1$, then $g \leq g_n$ ρ-a.e., for any $n \geq 1$.*

Next we discuss a property called uniform convexity which plays an important part in metric fixed point theory.

Definition 2.5 ([8])**.** Let ρ be a convex regular modular.

(i) Let $r > 0$ and $\varepsilon > 0$. Define

$$\delta_\rho(r, \varepsilon) = \inf \Big\{1 - \frac{1}{r}\, \rho\Big(\frac{f+g}{2}\Big);\ (f, g) \in D(r, \varepsilon)\Big\},$$

where

$$D(r, \varepsilon) = \{(f, g); f, g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r, \rho(f - g) \geq \varepsilon r\}.$$

ρ is said to be uniformly convex (UC) if for every $R > 0$ and $\varepsilon > 0$, we have $\delta_\rho(R, \varepsilon) > 0$.

(ii) ρ is said to be (UUC) if for every $s \geq 0, \varepsilon > 0$ there exists $\eta(s, \varepsilon) > 0$ such that $\delta_\rho(R, \varepsilon) > \eta(s, \varepsilon) > 0$, for $R > s$.

(iii) ρ is said to be strictly convex (SC), if for any g and h in L_ρ with $\rho(g) = \rho(h)$ and $\rho(\alpha\ g + (1 - \alpha)h) = \alpha\ \rho(g) + (1 - \alpha)\rho(h)$, for some $\alpha \in (0, 1)$, we must have $f = g$.

Note that the uniform convexity of ρ easily implies (SC).

Remark 2.6. It is known that, under suitable assumptions, the uniform convexity of the modular in Orlicz spaces is satisfied iff the Orlicz function is uniformly convex [10, 17]. Examples of Orlicz functions that do not satisfy the Δ_2 condition and are uniformly convex are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$ [14, 13].

Modular functions which are uniformly convex enjoys a property similar to reflexivity in Banach spaces.

Theorem 2.7 ([8, 10])**.** *Let ρ be a (UUC) convex regular modular. Then L_ρ has property (R), i.e. every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.*

Remark 2.8. Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded convex ρ-closed nonempty subset of L_ρ. Let $\{f_n\} \subset K$ be a monotone increasing sequence. Since order intervals in L_ρ are convex and ρ-closed, then the property (R) implies

$$\bigcap_{n \geq 1} \Big\{f \in K;\ f_n \leq f\ \ \rho - a.e.\Big\} \neq \emptyset.$$

In other words, there exists $f \in K$ such that $f_n \leq f$ ρ-a.e., for any $n \geq 1$. A similar conclusion holds for decreasing sequences.

The following lemma is useful throughout this work.

Lemma 2.9 ([7])**.** *Let ρ be a (UUC) convex regular modular. If there exists $R > 0$ and $\alpha \in (0,1)$ with $\limsup_{n\to\infty} \rho(f_n) \leq R$, $\limsup_{n\to\infty} \rho(g_n) \leq R$, and $\lim_{n\to\infty} \rho(\alpha\ f_n + (1-\alpha)\ g_n) = R$, then $\lim_{n\to\infty} \rho(f_n - g_n) \to 0$ holds.*

The concept of ρ-type functions will prove to be an important tool dealing with the existence of fixed points.

Definition 2.10 ([10])**.** Let ρ be a convex regular modular. Let C be a nonempty subset of L_ρ. A function $\tau : C \to [0,\infty]$ is called a ρ-type if there exists a sequence $\{g_m\}$ of elements of L_ρ such that

$$\tau(f) = \limsup_{m\to\infty} \rho(g_m - f),$$

for any $f \in C$. Let τ be a type. A sequence $\{f_n\}$ is called a minimizing sequence of τ in C if $\lim_{n\to\infty} \tau(f_n) = \inf\{\tau(f);\ f \in C\}$.

We have the following amazing result about type functions in MFS.

Lemma 2.11 ([7])**.** *Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded ρ-closed convex nonempty subset of L_ρ. Then any minimizing sequence of any ρ-type defined on K is ρ-convergent. Its limit is independent of the minimizing sequence.*

Before we finish this section, let us give the modular definitions of monotone Lipschitzian mappings. The definitions are straightforward generalizations of their norm and metric equivalents.

Definition 2.12. Let ρ be a convex regular modular. Let K be nonempty subset of L_ρ. A mapping $T : K \to K$ is said to be monotone if $T(f) \leq T(g)$ ρ-a.e. whenever $f \leq g$ ρ-a.e., for any $f, g \in K$. Moreover T is called monotone asymptotically nonexpansive if T is monotone and there exists $\{k_n\} \subset [1,+\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\rho(T^n(g) - T^n(h)) \leq k_n\ \rho(g - h),$$

for any g and h in K such that $g \leq h$ ρ-a.e., and $n \geq 1$. $g \in K$ is called a fixed point of T if and only if $T(g) = g$.

3. Monotone asymptotic nonexpansive mappings in modular function spaces

The fixed point theory for asymptotically nonexpansie mappings finds its root in the work of Goebel and Kirk [5]. Following the success of the fixed point theory of monotone mappings, the fixed point theorem of monotone asymptotically nonexpansie mappings was only proved recently [1].

Before we state our main result on monotone asymptotically nonexpansie mappings in MFS, recall that a map T is said to be ρ-continuous if $\{g_n\}$ ρ-converges to g implies $\{T(g_n)\}$ ρ-converges to $T(g)$. A similar result for asymptotically nonexpansive mapping in modular function spaces may be found in [7].

Theorem 3.1. *Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded ρ-closed convex nonempty subset of L_ρ. Let $T : K \to K$ be a ρ-continuous monotone asymptotically nonexpansive mapping. Assume there exists $f_0 \in K$ such that $f_0 \leq T(f_0)$ (resp. $T(f_0) \leq f_0$) ρ-a.e. Then T has a fixed point f such that $f_0 \leq f$ (resp. $f \leq f_0$) ρ-a.e.*

Proof. Without loss of generality assume $f_0 \leq T(f_0)$ ρ-a.e. Since T is monotone, the sequence $\{T^n(f_0)\}$ is monotone increasing. Remark 2.8 implies that $K_\infty = \{f \in K; f_n \leq f \;\; \rho - a.e.\}$ is not empty. Consider the ρ-type function $\varphi : K_\infty \to [0, +\infty)$ defined by

$$\varphi(h) = \limsup_{n\to\infty} \rho(T^n(f_0) - h), \;\; \text{for any } h \in K_\infty.$$

Let $\varphi_0 = \inf\{\varphi(h); h \in K_\infty\}$. Let $\{g_n\} \subset K_\infty$ be a minimizing sequence of φ. Lemma 2.11 implies that $\{g_n\}$ ρ-converges to $g \in K_\infty$. Let us prove that g is a fixed point of T. First notice that $\varphi(T^m(h)) \leq k_m\ \varphi(h)$, for any $h \in K_\infty$ and $m \geq 1$. In particular, we have $\varphi(T^m(g_n)) \leq k_m\ \varphi(g_n)$, for any $n, m \geq 1$. Clearly the sequence $\{T^{n+p}(g_n)\}$ is a minimizing sequence in K_∞, for any $p \in \mathbb{N}$. Again Lemma 2.11 will force $\{T^{n+p}(g_n)\}$ to ρ-converge to g, for any $p \in \mathbb{N}$. Since T is ρ-continuous and $\{T^n(g_n)\}$ is ρ-convergent to g, then $\{T^{n+1}(g_n)\}$ is ρ-convergent to $T(g)$ and g. Since the ρ-limit of any ρ-convergent sequence is unique, we must have $T(g) = g$. Since $g \in K_\infty$, we have $f_0 \leq g$ ρ-a.e., which completes the proof of Theorem 3.1. □

Next we discuss an iteration which will generate an approximate fixed point of monotone asymptotically nonexpansive mapping in MFS. This is useful because it allows for an algorithm with possible use in computational science.

Definition 3.2 ([16])**.** Let ρ be a convex regular modular. Let K be a convex nonempty subset of L_ρ. Let $T : K \to K$ be a mapping. Fix $f_0 \in K$ and $\alpha \in [0, 1]$. The modified Mann iteration is the sequence $\{f_n\}$ defined by

$$f_{n+1} = \alpha\ T^n(f_n) + (1 - \alpha)f_n, \tag{3.1}$$

for any $n \in \mathbb{N}$.

We start by proving some Lemmas which will be helpful.

Lemma 3.3. *Let ρ be a convex regular modular. Let K be a convex nonempty subset of L_ρ. Let $T : K \to K$ be a mapping. Let $f_0 \in K$ be such that $f_0 \leq T(f_0)$ (resp. $T(f_0) \leq f_0$) ρ-a.e. Let $\{t_n\} \subset [0, 1]$. Consider the modified Mann iteration sequence $\{f_n\}$ generated by f_0 and $\{t_n\}$. Let f be a fixed point of T such that $f_0 \leq f$ (resp. $f \leq f_0$) ρ-a.e. Then*

(i) *$f_0 \leq f_n \leq f$ (resp. $f \leq f_n \leq f_0$) ρ-a.e.,*
(ii) *$T^n(f_0) \leq T^n(f_n) \leq f$ (resp. $f \leq T^n(f_n) \leq T^n(f_0)$) ρ-a.e.,*

for any $n \in \mathbb{N}$.

Proof. Without loss of generality, assume $f_0 \leq T(f_0)$. Since T is monotone and f is a fixed point of T, we get (ii) from (i). Let us prove by induction (i). Indeed, we have $f_0 \leq T(f_0) \leq T(f) = f$ ρ-a.e. since T is monotone. Using the convexity of the order intervals, we conclude that $f_0 \leq f_1 \leq f$. Assume that $f_0 \leq f_n \leq f$ ρ-a.e. Again using the monotonicity of T, we get

$$f_0 \leq T^n(f_0) \leq T^n(f_n) \leq T^n(f) = f \ \ \rho - a.e.,$$

which implies by convexity of the order intervals that $f_0 \leq f_{n+1} \leq f$ ρ-a.e. By induction, we conclude that $f_0 \leq f_n \leq f$ ρ-a.e., for any $n \in \mathbb{N}$. □

Lemma 3.4. *Let ρ be a convex regular modular. Let K be a convex and ρ-bounded nonempty subset of L_ρ. Assume that the map $T : K \to K$ is monotone asymptotic nonexpansive with the associated constants $\{k_n\}$ satisfy $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $f_0 \in K$ be such that $f_0 \leq T(f_0)$ (resp. $T(f_0) \leq f_0$) ρ-a.e. Let $\alpha \in (0, 1)$. Consider the modified Mann iteration sequence $\{f_n\}$ generated by f_0 and α. Let f be a fixed point of T such that $f_0 \leq f$ (resp. $f \leq f_0$) ρ-a.e. Then $\lim\limits_{n\to\infty} \rho(f_n - f)$ exists.*

Proof. Without loss of generality, assume that $f_0 \leq T(f_0)$ ρ-a.e. From the definition of $\{f_n\}$, we have

$$\begin{array}{rcl} \rho(f_{n+1} - f) & \leq & \alpha\ \rho(T^n(f_n) - f) + (1 - \alpha)\ \rho(f_n - f) \\ & = & \alpha\ \rho(T^n(f_n) - T^n(f)) + (1 - \alpha)\ \rho(f_n - f), \end{array}$$

for any $n \geq 1$. Since T is monotone asymptotic nonexpansive, we get

$$\rho(f_{n+1} - f) \leq k_n\ \rho(f_n - f) = (k_n - 1)\ \rho(f_n - f) + \rho(f_n - f),$$

for any $n \geq 1$. Hence

$$\rho(f_{n+1} - f) - \rho(f_n - f) \leq (k_n - 1)\ \delta_\rho(K),$$

for any $n \in \mathbb{N}$, where $\delta_\rho(K) = \sup\{\rho(h - g);\ h, g \in K\}$ is the ρ-diameter of K. Hence

$$\rho(f_{n+m} - f) - \rho(f_n - f) \leq \delta_\rho(K) \sum_{i=0}^{m-1}(k_{n+i} - 1),$$

for any $n, m \geq 1$. If we let $m \to \infty$, we get

$$\limsup_{m\to\infty} \rho(f_m - f) \leq \rho(f_n - f) + \delta_\rho(K) \sum_{i=n}^{\infty}(k_i - 1),$$

for any $n \geq 1$. Next let $n \to \infty$, we get

$$\limsup_{m\to\infty} \rho(f_m - f) \leq \liminf_{n\to\infty} \rho(f_n - f) + \delta_\rho(K) \liminf_{n\to\infty} \sum_{i=n}^{\infty}(k_i - 1) = \liminf_{n\to\infty} \rho(f_n - f).$$

Therefore, we have $\limsup\limits_{m\to\infty} \rho(f_m - f) = \liminf\limits_{n\to\infty} \rho(f_n - f)$, which implies the desired conclusion. □

The next result shows that the sequence generated by the modified Mann iteration almost provide a fixed point. Similar results for such iteration in modular function spaces may be found in [2, 12].

Theorem 3.5. *Let ρ be a (UUC) convex regular modular. Let $K \subset L_\rho$ be a ρ-bounded ρ-closed convex nonempty subset. Let $T : K \to K$ be a ρ-continuous monotone asymptotically nonexpansive mapping with the associated constants $\{k_n\}$ satisfy $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $f_0 \in K$ be such that $f_0 \leq T(f_0)$ (resp. $T(f_0) \leq f_0$) ρ-a.e. Let $\alpha \in (0,1)$. Consider the modified Mann iteration sequence $\{f_n\}$ generated by f_0 and α. Then either $\{f_n\}$ ρ-converges to f or*

$$\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0.$$

Proof. Assume that $\{f_n\}$ does not ρ-converge to f. Let us prove that $\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0$. Without loss of generality, we assume $f_0 \leq T(f_0)$ ρ-a.e. Using Theorem 3.1, there exists f a fixed point of T such that $f_0 \leq f$ ρ-a.e. Using Lemma 3.4, we conclude that $\lim_{n\to\infty} \rho(f_n - f)$ exists. Set $R = \lim_{n\to\infty} \rho(f_n - f)$. Since $\{f_n\}$ does not ρ-converge to f, we have $R > 0$. We have

$$\limsup_{n\to\infty} \rho(T^n(f_n) - f) = \limsup_{n\to\infty} \rho(T^n(f_n) - T^n(f)) \leq \limsup_{n\to\infty} k_n \ \rho(f_n - f) = R.$$

On the other hand, we have $\rho(f_{n+1} - f) \leq \alpha \ \rho(T^n(f_n) - f) + (1-\alpha) \ \rho(f_n - f)$, for any $n \geq 1$. Let $\mathcal{U}$ be a non-trivial ultrafilter over $\mathbb{N}$. Hence

$$R = \lim_{\mathcal{U}} \rho(f_{n+1} - f) \leq \alpha \ \lim_{\mathcal{U}} \rho(T^n(f_n) - f) + (1-\alpha) \ R.$$

Since $\alpha \neq 0$, we get $\lim_{\mathcal{U}} \rho(T^n(f_n) - f) \geq R$. Hence

$$R \leq \liminf_{n\to\infty} \rho(T^n(f_n) - f) \leq \lim_{\mathcal{U}} \rho(T^n(f_n) - f) \leq \limsup_{n\to\infty} \rho(T^n(f_n) - f) \leq R.$$

So $\lim_{n\to\infty} \rho(T^n(f_n) - f) = R$. Using Lemma 2.9, we conclude that $\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0$, which finishes the proof of our claim. ☐

Remark 3.6. In fact, the modified Mann sequence $\{f_n\}$ is an approximate fixed point sequence of T under suitable conditions. Indeed, assume ρ satisfies the Δ_2-type condition and T is uniformly ρ-Lipschitzian, i.e. there exists $\ell > 0$ such that

$$\rho(T^n(g) - T^n(h)) \leq \ell \ \rho(g - h),$$

for any $g, h \in K$ and $n \geq 1$. In this case, we have

$$\lim_{n\to\infty} \rho(f_n - T^m(f_n)) = 0,$$

for any $m \geq 1$. Indeed, note that

$$\begin{array}{rcl} \rho(f_n - T(f_n)) & \leq & \omega(2) \ \rho\left(\dfrac{f_n - T(f_n)}{2}\right) \\ & \leq & \omega(2) \ \rho(f_n - T^n(f_n)) + \omega(2) \ \rho(T^n(f_n) - T(f_n)) \\ & \leq & \omega(2) \ \rho(f_n - T^n(f_n)) + \omega(2) \ \ell \ \rho(T^{n-1}(f_n) - f_n), \end{array}$$

for any $n \geq 2$. From

$$\begin{array}{rcl} \rho(T^{n-1}(f_n) - f_n) & \leq & \omega(2) \ \rho\left(\dfrac{T^{n-1}(f_n) - f_n}{2}\right) \\ & \leq & \omega(2)\rho(T^{n-1}(f_n) - T^{n-1}(f_{n-1})) + \omega(2)\rho(T^{n-1}(f_{n-1}) - f_n) \\ & \leq & \omega(2)\ell \ \rho(f_n - f_{n-1}) + \omega(2)\rho(T^{n-1}(f_{n-1}) - f_n), \end{array}$$

and $\rho(f_n - f_{n-1}) = \alpha\rho(f_{n-1} - T^{n-1}(f_{n-1}))$, $\rho(T^{n-1}(f_{n-1}) - f_n) = (1-\alpha)\ \rho(f_{n-1} - T^{n-1}(f_{n-1}))$, we get that

$$\rho(T^{n-1}(f_n) - f_n) \leq \omega(2)(\ell+1)\ \rho(f_{n-1} - T^{n-1}(f_{n-1})).$$

Hence

$$\rho(f_n - T(f_n)) \leq \omega(2)\ \rho(f_n - T^n(f_n)) + \omega^2(2)\ (\ell+1)^2\ \rho(f_{n-1} - T^{n-1}(f_{n-1})),$$

for any $n \geq 2$. Since $\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0$, we conclude that

$$\lim_{n\to\infty} \rho(f_n - T(f_n)) = 0,$$

i.e. $\{f_n\}$ is an approximate fixed point sequence of T. Finally let us fix $m \geq 1$. Then

$$\rho(f_n - T^m(f_n)) \leq \omega(m) \sum_{k=0}^{m-1} \rho(T^k(f_n) - T^{k+1}(f_n)) \leq \omega(m) \sum_{k=0}^{m-1} \ell\ \rho(f_n - T(f_n)),$$

which implies that $\rho(f_n - T^m(f_n)) \leq\ m\ \ell\ \omega(m)\ \rho(f_n - T(f_n))$, for any $m \geq 1$. Clearly, this implies

$$\lim_{n\to\infty} \rho(f_n - T^m(f_n)) = 0, \text{ for any } m \geq 1.$$

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