

## JAMES QUASI REFLEXIVE SPACE HAS THE FIXED POINT PROPERTY

M.A. KHAMSI

We prove that the classical sequence James space has the fixed point property. This gives an example of Banach space with a non-unconditional basis where the Maurey-Lin's method applies.

### INTRODUCTION

Let  $K$  be a nonempty weakly compact convex subset of a Banach space  $X$ . We say that  $K$  has the fixed point property (f.p.p.) if every non-expansive mapping  $T: K \rightarrow K$  (that is  $\|T(x) - T(y)\| \leq \|x - y\|$  for any  $x, y$  in  $K$ ) has a fixed point. We say that  $X$  has the fixed point property (f.p.p.) if every weakly convex compact subset of  $X$  has the f.p.p.

A theorem of Kirk [9] states that if  $K$  has normal structure, then it has the f.p.p. It was unknown whether the normal structure is essential. Karlovitz [7] answered the problem negatively.

Alspach [1] proved that  $L_1$  fails the f.p.p., proving that weak compactness is not sufficient to have the f.p.p. The purpose of this paper is to give a proof that the classical James space [5] has the f.p.p., using the beautiful works of Maurey [15] and Lin [12].

Let me point out that in [13], Lin proved positive results concerning the f.p.p. in Banach spaces with unconditional basis. Our paper shows that the ideas arising from Lin's paper are applicable in some Banach spaces with a "good" Schauder basis.

For more detailed history of the f.p.p., we suggest the reader consults [10] and [16] and the references listed therein.

### MAIN RESULT

First recall the definition of the James space  $J$ . This space consists of sequences  $x = (x_n)$  for which  $\text{Lim}(x_n) = 0$ , and  $\|x\|_J < \infty$  where

$$\|x\|_J = \text{Sup}\{[(x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \dots + (x_{p_{n-1}} - x_{p_n})^2 + (x_{p_n} - x_{p_1})^2]^{1/2}\}$$

and the supremum is taken over all positive integers  $n$  and all increasing sequences of positive integers  $\{p_1, p_2, \dots, p_n\}$ .

---

Received 8 March 1988

The author wishes to thank J.B. Baillon, G. Godefroy and B. Maurey for useful discussions regarding this paper.

---

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/89 \$A2.00+0.00.

**Remark.** Sometimes the term  $(x_{p_n} - x_{p_1})$  is dropped, and then we obtain a new space  $J_1$  which is isomorphic to  $J$ . In [8] it is proved that any weakly compact convex subset of  $J_1$  has the normal structure and therefore  $J_1$  has the f.p.p.

The space  $J$  was used to disprove several long-standing conjectures [14,(I) p.25, 103, 132], [14,(II) p.36, 39], [2, 3, 4] and [11].

For the proof of our result, we need one technical lemma, which seems to be new.

**LEMMA 1.**

- (1) For integers  $a \leq b$  we denote the interval of integers between  $a$  and  $b$  by  $F$ . Consider the natural projection  $P_F$  associated with the basis of  $J$ . Then:

$$\|I - P_F\|^2 \leq 2$$

- (2) Let  $u$  and  $v$  be defined by:

$$u = \sum_a^b \beta_i e_i \text{ and } v = \sum_c^d \alpha_i e_i \text{ with } a \leq b < c - 1 \text{ and } c \leq d, \text{ then}$$

$$\|u + v\| \leq \sqrt{2} \|u - v\|$$

**PROOF:** Since the proof of (1) and (2) uses the same techniques, we give only the proof of (1):

Let  $x$  be in  $J$  with  $\|x\| \leq 1$ , we have

$$(I - P_F)(x) = x_F = \sum_{i < a} x_i e_i + \sum_{i > b} x_i e_i = \sum_i y_i e_i$$

Let  $(p_i)$  denote a strictly increasing finite sequence of integers. There are two cases:

**First case. :**

$$\{p_i\} \cap F = \emptyset \text{ then:}$$

$$\sum_1^n (y_{p_i} - y_{p_{i+1}})^2 + (y_{p_n} - y_{p_1})^2 \leq \|x\|^2 \leq 1$$

**Second case. :**

$$\{p_i\} \cap F \neq \emptyset \text{ then:}$$

$$\sum_1^{n-1} (y_{p_i} - y_{p_{i+1}})^2 + (y_{p_n} - y_{p_1})^2$$

$$= \sum_{i \leq j} (x_{p_i} - x_{p_{i+1}})^2 + \sum_{i=k}^{i=n-1} (x_{p_i} - x_{p_{n+1}})^2 + x_j^2 + x_k^2 + (x_{p_n} - x_{p_1})^2$$

with  $j \leq a \leq b \leq k$ . But:

$$\begin{aligned} & \sum_{i=1}^{i=j} (x_{p_i} - x_{p_{i+1}})^2 + \sum_{i=k}^{i=n-1} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_n} - x_{p_1})^2 \\ & \leq \sum_{i=1}^{i=j} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_j} - x_{p_k})^2 \\ & \quad + \sum_{i=k}^{i=n-1} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_n} - x_{p_1})^2 \leq \|x\|^2 \leq 1 \end{aligned}$$

and  $x_i^2 + x_k^2 \leq 1$  (because the sequence  $(x_n)$  is in  $c_o$ )

We deduce that:

$$\|x_f\|^2 \leq 1 + 1 = 2.$$

■

Now we state the main theorem.

**THEOREM.** *Every weakly compact convex subset of  $J$  has the fixed point property.*

**PROOF:** Suppose that there exists a weakly compact, nonempty convex subset  $C$  of  $J$  and a non-expansive  $T: C \rightarrow C$  without fixed point. By Zorn's Lemma  $C$  contains a nonempty closed convex subset  $K$ ,  $T$ -invariant and minimal with respect to the inclusion. Our hypothesis on  $T$  implies that  $\text{diam } K > 0$ , without loss of generality we can assume that  $\text{diam } K = 1$ . It is easy to see that  $K$  contains a quasi-fixed sequence  $(x_n)$  (that is  $\text{Lim} \|x_n - T(x_n)\| = 0$ ). Using the fact that  $K$  is weakly compact, the sequence  $(x_n)$  has a subsequence which is weakly convergent. Since our problem is invariant by translation and by passing to a subsequence, we can assume that  $(x_n)$  converges weakly to 0.

The Karlovitz' Lemma [7] states that for any  $x$  in  $K$  we have:

$$(**) \quad \text{Lim} \|x_n - x\| = \text{diam } K = 1.$$

Since  $(x_n)$  converges weakly to 0 and satisfies  $(**)$  then there exists a subsequence  $(x'_n)$  and a sequence of blocks  $(u_n)$  such that:

- 1)  $\text{Lim} \|x'_n - u_n\| = 0,$
  - 2)  $\text{Lim} \|x'_{n+1} - x'_n\| = 1,$
- where  $u_n = \sum_{i=1_n}^{i=b_n} \beta_i^n e_i$  with  $a_n < b_n = a_{n+1}.$

Let  $P_n$  and  $Q_n$  denote the natural projections defined by:

$$P_n \left( \sum_i \beta_i e_i \right) = \sum_{i=a_n}^{i=b_n} \beta_i e_i \text{ and } Q_n \left( \sum_i \beta_i e_i \right) = \sum_{i>a_{n+1}} \beta_i e_i.$$

Then by the construction of  $(u_n)$  we have:

- i)  $\text{Lim} \|x'_n - P_n(x'_n)\| = 0$ ;
- ii)  $\text{Lim} \|x'_{n+2} - Q_n(x'_{n+2})\| = 0$  (because  $Q_n(u_{n+2}) = u_{n+2}$ );
- iii)  $\text{Lim} \|P_n(x)\| = \text{Lim} \|Q_n(x)\| = 0$  for every  $x$  in  $J$ .

Let  $\mathcal{U}$  denote a non-trivial ultrafilter on  $\mathbb{N}$ . The ultraproduct space  $\mathbf{J}$  of  $J$  is the quotient space of:

$$1_\infty(J) = \{(x_n); x_n \in J \text{ for all } n \in \mathbb{N} \text{ and } \|(x_n)\|_\infty = \sup \|x_n\| < \infty\}$$

by  $\mathcal{N} = \{(x_n) \in 1_\infty(J) \mid \text{Lim}_{\mathcal{U}} \|x_n\| = 0\}$ . We shall not distinguish between  $(x_n) \in 1_\infty(J)$  and the coset  $(x_n) + \mathcal{N} \in \mathbf{J}$ . Clearly,

$$\|(x_n)\|_{\mathbf{J}} = \text{Lim}_{\mathcal{U}} \|x_n\|_J.$$

It is also clear that  $J$  is isometric to a subspace of  $\mathbf{J}$  by the mapping  $x \rightarrow (x, x, \dots)$ . Hence, we may assume that  $J$  is a subspace of  $\mathbf{J}$ . We will write  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  for the general elements of  $\mathbf{J}$  and  $x, y, z$  for the general element of  $J$ . In  $\mathbf{J}$  we define:

$$\mathbf{K} = \{\mathbf{y} \in \mathbf{J}; \mathbf{y} = (y_n) \text{ with } y_n \in K\},$$

and

$$\mathbf{T}: \mathbf{K} \rightarrow \mathbf{K} \text{ with } \mathbf{T}(\mathbf{y}) = \mathbf{T}(y_n) = (T(y_n)).$$

Clearly  $\mathbf{K}$  is a closed convex set with  $\text{diam}(\mathbf{K}) = \text{diam}(K) = 1$ , and  $\mathbf{T}$  is a nonexpansive map on  $\mathbf{K}$ . Furthermore,  $\mathbf{T}$  has fixed points in  $\mathbf{K}$ . Indeed, if  $(x_n)$  is quasi fixed sequence for  $T$  in  $K$ , then  $\text{Lim} \|x_n - T(x_n)\| = 0$  and hence:

$$\|\mathbf{T}(x_n) - (x_n)\|_{\mathbf{J}} = \text{Lim}_{\mathcal{U}} \|x_n - T(x_n)\| = 0.$$

This means that  $\mathbf{T}(x_n) = (x_n)$ , that is  $(x_n)$  is a fixed point for  $\mathbf{T}$  in  $\mathbf{J}$ . Also, if  $\mathbf{T}(y_n) = (y_n)$ , then some subsequence of  $(y_n)$  is a quasi fixed sequence for  $\mathbf{T}$ . The Karlovitz Lemma can be stated in the space  $\mathbf{J}$  by the following:

LEMMA 2. [12]: Let  $(\mathbf{w}_n)$  be a quasi-fixed sequence for  $\mathbf{T}$ , then:

$$\text{Lim} \|\mathbf{w}_n - x\| = \text{diam}(K) = 1 \text{ for any } x \text{ in } K.$$

In other works, if  $\mathbf{W}$  is any nonempty closed  $\mathbf{T}$ -invariant convex subset of  $\mathbf{K}$ , we have:

$$\text{Sup}_{\mathbf{w} \in \mathbf{W}} \|\mathbf{w} - x\| = \text{diam}(K) = 1 \text{ for any } x \text{ in } K.$$

Define  $\mathbf{x}$  and  $\mathbf{y}$  by:  $\mathbf{x} = (x'_n)$  and  $\mathbf{y} = (x'_{n+2})$  by ii) we have:  $\mathbf{x} = (P_n(x'_n))$  and  $\mathbf{y} = (Q_n(x'_{n+2}))$  and by 2) we have:  $\|\mathbf{x} - \mathbf{y}\| = 1$ .

From Lemma 1, we deduce that:

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq 2 \|\mathbf{x} - \mathbf{y}\|^2$$

Let  $\mathbf{W} = \{\mathbf{w} \in \mathbf{K}, \exists x \in K \text{ s.t. } \|\mathbf{w} - x\| \leq 2^{-1/2} \& \text{Sup}(\|\mathbf{w} - \mathbf{x}\|, \|\mathbf{w} - \mathbf{y}\|) \leq 1/2\}$ .

Everything was done to ensure that  $\frac{\mathbf{x} + \mathbf{y}}{2}$  is in  $\mathbf{W}$ . Also it is easy to verify that  $\mathbf{W}$  is a closed  $\mathbf{T}$ -invariant convex subset of  $\mathbf{K}$ . Consider the projections  $\mathbf{P}$  and  $\mathbf{Q}$  defined on  $\mathbf{J}$  by:

$$\mathbf{P}(\mathbf{z}) = (P_n(z_n)) \text{ and } \mathbf{Q}(\mathbf{z}) = (Q_n(z_n)) \text{ where } \mathbf{z} = (z_n).$$

Since the basis of  $\mathbf{J}$  is bimonotone, we have:

$$\begin{aligned} \|\mathbf{P}\| &\leq \text{Sup} \|\mathbf{P}_n\| \leq 1 \\ \|\mathbf{Q}\| &\leq \text{Sup} \|\mathbf{Q}_n\| \leq 1, \\ \|\mathbf{P} + \mathbf{Q}\| &\leq \text{Sup} \|\mathbf{P}_n + \mathbf{Q}_n\| \leq 1, \\ \|\mathbf{I} - \mathbf{Q}\| &\leq 1. \end{aligned}$$

Invoking Lemma 1, we have:

$$\|\mathbf{I} - \mathbf{P}\|^2 \leq 2.$$

Choose  $\mathbf{w}$  in  $\mathbf{W}$  and  $x$  in  $K$  such that  $\|\mathbf{w} - x\| \leq 2^{-1/2}$ .

One has:

$$(*) \quad 2\mathbf{w} = (\mathbf{P} + \mathbf{Q})(\mathbf{w}) + (\mathbf{I} - \mathbf{P})(\mathbf{w}) + (\mathbf{I} - \mathbf{Q})(\mathbf{w}).$$

From the definitions of  $\mathbf{P}$  and  $\mathbf{Q}$ , we can directly derive the following:

$$\mathbf{P}(x) = \mathbf{Q}(x) = 0, \mathbf{P}(\mathbf{x}) = \mathbf{x} \text{ and } \mathbf{Q}(\mathbf{y}) = \mathbf{y}.$$

Using (\*) we deduce that  $2\mathbf{w} = (\mathbf{P} + \mathbf{Q})(\mathbf{w} - \mathbf{x}) + (\mathbf{I} - \mathbf{P})(\mathbf{w} - \mathbf{x}) + (\mathbf{I} - \mathbf{Q})(\mathbf{w} - \mathbf{y})$ .  
And then we have

$$2\|\mathbf{w}\| \leq \|\mathbf{P} + \mathbf{Q}\|\|\mathbf{w} - \mathbf{x}\| + \|\mathbf{I} - \mathbf{P}\|\|\mathbf{w} - \mathbf{x}\| + \|\mathbf{I} - \mathbf{Q}\|\|\mathbf{w} - \mathbf{y}\|.$$

And using all our previous inequalities, we obtain

$$2\|\mathbf{w}\| \leq 2^{-1/2} + 2^{-1/2} + 2^{-1}.$$

This implies

$$\sup_{\mathbf{w}} \|\mathbf{w}\| < 1,$$

which yields a contradiction to Lemma 2.

#### REFERENCES

- [1] D. Alspach, 'A fixed point free non expansive map', *Proc. Amer. Math. Soc.* **82** (1981), 423-424.
- [2] A. Andrew, 'James' quasi-reflexive space is not isomorphic to any subspace of its dual', *Israel J. Math.* **38** (1981), 276-282.
- [3] S. Bellenot, 'Transfinite duals of quasi-reflexive Banach spaces', *Trans. Amer. Math. Soc.* **273** (1982), 551-577.
- [4] P.G. Casazza, 'James quasi-reflexive space is primary', *Israel J. Math.* **26** (1977), 294-305.
- [5] R.C. James, 'A non-reflexive Banach space isometric with its second conjugate space', *Proc. Nat. Acad. Sci. U.S.A.* **37** (1951), 174-177.
- [6] R.C. James, 'Banach spaces quasi-reflexive of order one', *Studia. Math.* **60** (1977), 157-177.
- [7] L.A. Karlovitz, 'Existence of fixed points for non-expansive mappings in a space without normal structure', *Pacific J. Math.* **66** (1976), 153-159.
- [8] M.A. Khamsi, 'Normal structure for Banach spaces with Schauder decomposition' (to appear).
- [9] W.A. Kirk, 'A fixed point theorem for mappings which do not increase idstances', *Amer. Math. Monthly* **72** (1965), 1004-1006.
- [10] W.A. Kirk, *Fixed point theory for non-expansive mapping I, II: Lecture Notes in Math* 886, pp. 484-505 (Springer, Berlin, 1981). : *Contemp. Math.* **18**, pp. 121-140 (A.M.S., Providence RI).
- [11] B.L. Lin and R.H. Lohman, 'On generalized James quasi-reflexive Banach spaces', *Bull. Inst. Math. Acad. Sinica* **8** (1980), 389-399.
- [12] P.K. Lin, *Texas Functional Analysis Seminar 1982-1983* (The University of Texas Austin).
- [13] P.K. Lin, 'Unconditional bases and fixed points of non-expansive mappings', *Pacific J. Math.* **116** (1985), 69-76.
- [14] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol I and II (Springer, Berlin-Heidelberg-New York, 1977 and 1979).
- [15] B. Maurey, *Points fixes des contractions sur un convexe ferme de  $L_1$ : Seminaire d'analyse fonctionnelle*, pp. 80-81 (Ecole Polytechnique, Palaiseau).
- [16] S. Reich, 'The fixed point problem for non-expansive mappings, I, II', *Amer. Math. Monthly* **83** (1976), 266-268. **87**, pp. 292-294.

University of Southern California  
Department of Mathematics DRB 306  
1042W.36th Place  
Los Angeles, CA 90089-1113  
United States of America