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Goebel and Kirk fixed point theorem for multivalued asymptotically nonexpansive mappings

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ABSTRACT. We introduce the concept of a multivalued asymptotically nonexpansive mapping and establish Goebel and Kirk fixed point theorem for these mappings in uniformly hyperbolic metric spaces. We also define a modified Mann iteration process for this class of mappings and obtain an extension of some well-known results for singlevalued mappings defined on linear as well as nonlinear domains.

1. INTRODUCTION

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $T : X \to Y$ is called Lipschitzian if there exists a constant $k \geq 0$ such that

$$d_Y(T(x), T(y)) \leq k\ d_X(x, y),$$

for any $x, y \in X$. Using iterates of the mapping T, we introduce new classes of mappings. The ones which attracted serious attention were the classes of uniformly Lipschitzian mappings and asymptotically nonexpansive mappings. Recall that asymptotic nonexpansive mappings were introduced and studied in the fundamental paper of Goebel and Kirk [7]. The reader interested in an alternate proof of Goebel and Kirk's fixed point theorem and demiclosedness principle for asymptotically nonexpansive mappings is referred to Khamsi and Kirk [13].
Many contributions have been made in relation to this important class of mappings, we mention here a few of them:

(a) Demiclosedness principle for singlevalued asymptotically nonexpansive mappings on $CAT(0)$ is established by Nanjaras and Panyanak [23].
(b) The notion of total asymptotically nonexpansive mappings has been introduced by Alber et al. [1] and they have approximated their fixed point. Further studies have been made for this new class of mappings by Pansuwan and Sintunavarat [24], Chang et al. [4] and Panyanak [25].
(c) Dhompongsa et al. [5] have considered a homotopy result and an ultrapower approach to establish the existence of fixed points of nonexpansive set-valued mappings on $CAT(0)$ spaces and Banach spaces simultaneously.
(d) Zhang et al. [32] have studied strong convergence of multivalued Bregmann totally quasi-asymptotically nonexpansive mappings.

As, to the best of our knowledge, the case of multivalued asymptotically nonexpansive mappings has not been considered. In this work, we use the ideas developed in [12] to tackle problems about this class of mappings (see also the references [26, 28, 29, 30]).

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For more on metric fixed point theory, we strongly recommend the books [8, 13].

2. Basic definitions and results

Let T be a self-mapping on a subset A of a normed space E. We say that T is asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n \, \|x - y\|,$$

for all $x, y \in A$ and $n \geq 1$. In case, $k_n = 1$, for all $n \geq 1$, T is said to be nonexpansive. Goebel and Kirk fixed point theorem [7] states that if A is a bounded, closed and convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T on A has a fixed point (i.e. $T(x_0) = x_0$ for some x_0 in A). Goebel and Kirk fixed point theorem remains the only well-known existence result for these mappings. Therefore, iterative construction of fixed point of an asymptotically nonexpansive mapping becomes essential.

Throughout this work, (X, d) stands for a metric space. Let C be a nonempty subset of X. We denote by $N(C)$ the collection of all nonempty subsets of C, $\mathcal{C}(C)$ the collection of all nonempty closed subsets of C, and $CB(C)$ the collection of all nonempty closed and bounded subsets of C. For $A, B \in CB(X)$, set

$$H(A,B) = \max\Big\{ \sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B) \Big\},$$

where $d(x,A) = \inf\limits_{a\in A} d(x,a)$ is the distance of x to A. H is known as the generalized Pompeiu-Hausdorff distance induced by d. Note that for any A and B in $CB(X)$, $\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \varepsilon.$$

This remark allows us to avoid use of distance H which imposes restriction on the subsets to be bounded.

The concept of generalized orbits was introduced by Rus [28] and has been subsequently used by many authors (see, for example, [12, 26, 29, 30]). We present this concept here as it was considered in [12].

Definition 2.1. Let C be a nonempty subset of X. For the multivalued mapping $T : C \to N(X)$ and $x \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_0 = x$ and $x_{n+1} \in T(x_n)$, for any $n \geq 0$, will be called a generalized orbit of x.

If T is singlevalued, then generalized orbits coincide with the traditional definition of an orbit. It is clear that for a given $x \in X$, T may have many different generalized orbits generated by x.

Next we define the concept of a multivalued asymptotically nonexpansive mapping.

Definition 2.2. A multivalued mapping $T : X \to N(X)$ is called asymptotically nonexpansive mapping if there exists a sequence of positive numbers $\{k_n\}$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for any $x, y \in X$, and any generalized orbit $\{x_n\}$ of x, there exists a generalized orbit $\{y_n\}$ of y such that

$$d(x_{n+h}, y_h) \leq k_h \, d(x_n, y), \; n, h \in \mathbb{N}.$$

Note that we can always assume that $k_n \geq 1$, for any $n \in \mathbb{N}$.

We recall basics of a hyperbolic metric space.

In order to introduce convexity in metric spaces [22], the essential ingredient is the concept of metric segments. Let x and y be any two points in the metric space (X, d). The metric segment $[x, y]$ is an isometric image of the segment $[0, d(x, y)]$. We will assume that for any x and y in X, there exists a unique metric segment joining them. The unique point $z \in [x, y]$ defined by

$$d(x, z) = (1-t)\, d(x, y) \;\; and \;\; d(y, z) = t\, d(x, y),$$

for $t \in [0, 1]$, will be denoted by $t\, x \oplus (1-t)\, y$. A metric space (X, d) equipped with this class of segments is called a convex metric space. Moreover, if the following holds

$$d\Big(t\, a \oplus (1-t)\, x, t\, b \oplus (1-t)\, y\Big) \leq t\, d(a, b) + (1-t)(x, y),$$

for all a, b, x, y in X, and $t \in [0, 1]$, then X is called a hyperbolic metric space [27]. As in the linear case, a subset C of X will be convex if $[x, y] \subset C$ for any $x, y \in C$.
A natural example of hyperbolic metric spaces is given by normed vector spaces. Hadamard manifolds [3], the Hilbert open unit ball equipped with the hyperbolic distance [9] and CAT(0) metric spaces [2, 17, 18, 19, 20] are examples of nonlinear hyperbolic metric spaces.

Definition 2.3. For a hyperbolic metric space (X, d), we define the modulus of uniform convexity by

$$\delta_X(r, \varepsilon) = \inf\Big\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, z\Big); d(x, z) \leq r, d(y, z) \leq r, d(x, y) \geq r\varepsilon\Big\},$$

for any $r > 0$, $\varepsilon > 0$ and $x, y, z \in X$. X is said to be uniformly convex provided $\delta_X(r, \varepsilon) > 0$, for any $r > 0$ and $\varepsilon > 0$.

Let us recall the definition of a metric type function which plays a major role in metric fixed point theory. These functions are also known as asymptotic centers of a sequence. A function $\tau : X \to [0, \infty)$ is a type function if there exists a bounded sequence $\{x_n\}$ in X such that

$$\tau(x) = \limsup_{n\to\infty} d(x_n, x).$$

If X is hyperbolic, then any type function τ is convex and continuous.

We state important known results in a uniformly convex hyperbolic metric space.

Theorem 2.1. [14, 15] *Let (X, d) be a uniformly convex complete hyperbolic metric space.*

(i) X satisfies the property (R), i.e. for any decreasing sequence of nonempty closed, convex and bounded subsets $\{K_n\}$ in X, we have $\bigcap K_n \neq \emptyset$.
(ii) Let C be a nonempty closed and convex subset of X. Any type function $\tau : X \to [0, \infty)$ has a unique minimum point z in C, i.e.

$$\tau(z) = \inf\{\tau(x); x \in C\}.$$

Moreover, any minimizing sequence $\{z_n\}$ in C, i.e. $\lim_{n\to\infty} \tau(z_n) = \tau(z)$, is convergent.

(iii) Let $R > 0$ and $z \in X$. Assume that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n\to\infty} d(x_n, z) \le R$, $\limsup_{n\to\infty} d(y_n, z) \le R$, and

$$\lim_{n\to\infty} d(\alpha x_n \oplus (1-\alpha)\, y_n, z) = R,$$

then we must have, $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Let $T : C \to N(C)$ be a multivalued mapping. A point $x \in C$ is called a *fixed point* of T if $x \in T(x)$. A generalization of Goebel and Kirk fixed point theorem for nearly uniformly convex Banach spaces in given in [21] and a demiclosed principle for asymptotically nonexpansive mappings on a subclass of metric spaces, namely $CAT(0)$ spaces, has been established in [20]. We intend to give a multivalued version of Goebel and Kirk's fixed point theorem for asymptotically nonexpansive mappings on a very general nonlinear domain. This result will be new in the literature.

A multivalued mapping $T : C \to N(C)$ is H-continuous if whenever $\{x_n\}$ converges to x in C, we have

$$\lim_{n\to\infty} d(a_n, T(x)) = 0,$$

for any sequence $\{a_n\}$ such that $a_n \in T(x_n)$, for any $n \in \mathbb{N}$. In [10, 16], it is proved that H-continuity is equivalent to the lower and upper semi-continuity of T when T is compact-valued. Note that if $T : C \to N(C)$ is asymptotically nonexpansive, then T is H-continuous. Indeed, let $\{k_n\}$ be the Lipschitz sequence of positive numbers associated with T. Let $x, y \in C$. Let $\{x_n\}$ be a generalized orbit of x. Then there exists a generalized orbit $\{y_n\}$ of y such that

$$d(x_{n+h}, y_h) \le k_h\, d(x_n, y),\ n, h \in \mathbb{N}.$$

In particular, we have $d(x_1, y_1) \le k_1\, d(x, y)$ which implies

$$d(x_1, T(y)) \le\ d(x_1, y_1) \le\ k_1\, d(x, y),$$

for any $x_1 \in T(x)$. Clearly, this implies that T is H-continuous.

Theorem 2.2. *Let (X, d) be a complete uniformly convex hyperbolic metric space. Let C be a nonempty closed, bounded and convex subset of X. Let $T : C \to \mathcal{C}(C)$, i.e. $T(x)$ is nonempty and closed subset of C, for any $x \in C$. If T is asymptotically nonexpansive, then T has a fixed point.*

Proof. Let $x_0 \in C$ and $\{x_n\}$ be a generalized orbit of x_0. Since C is bounded, the sequence $\{x_n\}$ is bounded. Consider the type function generated by $\{x_n\}$, i.e. $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$. By (ii) of Theorem 2.1, τ has a unique minimum point z in C. Since T is asymptotically nonexpansive, there exists a sequence of positive numbers $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$ and a generalized orbit $\{z_n\}$ of z such that

$$d(x_{n+h}, z_h) \le k_h\, d(x_n, z),\ n, h \in \mathbb{N}.$$

This will imply $\tau(z_h) \le k_h\, \tau(z)$, for any $h \in \mathbb{N}$. Since $\lim_{n\to\infty} k_n = 1$, we conclude that $\{z_n\}$ is a minimizing sequence of τ. Using again (ii) of Theorem 2.1, we conclude that $\{z_n\}$ converges to z. Since T is H-continuous and $z_{n+1} \in T(z_n)$, for any $n \in \mathbb{N}$, therefore we get

$$\lim_{n\to\infty} d(z_{n+1}, T(z)) = 0.$$

As $T(z)$ is closed, and $\{z_n\}$ converges to z, so we conclude that $z \in T(z)$, i.e. z is a fixed point of T. □

Once an analogue of Goebel and Kirk's fixed point theorem is established for multivalued asymptotically nonexpansive mappings, it is natural to ask whether Schu's iterative approximation [31] (through modified Mann iteration process), may be extended to the multivalued case. Let (X,d) be a complete uniformly convex hyperbolic metric space. Let C be a nonempty closed, bounded and convex subset of X. Let $T : C \to \mathcal{C}(C)$ be asymptotically nonexpansive. Assume that $p \in C$ is a fixed point of T such that $T(p) = \{p\}$. Fix $\alpha \in (0,1)$ and $x_1 \in C$. Since T is asymptotically nonexpansive, there exists $\{k_n\}$ with $\lim\limits_{n\to\infty} k_n = 1$. We will assume that $k_n \geq 1$, for any $n \in \mathbb{N}$. Let $\{x_n^1\}$ be a generalized orbit of x_1. Set $x_2 = \alpha\, x_1 \oplus (1-\alpha)\, x_1^1$. Let $\{x_n^2\}$ be a generalized orbit of x_2 such that

$$d(x_{n+h}^1, x_h^2) \leq\ k_h\, d(x_n^1, x_2), \quad for\ n,h \in \mathbb{N}.$$

By induction, we construct a sequence $\{x_n\}$ in C and for any $m \geq 1$, a sequence $\{x_n^m\}$ which is a generalized orbit of x_m such that

$$d(x_{n+h}^{m-1}, x_h^m) \leq\ k_h\, d(x_n^{m-1}, x_m), \quad for\ n,h \in \mathbb{N},$$

and

(MMI) $$x_{m+1} = \alpha\, x_m \oplus (1-\alpha)\, x_m^m.$$

In view of $T(p) = \{p\}$, we have $d(x_{n+h}^m, p) \leq\ k_h\, d(x_n^m, p)$, for any $n,h \in \mathbb{N}$ and $m \in \mathbb{N}$. Using (MMI) and the hyperbolicity of X, we get

$$\begin{aligned} d(x_{m+1},p) &\leq \alpha\, d(x_m,p) + (1-\alpha)\, d(x_m^m,p) \\ &\leq \alpha\, d(x_m,p) + (1-\alpha)\, k_m\, d(x_m,p) \\ &\leq k_m\, d(x_m,p), \end{aligned}$$

since $k_m \geq 1$, for any $m \in \mathbb{N}$. This will imply

$$d(x_{m+1},p) - d(x_m,p) \leq (k_m - 1)\, d(x_m,p) \leq (k_m-1)\, \delta(C),$$

for any $m \in \mathbb{N}$, where $\delta(C) = \{d(a,b);\ a,b \in C\}$ is the diameter of C. Assume that $\sum\limits_m (k_m - 1)$ is convergent. Then we have

$$d(x_{m+h},p) - d(x_m,p) \leq \delta(C)\ \sum_{i=m}^{m+h-1} (k_i - 1),$$

for any $m,h \in \mathbb{N}$. If we let h go to infinity, we get

$$\limsup_{n\to\infty}\ d(x_n,p) - d(x_m,p) \leq \delta(C)\ \sum_{i=m}^{\infty} (k_i-1),$$

for any $m \in \mathbb{N}$. Now we let m go to infinity to get

$$\limsup_{n\to\infty}\ d(x_n,p) \leq \liminf_{m\to\infty} d(x_m,p),$$

which implies that $\{d(x_n,p)\}$ is convergent. Set $R = \lim\limits_{n\to\infty} d(x_n,p)$. Since $\alpha \in (0,1)$, we easily deduce that $\lim\limits_{n\to\infty} d(x_n^n,p) = R$ as well. If $R = 0$, then we have $\lim\limits_{n\to\infty} d(x_n^n, x_n) = 0$. Otherwise, assume $R > 0$. Then we have

$$\lim_{n\to\infty} d(x_{n+1},p) = \lim_{n\to\infty} d(\alpha x_n \oplus (1-\alpha)x_n^n, p) = R.$$

Using (iii) of Theorem 2.1, we conclude that $\lim\limits_{n\to\infty} d(x_n^n, x_n) = 0$. In fact, from the choice of our generalized orbits, we have

$$\lim_{n\to\infty} d(x_1^n, x_n) = 0,$$

i.e., $\{x_n\}$ is an approximate fixed point sequence of T since $x_1^n \in T(x_n)$. Indeed, we have

$$\begin{array}{lll} d(x_n, x_1^n) & \leq & d(x_n, x_n^n) + d(x_n^n, x_n^{n-1}) + d(x_n^{n-1}, x_1^n) \\ & \leq & d(x_n, x_n^n) + k_n\, d(x_n, x_{n-1}) + k_1\, d(x_{n-1}^{n-1}, x_n) \\ & \leq & d(x_n, x_n^n) + k_n\, (1-\alpha) d(x_{n-1}^{n-1}, x_{n-1}) + k_1\, \alpha d(x_n^{n-1}, x_{n-1}) \\ & \leq & d(x_n, x_n^n) + \Big(\sup\limits_{m\in\mathbb{N}} k_m\Big)\, d(x_{n-1}^{n-1}, x_{n-1}), \end{array}$$

for any $n \geq 1$. This clearly implies

$$\lim_{n\to\infty} d(x_1^n, x_n) = 0,$$

as claimed.

We summarize what we have just proved.

Theorem 2.3. *Let (X, d) be a complete uniformly convex hyperbolic metric space. Let C be a nonempty closed, bounded and convex subset of X. Let $T : C \to \mathcal{C}(C)$ be an asymptotically nonexpansive mapping. Let $\{k_n\}_{n\in\mathbb{N}}$ be the Lipschitz sequence associated with T and assume that $\sum\limits_{n\in\mathbb{N}} (k_n - 1)$ is convergent. Fix $x_1 \in C$ and $\alpha \in (0, 1)$. Consider the sequence $\{x_n\}$ generated by (MMI). Then*

$$\lim_{n\to\infty} d(x_1^n, x_n) = 0,$$

which implies $\lim\limits_{n\to\infty} d(x_n, T(x_n)) = 0$, i.e. $\{x_n\}$ is an approximate fixed point sequence of T.

As a consequence of Theorem 2.3, we obtain a result amazingly similar to Theorem 2.2 in [31] for multivalued mappings. First, we give a definition of what it means for an iterate of T to be compact. Let $m \geq 1$ be fixed. We say that $T : C \to \mathcal{C}(C)$ is m-compact if there exists a nonempty compact subset K of C such that for any $x \in C$ and any generalized orbit $\{x_n\}$ of x, we have $x_n \in K$ for any $n \geq m$.

Theorem 2.4. *Let (X, d) be a complete uniformly convex hyperbolic metric space. Let C be a nonempty closed, bounded and convex subset of X. Let $T : C \to \mathcal{C}(C)$ be an asymptotically nonexpansive mapping which is m-compact for some $m \geq 1$. Let $\{k_n\}_{n\in\mathbb{N}}$ be the Lipschitz sequence associated with T and assume that $\sum\limits_{n\in\mathbb{N}} (k_n - 1)$ is convergent. Fix $x_1 \in C$ and $\alpha \in (0, 1)$. Consider the sequence $\{x_n\}$ generated by (MMI). Then $\{x_n\}$ has a subsequence which converges to a fixed point of T.*

Proof. Let K be a nonempty compact subset of C associated with the m-compactness of T. For $n \geq m$, we have $x_n^n \in K$. Hence, there exists a subsequence $\{x_{\phi(n)}\}$ of $\{x_n\}$ such that $\{x_{\phi(n)}^{\phi(n)}\}$ converges to some $z \in K$. Using Theorem 2.3, we get

$$\lim_{n\to\infty} d(x_{\phi(n)}, x_{\phi(n)}^{\phi(n)}) = \lim_{n\to\infty} d(x_{\phi(n)}, x_1^{\phi(n)}) = 0.$$

Hence $\{x_{\phi(n)}\}$ and $\{x_1^{\phi(n)}\}$ also converge to z. Since $x_1^{\phi(n)} \in T(x_{\phi(n)})$, we get

$$d(x_1^{\phi(n)}, T(z)) \leq\ k_1\, d(x_{\phi(n)}, z),$$

for any $n \in \mathbb{N}$. Hence $d(z, T(z)) = 0$, i.e. $z \in T(z)$ since $T(z)$ is closed. □

Remark 2.1. We observe that:

(i) Theorem 2.2 provides multivalued version of Goebel and Kirk fixed point theorem on a very general nonlinear domain, namely uniformly convex hyperbolic metric space.

(ii) Theorem 2.2 extends ([6], Corollary 2.1) and provides an analogue of ([11], Theorem 3.1) for multivalued mappings;

(iii) Theorem 2.4 generalizes, Theorem 2.2 of Schu [31] for multivalued mappings on a nonlinear domain.

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