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Strong convergence of a general iteration scheme in $CAT(0)$ spaces

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ABSTRACT

We introduce and study strong convergence of a general iteration scheme for a finite family of asymptotically quasi-nonexpansive maps in convex metric spaces and $CAT(0)$ spaces. The new iteration scheme includes modified Mann and Ishikawa iterations, the three-step iteration scheme of Xu and Noor and the scheme of Khan, Domlo and Fukhar-ud-din as special cases in Banach spaces. Our results are refinements and generalizations of several recent results from the current literature.

1. Introduction and basic definitions

Let T be a self-map on a nonempty subset K of a metric space (X, d). Denote by $F(T) = \{x \in K : T(x) = x\}$ the set of fixed points of T.

The map T is said to be: (i) uniformly L-Lipschitzian if for $L > 0$, we have $d(T^n x, T^n y) \leq L\, d(x, y)$ for $x, y \in K$, and $n \geq 1$; (ii) asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n\, d(x, y)$ for $x, y \in K$, and $n \geq 1$; and (iii) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, p) \leq k_n\, d(x, p)$ for $x \in K$, $p \in F(T)$, and $n \geq 1$.

If $k_n = 1$ for $n \geq 1$ in the above definitions (ii), (iii), then T becomes a nonexpansive and a quasi-nonexpansive map, respectively.

Various iteration processes have been studied for an asymptotically nonexpansive map T on a convex subset K of a normed space E. Schu [2] considered the following modified Mann iterations:

$$x_{n+1} = (1 - a_n)x_n + a_n T^n x_n, \quad n \geq 1, \tag{1.1}$$

where $0 < a_n < 1$.

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Fukhar-ud-din and Khan [3] have studied the modified Ishikawa iterations:

$$x_{n+1} = (1 - a_{n(1)})x_n + a_{n(1)}T^n((1 - a_{n(2)})x_n + a_{n(2)}T^n x_n), \quad n \geq 1 \tag{1.2}$$

where $0 \leq a_{n(1)}, a_{n(2)} \leq 1$, such that $\{a_{n(1)}\}$ is bounded away from 0 and 1 and $\{a_{n(2)}\}$ is bounded away from 1.

Xu and Noor [4] introduced and studied a three-step iteration scheme. Khan et al. [5] have defined a general iteration scheme for a family of maps which extends the scheme of Khan and Takahashi [6] and the three-step iteration scheme of Xu and Noor [4] simultaneously, as follows:

Throughout this paper, we will use $I = \{1, 2, \ldots, k\}$, where $r \geq 1$. Suppose that $a_{in} \in [0, 1]$, $n \geq 1$ and $i \in I$. Let $\{T_i : i \in I\}$ be a family of asymptotically quasi-nonexpansive self-maps of K. Let $x_1 \in K$. The scheme introduced in [5] is

$$\begin{aligned}
x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n\, y_{(k-1)n},\\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n\, y_{(k-2)n},\\
y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n\, y_{(k-3)n},\\
&\cdots\cdots\cdots\cdots\\
y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n\, y_{1n},\\
y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n\, y_{0n},
\end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ for all n.

Very recently, inspired by the scheme (1.3) and the work in [5], Xiao et al. [7] have introduced an $(r + 1)$-step iteration scheme with error terms and studied its strong convergence under weaker boundary conditions.

One of the most interesting aspects of metric fixed point theory is to extend a linear version of a known result to the nonlinear case in metric spaces. To achieve this, Takahashi [8] introduced a convex structure in a metric space (X, d). A map $W: X^2 \times [0, 1] \to X$ is a convex structure in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space together with a convex structure W is known as a convex metric space. A nonempty subset K of a convex metric space is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see [8]). A hyperconvex metric space is another example of a convex metric space. For more on these spaces and their applications, we refer the reader to [9,10].

Let (X, d) be a metric space. A *geodesic* from x to y in X is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists [11].

A geodesic metric space is said to be a *CAT*(0) space if all geodesic triangles of appropriate size satisfy the following *CAT*(0) comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the *CAT*(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete *CAT*(0) spaces are often called *Hadamard spaces* (see [12]). If x, y_1, y_2 are points of a *CAT*(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the *CAT*(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Titz [13]. The above inequality has been extended by Khamsi and Kirk [14] as

$$d^2(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y), \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$. The inequality (CN*) also appeared in [15].

Let us recall that *a geodesic metric space is a CAT*(0) *space if and only if it satisfies the (CN) inequality* (see [11], p. 163). Moreover, if X is a *CAT*(0) metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y)$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1 - \alpha)y : \alpha \in [0, 1]\}$.

In view of the above inequality, $CAT(0)$ spaces have Takahashi's convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y$. A subset K of a $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

The existence of fixed (common fixed) points of one map (or two maps or a family of maps) is not known in many situations. So the approximation of fixed (common fixed) points of one or more nonexpansive, asymptotically nonexpansive, or asymptotically quasi-nonexpansive maps by various iterations have been extensively studied, in Banach spaces, convex metric spaces and $CAT(0)$ spaces (see [2–7,16–21]).

We now translate the scheme (1.3) from the normed space setting to the more general setup of convex metric spaces as follows:

$$x_1 \in K, \qquad x_{n+1} = U_{n(r)}x_n, \quad n \geq 1, \tag{1.4}$$

where

$$\begin{aligned}
&U_{n(0)} = I \text{ (the identity map)},\\
&U_{n(1)}x = W(T_1^n U_{n(0)}x, x, a_{n(1)}),\\
&U_{n(2)}x = W(T_2^n U_{n(1)}x, x, a_{n(2)}),\\
&\ldots\ldots\\
&U_{n(r-1)}x = W(T_{r-1}^n U_{n(r-2)}x, x, a_{n(r-1)})\\
&U_{n(r)}x = W(T_r^n U_{n(r-1)}x, x, a_{n(r)}),
\end{aligned}$$

where $0 \leq a_{n(i)} \leq 1$, for $i \in I$.

In a convex metric space, the scheme (1.4) provides analogues of:

(i) the scheme (1.1) if $r = 1$ and $T_1 = T$;
(ii) the scheme (1.2) if $r = 2$, $T_1 = T_2 = T$ and
(iii) the Xu and Noor [4] iteration scheme if $r = 3$, $T_1 = T_2 = T_3 = T$.

This scheme becomes the scheme (1.3) if we choose a special convex metric space, namely, a normed space.

In this paper, we establish theorems of strong convergence, for the

iteration scheme (1.4), to a common fixed point of a finite family of asymptotically quasi-nonexpansive maps, where the underlying space is either a convex metric space or a $CAT(0)$ space. Our work extends as well as refines several comparable results given in [2–7,16–18].

In the sequel, it is assumed that $F = \bigcap_{i=1}^r F(T_i) \neq \phi$.

2. Results for convex metric spaces

We begin with a technical result.

Lemma 2.1. *Let K be nonempty convex subset of a convex metric space X and let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive self-maps of K with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i)-1) < \infty$. Then for the iteration scheme $\{x_n\}$ in (1.4), we have*

$$\text{(i): } d(x_{n+1}, p) \leq k_n^r d(x_n, p), \quad \text{where } k_n = \max_{1 \leq i \leq r} k_n(i);$$

$$\text{(ii): } d(x_{n+m}, p) \leq s\, d(x_n, p), \quad \text{for } m \geq 1, n \geq 1, p \in F \text{ and for some } s > 0.$$

Proof. (i) It is clear that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ if and only if $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$.

Now for any $p \in F$, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(T_r^n U_{n(r-1)}x_n, x_n, a_{n(r)}), p)\\
&\leq a_{n(r)} d(T_r^n U_{n(r-1)}x_n, p) + (1 - a_{n(r)}) d(x_n, p)\\
&\leq a_{n(r)} k_n d(U_{n(r-1)}x_n, p) + (1 - a_{n(r)}) d(x_n, p)\\
&\leq a_{n(r)} a_{n(r-1)} k_n^2 d(U_{n(r-2)}x_n, p) + (1 - a_{n(r)}) d(x_n, p) + a_{n(r)}(1 - a_{n(r-1)}) d(x_n, p)\\
&\leq a_{n(r)} a_{n(r-1)} k_n^2 d(U_{n(r-2)}x_n, p) + (1 - a_{n(r)}) d(x_n, p) + a_{n(r)}(1 - a_{n(r-1)}) k_n^2 d(x_n, p)\\
&= a_{n(r)} a_{n(r-1)} k_n^2 d(U_{n(r-2)}x_n, p) + (1 - a_{n(r)} a_{n(r-1)}) k_n^2 d(x_n, p)\\
&\qquad \ldots\ldots\ldots\\
&\leq a_{n(r)} a_{n(r-1)} a_{n(r-2)} \ldots a_{n(1)} k_n^r d(p, U_{n(0)}x_n) + (1 - a_{n(r)} a_{n(r-1)} a_{n(r-2)} \ldots a_{n(1)}) k_n^r d(x_n, p).
\end{aligned}$$

That is,

$$d(x_{n+1}, p) \leq k_n^r d(x_n, p). \tag{2.1}$$

(ii) If $x \geq 1$, then $x \leq \exp(x-1)$. Therefore, it follows from (2.1) that

$$\begin{aligned} d(x_{n+m}, p) &\leq k_{n+m-1}^r d(x_{n+m-1}, p) \\ &\leq \exp((rk_{n+m-1} - r)d(x_{n+m-1}, p)) \\ &\leq \exp((rk_{n+m-1} - r)[k_{n+m-2}^r d(x_{n+m-2}, p)]) \\ &\leq \exp((rk_{n+m-1} + rk_{n+m-2} - 2r)d(x_{n+m-2}, p)) \\ &\quad \cdots\cdots\cdots \\ &\leq \exp\left(r \sum_{i=n}^{n+m-1} k_i - mr\right) d(x_n, p) \\ &\leq \exp\left(r \sum_{i=n}^{\infty} k_i - r\right) d(x_n, p) \\ &\leq s\, d(x_n, p), \end{aligned}$$

where $s = \exp(r \sum_{i=1}^{\infty} k_i - r)$.

That is,

$$d(x_{n+m}, p) \leq s\, d(x_n, p) \tag{2.2}$$

for $m \geq 1$, $n \geq 1$, $p \in F$ and for some $s > 0$. □

We need the following lemma for further development.

Lemma 2.2 *(See [5], Lemma 1.1)*. *Let $\{a_n\}$ and $\{u_n\}$ be positive sequences of real numbers such that $a_{n+1} \leq (1+u_n)a_n$ and $\sum_{n=1}^{\infty} u_n < +\infty$.*

Then:

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *if* $\liminf_{n\to\infty} a_n = 0$, *then from* (i), *we get* $\lim_{n\to\infty} a_n = 0$.

We now state and prove the main theorem of this section.

Theorem 2.1. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive self-maps of K with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then the iteration scheme $\{x_n\}$ in (1.4) converges to $p \in F$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From (2.1), we have that

$$d(x_{n+1}, F) \leq k_n^r\, d(x_n, F).$$

We have $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, so $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 2.2. Now $\liminf_{n\to\infty} d(x_n, F) = 0$ reveals that $\lim_{n\to\infty} d(x_n, F) = 0$. Hereafter, we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists an integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{3s} \quad \text{for all } n \geq n_0,$$

where s is as in Lemma 2.1(ii). In particular,

$$d(x_{n_0}, F) < \frac{\varepsilon}{3s}.$$

That is,

$$\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{3s}.$$

So there must exist $p^* \in F$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{2s}.$$

Now, for $n \geq n_0$, we have from the inequality (2.2) that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2sd(x_{n_0}, p^*) < 2s\,\frac{\varepsilon}{2s} = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in X. As X is complete and K is closed, $\{x_n\}$ must converge to a point $q \in K$. We claim that $q \in F$. Indeed, let $\varepsilon' > 0$. Since $\lim_{n\to\infty} x_n = q$, there exists an integer $n_1 \geq 1$ such that

$$d(x_n, q) < \frac{\varepsilon'}{2k_1}, \tag{2.3}$$

for all $n \geq n_1$. Also $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists an integer $n_2 \geq 1$ such that

$$d(x_n, F) < \frac{\varepsilon'}{7k_1}$$

for all $n \geq n_2$. Hence there exists $p' \in F$ such that

$$d(x_{n_j}, p') < \frac{\varepsilon'}{6k_1}. \tag{2.4}$$

Using (2.3) and (2.4), we have, for any fixed $i \in I$,

$$\begin{aligned} d(T_i q, q) &\leq d(T_i q, p') + d(p', T_i x_{n_j}) + d(T_i x_{n_j}, p') + d(x_{n_j}, p') + d(x_{n_j}, q) \\ &\leq k_1 d(q, p') + 2k_1 d(x_{n_j}, p') + d(x_{n_j}, q) \\ &\leq k_1 d(q, x_{n_j}) + k_1 d(x_{n_j}, p') + 2k_1 d(x_{n_j}, p') + d(x_{n_j}, q) \\ &< k_1 \frac{\varepsilon'}{2k_1} + 3k_1 \frac{\varepsilon'}{6k_1} = \varepsilon'. \end{aligned}$$

That is, $d(T_i q, q) < \varepsilon'$, for any arbitrary ε'. Therefore, we have $d(T_i q, q) = 0$. Hence q is a common fixed point of $\{T_i, i \in I\}$. □

Note that every quasi-nonexpansive map is asymptotically quasi-nonexpansive, so we have:

Corollary 2.1. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $\{T_i : i \in I\}$ be a finite family of quasi-nonexpansive self-maps of K. Define the iteration scheme $\{x_n\}$ as*

$$x_1 \in K, \qquad x_{n+1} = U_{n(r)} x_n, \quad n \geq 1,$$

where

$$\begin{aligned} &U_{n(0)} = I \text{ (the identity map)}, \\ &U_{n(1)} x = W(T_1 U_{n(0)} x, x, a_{n(1)}), \\ &U_{n(2)} x = W(T_2 U_{n(1)} x, x, a_{n(2)}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &U_{n(r-1)} x = W(T_{r-1} U_{n(r-2)} x, x, a_{n(r-1)}), \\ &U_{n(r)} x = W(T_r U_{n(r-1)} x, x, a_{n(r)}), \end{aligned}$$

where $0 \leq a_{n(i)} \leq 1$, for $i \in I$. Then sequence $\{x_n\}$ converges to $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Since an asymptotically nonexpansive map is an asymptotically quasi-nonexpansive, so we get the following extension of Theorem 2.5 in [7].

Corollary 2.2. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $\{T_i : i \in I\}$ be a finite family of asymptotically nonexpansive self-maps of K with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then the sequence $\{x_n\}$ in (1.4) converges to $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Recall that a map $T : K \to K$ (a subset of a metric space) is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 2.2. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $\{T_i : i \in I\}$ be a finite family of asymptotically nonexpansive self-maps of K with sequences $\{k_n(i)\} \subset [1, \infty)$ for each $i \in I$, respectively, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$. Then $\{x_n\}$ in (1.4) converges to $p \in F$ provided $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, for each $i \in I$, and one member of the family $\{T_i : i \in I\}$ is semi-compact.*

Proof. Without loss of generality, we assume that T_1 is semi-compact. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in K$. Hence, for any $i \in I$, we have

$$\begin{aligned} d(q, T_i q) &\leq d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i q) \\ &\leq (1 + k_{n_j}) d(q, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) \to 0. \end{aligned}$$

Thus $q \in F$. By Lemma 2.2, $x_n \to q$. □

3. Convergence in $CAT(0)$ spaces

The scheme (1.4) in $CAT(0)$ spaces is translated as follows:

$$x_1 \in K, \qquad x_{n+1} = U_{n(r)}x_n, \quad n \geq 1, \tag{3.1}$$

where

$$\begin{aligned}
&U_{n(0)} = I, \text{ the identity map,}\\
&U_{n(1)}x = a_{n(1)}T_1^n U_{n(0)}x \oplus (1 - a_{n(1)})x,\\
&U_{n(2)}x = a_{n(2)}T_2^n U_{n(1)}x \oplus (1 - a_{n(2)})x,\\
&\ldots\\
&U_{n(r-1)}x = a_{n(r-1)}T_{r-1}^n U_{n(r-2)}x \oplus (1 - a_{n(r-1)})x,\\
&U_{n(r)}x = a_{n(r)}T_r^n U_{n(r-1)}x \oplus (1 - a_{n(r)})x,
\end{aligned}$$

where $0 \leq a_{n(i)} \leq 1$ for each $i \in I$.

We prove some lemmas needed for the development of our main theorem in this section.

Lemma 3.1. *Let K be a nonempty bounded closed convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I\}$ be a family of uniformly L-Lipschitzian self-maps on K. Then for $\{x_n\}$ in (3.1) with $\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0$, we have*

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \textit{for each } i \in I.$$

Proof. Denote $d(x_n, T_i^n x_n)$ by $c_n^{(i)}$ for each $i \in I$. Then

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, U_{n(r)}x_n)\\
&= d(x_n, a_{n(r)}T_r^n U_{n(r-1)}x_n \oplus (1 - a_{n(r)})x_n)\\
&\leq d(x_n, T_r^n x_n) + d(T_r^n x_n, T_r^n U_{n(r-1)}x_n)\\
&\leq c_n^{(r)} + L\, d(x_n, U_{n(r-1)}x_n)\\
&\leq c_n^{(r)} + L\{a_{n(r-1)}d(x_n, T_{r-1}^n U_{n(r-2)}x_n) + (1 - a_{n(r-1)})d(x_n, x_n)\}\\
&\leq c_n^{(r)} + La_{n(r-1)}d(x_n, T_{r-1}^n U_{n(r-2)}x_n)\\
&\leq c_n^{(r)} + La_{n(r-1)}\{d(x_n, T_{r-1}^n x_n) + d(T_{r-1}^n x_n, T_{r-1}^n U_{n(r-2)}x_n)\}\\
&\leq c_n^{(r)} + Lc_n^{(r-1)} + L^2 d(x_n, U_{n(r-2)}x_n).
\end{aligned}$$

Continuing in this way, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq c_n^{(r)} + Lc_n^{(r-1)} + L^2 c_n^{(r-2)} + \cdots + L^r d(x_n, T_1^n x_n)\\
&\leq c_n^{(r)} + Lc_n^{(r-1)} + L^2 c_n^{(r-2)} + \cdots + L^r c_n^{(1)}.
\end{aligned} \tag{3.2}$$

Taking the lim sup on both sides, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.3}$$

Further, observe that

$$\begin{aligned}
d(x_n, T_i x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1}x_{n+1}) + d(T_i^{n+1}x_{n+1}, T_i^{n+1}x_n) + d(T_i^{n+1}x_n, T_i x_n)\\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1}x_{n+1}) + L\, d(x_{n+1}, x_n) + L\, d(x_n, T_i^n x_n)\\
&= (1 + L)d(x_n, x_{n+1}) + c_{n+1}^{(i)} + Lc_n^{(i)}.
\end{aligned} \tag{3.4}$$

Taking the lim sup on both sides in (3.4) and using (3.3) and $\lim_{n\to\infty} c_n^{(i)} = 0$, we get that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \text{for each } i \in I. \qquad \square$$

Lemma 3.2. *Let K be a nonempty bounded closed convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I\}$ be a family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-maps on K with sequences $\{k_n(i)\} \subset [1, \infty)$, such that $\sum_{n=1}^{\infty}(k_n(i) - 1) < \infty$ for each $i \in I$. Then for the sequence $\{x_n\}$ in (3.1) with $0 < \delta \leq a_{n(i)} \leq 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$, we have*

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \textit{for each } i \in I.$$

Proof. Take $p \in F$ and apply the inequality (CN*) to the scheme(3.1) to get

$$\begin{aligned} d^2(x_{n+1},p) &= d^2(a_{n(r)}T_r^nU_{n(r-1)}x_n \oplus (1-a_{n(r)})x_n, p) \\ &\le a_{n(r)}d^2(T_r^nU_{n(r-1)}x_n, p) + (1-a_{n(r)})d^2(x_n,p) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n) \\ &\le a_{n(r)}k_n^2\, d^2(U_{n(r-1)}x_n, p) + (1-a_{n(r)})d^2(x_n,p) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n) \\ &= a_{n(r)}k_n^2\, d^2(a_{n(r-1)}T_{r-1}^nU_{n(r-2)}x_n \oplus (1-a_{n(r-1)})x_n, p) \\ &\quad + (1-a_{n(r)})d^2(x_n,p) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n) \\ &\le a_{n(r)}k_n^2\, [a_{n(r-1)}d^2(p, T_{r-1}^nU_{n(r-2)}x_n) + (1-a_{n(r-1)})d^2(p,x_n) \\ &\quad - a_{n(r-1)}(1-a_{n(r-1)})d^2(x_n, T_{r-1}^nU_{n(r-2)}x_n)] \\ &\quad + (1-a_{n(r)})d^2(x_n,p) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n). \end{aligned}$$

That is,

$$\begin{aligned} d^2(x_{n+1},p) &\le a_{n(r)}a_{n(r-1)}(k_n^2)^2d^2(U_{n(r-2)}x_n, p) + [a_{n(r)}(1-a_{n(r-1)})k_n^2 + (1-a_{n(r)})]d^2(x_n,p) \\ &\quad - a_{n(r)}a_{n(r-1)}(1-a_{n(r-1)})d^2(x_n, T_{r-1}^nU_{n(r-2)}x_n) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n). \end{aligned}$$

After applying the inequality (CN*) to the scheme (3.1) r times, we get

$$\begin{aligned} d^2(x_{n+1},p) &\le \left[\prod_{i=1}^r a_{n(i)} + \left\{\prod_{i=2}^r a_{n(i)} - \prod_{i=1}^r a_{n(i)}\right\} + \left\{\Pi_{i=3}^r a_{n(i)} - \prod_{i=2}^r a_{n(i)}\right\}\right. \\ &\quad \left. + \cdots + \{a_{n(r)} - a_{n(r)}a_{n(r-1)}\}\right](k_n^2)^r d^2(x_n,p) \\ &\quad - (1-a_{n(1)})\prod_{i=1}^r a_{n(i)}d^2(x_n, T_1^nx_n) \\ &\quad - (1-a_{n(2)})\prod_{i=2}^r a_{n(i)}d^2(x_n, T_2^nU_{n(1)}x_n) \\ &\quad \cdots \\ &\quad - (1-a_{n(r)})a_{n(r)}d^2(x_n, T_r^nU_{n(r-1)}x_n). \end{aligned}$$

From the above computation, we have the following r inequalities:

$$d^2(x_{n+1},p) \le (k_n^2)^r d^2(x_n,p) - (1-a_{n(1)})\prod_{i=1}^r a_{n(i)}d^2(x_n, T_1^nx_n) \tag{1}$$

$$d^2(x_{n+1},p) \le (k_n^2)^r d^2(x_n,p) - (1-a_{n(2)})\prod_{i=2}^r a_{n(i)}d^2(x_n, T_2^nU_{n(1)}x_n) \tag{2}$$

$$\cdots$$

$$d^2(x_{n+1},p) \le (k_n^2)^r d^2(x_n,p) - a_{n(r)}a_{n(r-1)}(1-a_{n(r-1)})d^2(x_n, T_{r-1}^nU_{n(r-2)}x_n) \tag{r-1}$$

$$d^2(x_{n+1},p) \le (k_n^2)^r d^2(x_n,p) - a_{n(r)}(1-a_{n(r)})d^2(x_n, T_r^nU_{n(r-1)}x_n). \tag{r}$$

Using $\delta \le a_{n(i)} \le 1-\delta$ in the above (1)–(r) inequalities and then arranging the terms, we have

$$\delta^{r+1}d^2(x_n, T_1^nx_n) \le (k_n^2)^r d^2(x_n,p) - d^2(x_{n+1},p) \tag{1*}$$

$$\delta^r d^2(x_n, T_2^nU_{n(1)}x_n) \le (k_n^2)^r d^2(x_n,p) - d^2(x_{n+1},p) \tag{2*}$$

$$\cdots$$

$$\delta^2 d^2(x_n, T_r^nU_{n(r-1)}x_n) \le (k_n^2)^r d^2(x_n,p) - d^2(x_{n+1},p). \tag{r*}$$

The sequence $\{d(x_n,p)\}$ is convergent and $k_n \to 1$; therefore from the inequalities (1*)–(r*), we deduce

$$\lim_{n\to\infty} d(x_n, T_i^nU_{n(i-1)}x_n) = 0 \quad \text{for } i \in I. \tag{3.5}$$

Further,

$$\begin{aligned} d(x_n, T_2^n x_n) &\le d(x_n, T_2^n U_{n(1)} x_n) + d(T_2^n U_{n(1)} x_n, T_2^n x_n) \\ &\le d(x_n, T_2^n U_{n(1)} x_n) + L\, d(a_{n(1)} T_1^n x_n \oplus (1 - a_{n(1)}) x_n, x_n) \\ &\le d(x_n, T_2^n U_{n(1)} x_n) + L\, a_{n(1)} d(T_1^n x_n, x_n) \end{aligned}$$

together with (3.5) (for $i = 2$) gives that

$$\lim_{n\to\infty} d(x_n, T_2^n x_n) = 0.$$

Similar computations show that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0 \quad \text{for each } i \in I. \tag{3.6}$$

Finally, by Lemma 3.1, we get

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \text{for each } i \in I. \quad \square$$

For further analysis, we need the following concept:

A family of self-maps $\{T_i : i \in I\}$ on a subset K of a metric space (X, d) with at least one common fixed point is said to satisfy Condition (AV) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F)) \le \frac{1}{r} \sum_{i=1}^{r} d(x, T_i x)$$

for all $x \in K$.

Theorem 3.1. *Let K be a nonempty closed convex subset of a $CAT(0)$ space. Let $\{T_i : i \in I\}$ be a family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-maps on K with sequences $\{k_n(i)\} \subset [1, \infty)$, such that $\sum_{n=1}^{\infty} (k_n(i) - 1) < \infty$ for each $i \in I$. If $\{T_i : i \in I\}$ satisfies the Condition (AV), then the sequence $\{x_n\}$ in (3.1) with $0 < \delta \le a_{n(i)} \le 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$ converges to a common fixed point of $\{T_i : i \in I\}$.*

Proof. Immediate from Lemma 3.2 and Theorem 2.1. $\square$

Recall that a convex metric space X is uniformly convex [8] if for $\epsilon > 0$ and $r_0 > 0$, there exists $\alpha(\epsilon) > 0$ such that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \le r_0(1 - \alpha) \tag{3.7}$$

whenever $x, y, z \in X$, $d(z, x) \le r_0$, $d(z, y) \le r_0$, $d(x, y) \ge r_0\epsilon$.

$CAT(0)$ spaces provide wonderful examples of uniformly convex metric spaces (see [22]). The results obtained in Section 3, extend nicely to uniformly convex metric spaces which satisfy an inequality like (CN*), satisfied by $CAT(0)$ spaces.

Remark 3.1. (1) The approximation results about

(i) modified Mann iterations in Hilbert spaces [2],
(ii) modified Ishikawa iterations in Banach spaces [3,6,17], and
(iii) the three-step iteration scheme in uniformly convex Banach spaces from [4,5] are immediate consequences of our results.

(2) Theorem 3.1 extends and improves Theorem 5.7 of Nanjaras and Panyanak [18] to the case of an asymptotically quasi-nonexpansive map defined on an unbounded domain in a $CAT(0)$ space.

(3) The conclusion of Theorem 3.1 also holds if one of the maps in $\{T_i : i \in I\}$ is semi-compact. This, in turn, generalizes Theorem 3.2 of Xiao et al. [7] in the setup of $CAT(0)$ spaces.

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