Nonlinear Analysis 74 (2011) 5956–5961

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Banach operator pairs and common fixed points in hyperconvex metric spaces

N. Hussain [a], M.A. Khamsi [b,*], A. Latif [a]

[a] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[b] Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA

ARTICLE INFO

Article history:
Received 17 March 2011
Accepted 24 May 2011
Communicated by Ravi Agarwal

MSC:
primary 06F30
46B20
47E10

Keywords:
Banach operator pair
Fixed point
Hyperconvex metric space
Nearest point projection
Nonexpansive mapping

ABSTRACT

The purpose of this paper is to establish DeMarr's well-known theorem for an arbitrary family of symmetric Banach operator pairs in hyperconvex metric spaces without the compactness assumption. We also give necessary and sufficient criteria for the existence of a common fixed point of a semigroup of isometric mappings. As an application, several results on the invariant best approximation are proved.

1. Introduction

The celebrated result of the existence of a common fixed point for a nonexpansive commutative family was first established by DeMarr [1] under the assumption that C is a compact convex subset of a normed space X. In 1965, Browder [2] obtained the corresponding result under the assumption that C is a bounded, closed and convex subset of a uniformly convex Banach space X. In 1992, Khamsi et al. [3] established the above mentioned results for a finite as well as an arbitrary commutative family of maps in hyperconvex metric spaces. Recently, Espinola and Hussain [4] proved DeMarr's theorem in uniformly convex metric spaces of type (T). More recently, Chen and Li [5] have introduced the concept of symmetric Banach operator pairs and extended DeMarr's result to the family of these operators.

In this paper we establish DeMarr's result for an arbitrary family of symmetric Banach operator pairs in hyperconvex metric spaces without the compactness assumption. We also give necessary and sufficient criteria for a common fixed point of a semigroup of isometric mappings. As an application, several results on the invariant best approximation are proved.

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [6] who proved that a hyperconvex space is a nonexpansive absolute retract, i.e., it is a nonexpansive retract of any metric space in which it is isometrically embedded. The interest in these spaces goes back to the results of Sine [7] and Soardi [8] who proved independently that the fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many interesting results have appeared in the setting of hyperconvex spaces (see [9–11]).

* Corresponding author. Tel.: +1 915 747 6763; fax: +1 915 747 6502.
E-mail addresses: nhusain@kau.edu.sa (N. Hussain), mohamed@utep.edu (M.A. Khamsi), alatif@kau.edu.sa (A. Latif).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.05.072

2. Banach operator pairs

A metric space H is said to be hyperconvex [6] if given any family $\{x_\alpha\}$ of points of H and any family $\{r_\alpha\}$ of nonnegative real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\bigcap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$.

An *admissible* subset of a metric space H is a set of the form

$$\bigcap_i B(x_i; r_i)$$

where $\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in H$ with respective radii r_i. It is quite easy to see that an admissible subset of a hyperconvex metric space is hyperconvex.

Sine [7] and Soardi [8] proved that if H is a bounded hyperconvex metric space and $T : H \to H$ is nonexpansive, i.e. $d(T(x), T(y)) \leq d(x, y)$ for any $x, y \in H$, then there exists a fixed point $x \in H$, i.e. $T(x) = x$. Moreover the fixed point set $\mathrm{Fix}(T)$ is hyperconvex and consequently is a nonexpansive retract of H.

Definition 2.1. The ordered pair (S, T) of two self-maps of a metric space H is called a Banach operator pair if the set $\mathrm{Fix}(T)$ is S-invariant, namely $S(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(T)$.

Obviously a commuting pair (S, T) is a Banach operator pair but not conversely in general; see [12–14] and Examples 2.1 and 4.1.

Lin and Sine [15] showed that if $\mathcal{T}$ is a commuting family of nonexpansive maps on any hyperconvex space which has a nonempty set of common fixed points, then there is a nonexpansive retraction onto $\mathrm{Fix}(\mathcal{T})$ which commutes with every member of $\mathcal{T}$. But this is not true in general for continuous, noncommuting maps as is clear from the following example.

Example 2.1. Note that $[0, 1]$ is a compact hyperconvex metric space. Let $T : [0, 1] \to [0, 1]$ and $S : [0, 1] \to [0, 1]$ be defined as

$$T(x) = x^2 \quad \text{and} \quad S(x) = 1 - x.$$

Then T and S do not commute. (S, T) is a Banach operator pair while (T, S) is not. It is worth mentioning that $\mathrm{Fix}(T)$ is not a continuous retract of $[0, 1]$.

3. Common fixed points for Banach operator pairs

The study of a common fixed point of commuting mappings is an old one. In fact right after the first fixed point result was published, the common fixed point problem was investigated. DeMarr's results [1] may be considered as one of the pioneer works in this direction (see also [16–19]). This problem becomes more challenging and seems to be of vital interest in view of a historically significant and negatively settled problem that a pair of commuting continuous self-mappings on the unit interval $[0, 1]$ need not have a common fixed point [20]. Since then, many fixed point theorists have attempted to find weaker forms of commutativity that may ensure the existence of a common fixed point for a pair of self-mappings on a metric space. In this context, the notions of weakly compatible mappings [21] and Banach operator pairs [12,14] have been of significant interest for generalizing results in metric fixed point theory for single-valued mappings.

We first prove the following fixed point result.

Lemma 3.1. *Let H be a hyperconvex metric space. Let $T : H \to H$ be continuous such that $\overline{T(H)}$ is compact. Then there exists $K \subset H$ compact and hyperconvex such that $T(K) \subset K$. Moreover $\mathrm{Fix}(T)$ is not empty and is compact.*

Proof. Define $K_0 = \overline{T(H)}$. Then any hyperconvex hull K of K_0 in H will be compact. The first part follows from $T(K) \subset T(H) \subset K_0 \subset K$. Next we know from the main result of [22] that T has a fixed point in K. This implies that $\mathrm{Fix}(T)$ is not empty. The compactness of $\mathrm{Fix}(T)$ follows from the fact that it is a closed subset of $\overline{T(H)}$. □

Definition 3.1. The mapping $T : H \to H$ is called an R-map if $\mathrm{Fix}(T)$ is a continuous retract of H.

Note that the fixed point set of continuous functions defined on any compact hyperconvex metric spaces may not be a continuous retract (see the example above).

Theorem 3.1. *Let H be a hyperconvex metric space. Let $T : H \to H$ be a continuous R-map such that $\overline{T(H)}$ is compact. Let $S : H \to H$ be continuous such that (S, T) is a Banach operator pair. Then $F(S, T)$ is not empty.*

Proof. From the above lemma, we know that $\mathrm{Fix}(T)$ is not empty and is a compact subset of H. Since T is an R-map, then there exists a continuous retract $R : H \to \mathrm{Fix}(T)$. Since (S, T) is a Banach pair of operators, then $S(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$. Hence $S \circ R : H \to H$ is continuous and such that $S \circ R(H) \subset \mathrm{Fix}(T)$. Hence $\overline{S \circ R(H)}$ is compact. The above lemma implies the existence a fixed point of $S \circ R$. Clearly such a fixed point is a fixed point of S which belongs to $\mathrm{Fix}(T)$. Hence $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) = F(S, T)$ is not empty. □

Note that if T is nonexpansive and H is bounded, then $\text{Fix}(T)$ is hyperconvex which implies that $\text{Fix}(T)$ is a nonexpansive absolute retract of H.

If both operators are nonexpansive, then we have a result in hyperconvex metric spaces which does not involve compactness and provides a nonlinear version of a recent result (Theorem 2.1) from [5].

Theorem 3.2. *Let H be a bounded hyperconvex metric space. Let $T : H \to H$ be nonexpansive. Let $S : H \to H$ be nonexpansive and such that (S, T) is a Banach operator pair. Then $F(S, T)$ is not empty.*

Another similar result which does not involve compactness nor boundedness is found in complete trees [23].

Theorem 3.3. *Let H be a complete $\mathbb{R}$-tree, and suppose that K is a closed convex subset of H which does not contain a geodesic ray. Let $S, T : H \to H$ be nonexpansive maps such that (S, T) is a Banach operator pair. Then $F(S, T)$ is not empty.*

4. The fixed point of a Banach operator family

The aim of this section is to study the class of symmetric Banach operator pairs which properly contains the class of commuting maps and is different from the class of weakly compatible maps. We will prove DeMarr's result for this newly defined class in the setting of hyperconvex metric space without the compactness of the domain. We also characterize necessary and sufficient conditions for a common fixed point of an invertible semigroup of isometric mappings.

Definition 4.1. Let T and S be two self-maps of a metric space H. The pair (S, T) is called a symmetric Banach operator pair if both (S, T) and (T, S) are Banach operator pairs, i.e., $T(\text{Fix}(S)) \subseteq \text{Fix}(S)$ and $S(\text{Fix}(T)) \subseteq \text{Fix}(T)$.

It is easy to see that the pair (S, T) is a symmetric Banach operator pair if and only if T and S are commuting on $\text{Fix}(T) \bigcup \text{Fix}(S)$. The maps S and T are called weakly compatible [21] if they commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$. We say that S and T are nontrivially weakly compatible if and only if they are weakly compatible and $S(x) = T(x)$ for at least one $x \in X$. The author in [21] obtained the common fixed point of the family

$$K_g = \{f : X \to X;\ f \text{ is continuous and nontrivially weakly compatible with } g\}$$

in the setting of the Hausdorff topological space X. It is worth to note here that the class of symmetric Banach operator pairs is different from that of weakly compatible maps as is clear from the following example.

Example 4.1. Consider $X = \mathbb{R}^2$ with the usual norm. Define T and S on X as follows:

$$T(x, y) = \left(x^3 + x - 1, \frac{\sqrt[3]{x^2 + y^3 - 1}}{3}\right),$$

$$S(x, y) = \left(x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1}\right).$$

Then

$$\text{Fix}(T) = \{(1, 0)\}; \qquad \text{Fix}(S) = \{(1, y) : y \in R^1\};$$
$$C(S, T) = \{(x, y) : y = \sqrt[3]{1 - x^2},\ x \in R^1\};$$
$$T(\text{Fix}(S)) = \{T(1, y) : y \in R^1\} = \left\{\left(1, \frac{y}{3}\right) : y \in R^1\right\} \subseteq \{(1, y) : y \in R^1\} = \text{Fix}(S)$$
$$S(\text{Fix}(T)) = S(\{(1, 0)\}) = \{(1, 0)\} = \text{Fix}(T).$$

Obviously, (S, T) is a symmetric Banach operator pair. It is easy to see that T and S do not commute on the set of coincidence points, $C(S, T)$, of S and T. Thus S and T are not weakly compatible and hence not commuting.

Let H be a hyperconvex metric space and $\mathcal{T}$ be a family of mappings defined on H. Then the family $\mathcal{T}$ has a common fixed point if it is the fixed point of each member of $\mathcal{T}$.

Theorem 4.1. *Let H be a hyperconvex metric space. Let $\mathcal{T}$ be a family of nonexpansive mappings defined on H. Assume that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point provided one map from $\mathcal{T}$ has a bounded nonempty fixed point set. Moreover the common fixed point set $F(\mathcal{T})$ is hyperconvex.*

Proof. Let $T_0 \in \mathcal{T}$ be the map for which $F(T_0) = H_0$ is nonempty and bounded. Clearly H_0 is hyperconvex since T_0 is nonexpansive. Since any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair, then for any $T \in \mathcal{T}$, we have $T(H_0) \subset H_0$. Since H_0 is bounded, T has a fixed point in H_0. The fixed point set of T in H_0 is $F(T) \cap F(T_0)$ and is hyperconvex. Suppose that $S \in \mathcal{T}$. Then $S\Big(F(T) \cap F(T_0)\Big) \subset F(T) \cap F(T_0)$. Since $F(T) \cap F(T_0)$ is bounded and hyperconvex, then S has a fixed point in $F(T) \cap F(T_0)$. The fixed point set of S in $F(T) \cap F(T_0)$ is $F(S) \cap F(T) \cap F(T_0)$ which is bounded and hyperconvex. By induction, we will prove that any finite subfamily $T_1, \ldots, T_n$ of $\mathcal{T}$ has a nonempty common fixed point set $F(T_1) \cap \cdots \cap F(T_n) \cap H_0$ in H_0 which is hyperconvex. Baillon's main theorem [24] implies that $\bigcap_{T \in \mathcal{T}} F(T) \cap H_0$ is not empty and is hyperconvex. Since

$$F(\mathcal{T}) \cap H_0 = F(\mathcal{T})$$

we conclude that $F(\mathcal{T})$ is not empty and is hyperconvex. □

Remark 4.1. One may relax the boundedness assumption and obtain a similar conclusion when H is an $\mathbb{R}$-tree. This is an extension of Theorem 4.3 of [23].

Note that in DeMarr's result, the domain is assumed to be compact, as in [5, Theorem 3.5].

Lemma 4.1. *Let H be a hyperconvex metric space. Let $\mathcal{T}$ be a family of nonexpansive mappings defined on H. Let τ be a topology on H for which the closed balls are τ-closed. Assume that there exists a bounded subset $A \subset H$ such that $A \subset \overline{T(A)}^{\tau}$, for any $T \in \mathcal{T}$, where $\overline{T(A)}^{\tau}$ is the τ-closure of $T(A)$. Assume that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point.*

Proof. Define $\delta = \mathrm{diam}(A)$. Consider the subset

$$C = \bigcap_{a\in A} B\left(a, \delta\right).$$

Clearly we have $A \subset C$. Suppose that $T \in \mathcal{T}$; then

$$T(C) \subset \bigcap_{a\in A} B\left(T(a), \delta\right),$$

since T is nonexpansive. This implies

$$T(A) \subset \bigcap_{c\in T(C)} B\left(c, \delta\right).$$

Since the closed balls are τ-closed, we get

$$\overline{T(A)}^{\tau} \subset \bigcap_{c\in T(C)} B\left(c, \delta\right).$$

Our assumption implies

$$A \subset \bigcap_{c\in T(C)} B\left(c, \delta\right).$$

Hence

$$T(C) \subset \bigcap_{a\in A} B\left(a, \delta\right) = C.$$

Since C is a bounded hyperconvex metric space, the theorem above implies that $\mathcal{T}$ has a common fixed point. □

Definition 4.2. Let H be a metric space. Let $\mathcal{T}$ be a family of mappings defined on H. The family $\mathcal{T}$ is called a semigroup if $S \circ T \in \mathcal{T}$ whenever $S, T \in \mathcal{T}$. We will call the semigroup $\mathcal{T}$ an invertible semigroup if and only if any element in $\mathcal{T}$ is invertible and $T^{-1} \in \mathcal{T}$ for any $T \in \mathcal{T}$. For any $x \in H$, define the orbit of x by

$$\mathcal{T}(x) = \{T(x); T \in \mathcal{T}\}.$$

Theorem 4.2. *Let H be a hyperconvex metric space. Let $\mathcal{T}$ be an invertible semigroup of isometric mappings defined on H such that any two mappings from $\mathcal{T}$ form a symmetric Banach operator pair. Then the family $\mathcal{T}$ has a common fixed point if and only if $\bigcap_{T\in\mathcal{T}} T(M)$ is not empty and $\mathcal{T}$-orbits are bounded.*

Proof. Clearly if $\mathcal{T}$ has a fixed point, then we have that $\bigcap_{T\in\mathcal{T}} T(M)$ is not empty and $\mathcal{T}$-orbits are bounded. So let us assume that $\bigcap_{T\in\mathcal{T}} T(M)$ is not empty and $\mathcal{T}$-orbits are bounded. Suppose that $x \in \bigcap_{T\in\mathcal{T}} T(M)$. The orbit $\mathcal{T}(x)$ is bounded. Set $A = \mathcal{T}(x)$. Note that $T(A) = A$ for any $A \in \mathcal{T}$. Indeed, by definition of the orbit $\mathcal{T}(x)$, we have $T(A) \subset A$. Suppose that $a \in A$; then there exists $S \in \mathcal{T}$ such that $a = S(x)$. Clearly we have $a = T\left(T^{-1} \circ S(x)\right)$. Since $T^{-1} \circ S \in \mathcal{T}$, we conclude that $a \in T(A)$. Next we consider the admissible subset $C = \bigcap_{a\in A} B(a, \delta)$, where $\delta = \mathrm{diam}(A)$. Obviously $A \subset C$ and C is bounded and hyperconvex. As in the proof of the lemma above, one can easily show that $T(C) \subset C$, for any $T \in \mathcal{T}$. So from the above theorem we conclude that $\mathcal{T}$ has a common fixed point and its fixed point set $\mathrm{Fix}(\mathcal{T})$ is hyperconvex. □

5. The invariant approximation

In [25] Al-Thagafi proved some results on invariant approximations for commuting maps in normed spaces. Hussain [13], Jungck and Hussain [26], Pathak and Hussain [14,27], and Khan and Akbar [28] have extended the work of Al-Thagafi [25] to compatible, C_q-commuting maps, Banach operator pairs and $\mathcal{P}$-operator pairs.

Let H be a hyperconvex metric space. Let C be a nonempty admissible subset of H. Then we know that for any $x \in H$, the subset

$$P_C(x) = \{y \in C; d(x, y) = d(x, C) = \inf_{c \in C} d(x, c)\}$$

is not empty, i.e. C is a proximinal set. In fact this conclusion is still valid even when C is an externally hyperconvex subset of H. Clearly we have

$$P_C(x) = C \cap B(x, d(x, C)).$$

This shows that $P_C(x)$ is an admissible subset of H. In particular, $P_C(x)$ is hyperconvex. In [25], the author obtained invariant approximation results in the linear case. We have here similar results for hyperconvex metric spaces.

Theorem 5.1. *Let H be a hyperconvex metric space and C be a nonempty admissible subset of H. Let $T : H \to H$ be a mapping such that* $\mathrm{Fix}(T)$ *is not empty. Suppose that $x \in \mathrm{Fix}(T)$. Assume that the restriction of T to $B(x, d(x, C))$ is nonexpansive and $T(C) \subset C$. Then we have:*

(i) $T(P_C(x)) \subset P_C(x)$;
(ii) $\mathrm{Fix}(T) \cap P_C(x)$ *is not empty and is hyperconvex.*

Proof. Since $x \in \mathrm{Fix}(T)$ and the restriction of T to $B(x, d(x, C))$ is nonexpansive, we get $T(B(x, d(x, C))) \subset B(x, d(x, C))$. Using the assumption $T(C) \subset C$, we conclude that

$$T(P_C(x)) \subset P_C(x).$$

The restriction of T to $P_C(x)$ is nonexpansive. Since $P_C(x)$ is a bounded hyperconvex subset of H, we deduce that T has a fixed point in $P_C(x)$. Since the fixed point set of a nonexpansive map defined on a hyperconvex metric space is hyperconvex, we conclude that $\mathrm{Fix}(T) \cap P_C(x)$ is hyperconvex. □

The conclusion of the above theorem improves Al-Thagafi's result since we do not assume compactness. Next we look at the above conclusion for Banach operator pairs.

Theorem 5.2. *Let H be a hyperconvex metric space and C be a nonempty admissible subset of H. Let $T : H \to H$ be a mapping such that* $\mathrm{Fix}(T)$ *is not empty. Suppose that $x \in \mathrm{Fix}(T)$. Let $S : H \to H$ be a mapping such that (S, T) is a Banach operator pair on $B(x, d(x, C))$. Assume that the restrictions of T and S to $B(x, d(x, C))$ are nonexpansive, $T(C) \subset C$ and $S(C) \subset C$. Then* $\mathrm{Fix}(S, T) \cap P_C(x)$ *is not empty and is hyperconvex.*

Proof. From the above theorem we know that $\mathrm{Fix}(T) \cap P_C(x)$ is nonempty and hyperconvex. Note that $P_C(x)$ is a subset of $B(x, d(x, C))$. Hence $\mathrm{Fix}(T) \cap P_C(x)$ is bounded. Since $S\Big(\mathrm{Fix}(T) \cap P_C(x)\Big) \subset \mathrm{Fix}(T) \cap P_C(x)$ and is nonexpansive, we conclude that S has a fixed point in $\mathrm{Fix}(T) \cap P_C(x)$, i.e.

$$\mathrm{Fix}(S, T) \cap P_C(x) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \cap P_C(x) \neq \emptyset.$$

The fact that $\mathrm{Fix}(S, T) \cap P_C(x)$ is hyperconvex follows from known results. □

Acknowledgments

The authors gratefully acknowledge the financial support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) represented by the Unit of Research Groups through the grant number (11/31/Gr) for the group entitled Nonlinear Analysis and Applied Mathematics.

The authors thank the referee for pointing out some oversights and calling attention to some related literature.

References

[1] R. DeMarr, Common fixed points for commuting contraction mappings, Pacific J. Math. 13 (1963) 1139–1141.
[2] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA 54 (1965) 1041–1054.
[3] M.A. Khamsi, M. Lin, R. Sine, On the fixed points of commuting nonexpansive maps in hyperconvex spaces, J. Math. Anal. Appl. 168 (1992) 372–380.
[4] R. Espinola, N. Hussain, Common fixed points for multimaps in metric spaces, Fixed Point Theory Appl. 2010 (2010) Article ID 204981, 14 pages.
[5] J. Chen, Z. Li, Banach operator pair and common fixed points for nonexpansive maps, Nonlinear Anal. (2011) doi:10.1016/j.na.2010.12.010.
[6] N. Aronszajn, P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956) 405–439.
[7] R.C. Sine, On linear contraction semigroups in sup norm spaces, Nonlinear Anal. 3 (1979) 885–890.
[8] P. Soardi, Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73 (1979) 25–29.

[9] A.G. Aksoy, M.A. Khamsi, Nonstandard Methods in Fixed Point Theory, Springer-Verlag, New York, Berlin, 1990.
[10] R. Espinola, M.A. Khamsi, in: W.A. Kirk, B. Sims (Eds.), Introduction to Hyperconvex Spaces, in: Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, 2001.
[11] M.A. Khamsi, W.A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, John Wiley & Sons, New York, Brisbane, Toronto, 2001.
[12] J. Chen, Z. Li, Common fixed points for Banach operator pairs in best approximation, J. Math. Anal. Appl. 336 (2007) 1466–1475.
[13] N. Hussain, Common fixed points in best approximation for Banach operator pairs with Ciric type I-contractions, J. Math. Anal. Appl. 338 (2008) 1351–1362.
[14] H.K. Pathak, N. Hussain, Common fixed points for Banach operator pairs with applications, Nonlinear Anal. 69 (2008) 2788–2802.
[15] M. Lin, R.C. Sine, Retractions on the fixed point set of semigroups of nonexpansive maps in hyperconvex spaces, Nonlinear Anal. 15 (1990) 943–954.
[16] L.P. Belluce, W.A. Kirk, Fixed-point theorems for families of contraction mappings, Pacific J. Math. 18 (1966) 213–217.
[17] R.E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappings, Pacific J. Math. 53 (1974) 59–71.
[18] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976) 261–263.
[19] T.C. Lim, A fixed point theorem for families of nonexpansive mappings, Pacific J. Math. 53 (1974) 487–493.
[20] J.P. Huneke, On common fixed points of commuting continuous functions on an interval, Trans. Amer. Math. Soc. 139 (1969) 371–381.
[21] G. Jungck, Common fixed point theorems for compatible self maps of Hausdorff topological spaces, Fixed Point Theory Appl. 2005 (2005) 355–363.
[22] M.A. Khamsi, KKM and Ky Fan theorems in hyperconvex metric spaces, J. Math. Anal. Appl. 204 (1996) 298–306.
[23] R. Espinola, W.A. Kirk, Fixed point theorems in R-trees with applications to graph theory, Topology Appl. 153 (2006) 1046–1055.
[24] J.B. Baillon, Nonexpansive mappings and hyperconvex spaces, Contemp. Math. 72 (1988) 11–19.
[25] M.A. Al-Thagafi, Common fixed points and best approximation, J. Approx. Theory 85 (1996) 318–323.
[26] G. Jungck, N. Hussain, Compatible maps and invariant approximations, J. Math. Anal. Appl. 325 (2007) 1003–1012.
[27] H.K. Pathak, N. Hussain, Common fixed points for $\mathscr{P}$-operator pair with applications, Appl. Math. Comput. 217 (2010) 3137–3143.
[28] A.R. Khan, F. Akbar, Best simultaneous approximations, asymptotically nonexpansive mappings and variational inequalities in Banach spaces, J. Math. Anal. Appl. 354 (2009) 469–477.