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FIXED POINT THEORY IN MODULAR FUNCTION SPACES

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1. INTRODUCTION

THE PURPOSE of this paper is to give an outline of a fixed point theory for nonexpansive mappings defined on some subsets of modular function spaces. The theory of nonexpansive mappings (i.e. mappings with Lipschitz constant 1) on convex subsets of Banach spaces has been well developed since the 1960s [4, 11, 22]. Progress is much less impressive, however, if we consider the theory of nonexpansive mappings acting on other metric spaces, e.g. F-spaces. As a matter of fact, we are able to note only a few results. Goebel *et al.* [5] considered nonexpansive mappings with respect to the hyperbolic metric in the Hilbert unit ball (for more information see Goebel and Reich [4]). In another direction Lami Dozo and Turpin [16] gave several fixed point theorems for nonexpansive mappings in Musielak–Orlicz spaces. Musielak–Orlicz spaces are examples of modular function spaces [15], i.e. function F-spaces defined by means of function modulars. Since F-norms induced by the function modulars are defined indirectly, it is much more convenient and natural to consider mappings that are nonexpansive in the modular sense than to use the respective F-norms. The function modulars are functional that lack homogeneity and subadditivity and, therefore, it might be surprising that we are able to use techniques involving asymptotic centers, normal structure and uniform convexity to obtain fixed point theorems.

The paper is divided into three sections. Section 1 is preliminary. In Section 2 we prove several fixed point theorems using constructive methods, while in Section 3 we introduce the notions of normal structure and uniform convexity in the modular sense and apply them to fixed point theory.

SECTION 1

We begin by recalling some basic concepts of the theory of modular spaces. For more information we refer the reader to the book by Musielak [18].

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Definition 1.1. Let $\mathfrak{X}$ be an arbitrary vector space.

(a) A functional $\rho: \mathfrak{X} \to [0, +\infty]$ is called a modular if for arbitrary $x, y \in \mathfrak{X}$,

(1) $\rho(x) = 0$ if and only if $x = 0$

(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$

(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$.

(b) If (3) is replaced by

(3′) $\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \ge 0$, $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, the modular ρ is called an s-convex modular. 1-convex modulars are called convex modulars.

(c) A modular ρ defines a corresponding modular space, i.e. the vector space $\mathfrak{X}_\rho$ given by

$$\mathfrak{X}_\rho = \{x \in \mathfrak{X} : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Note that in general there is no reason to expect the subadditivity of a modular ρ. Nevertheless, in view of (3) from definition 1.1, there holds

$$\rho(x + y) = \rho(\tfrac{1}{2}(2x) + \tfrac{1}{2}(2y)) \le \rho(2x) + \rho(2y).$$

Let us also recall the following definitions.

Definition 1.2. A functional $\|\cdot\|: \mathfrak{X} \to [0, +\infty]$ is called an F-norm if for arbitrary $x, y \in \mathfrak{X}$,

(1) $\|x\| = 0$ if and only if $x = 0$

(2) $\|\alpha x\| = \|x\|$ if α is a scalar and $|\alpha| = 1$

(3) $\|x + y\| \le \|x\| + \|y\|$

(4) $\|\alpha_k x_k - \alpha x\| \to 0$ if $\alpha_k \to \alpha$ and $\|x_k - x\| \to 0$, where $\{x_k\}$ is a sequence of elements from $\mathfrak{X}$.

Definition 1.3. The linear metric space $(\mathfrak{X}, d)$, where $d(x, y) = \|x - y\|$, is called an F-space if d is a complete metric.

A modular space $\mathfrak{X}_\rho$ can be equipped with an F-norm defined by

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho(\alpha^{-1}x) \le \alpha\}.$$

In the case of an s-convex modular ρ, the formula

$$\|x\|_\rho = \inf\left\{\alpha^s > 0 : \rho\left(\frac{x}{\alpha}\right) \le 1\right\}$$

defines an s-norm (i.e. an F-norm with the additional property $\|\alpha x\| = |\alpha|^s\|x\|$). For $s = 1$ this norm is frequently called the Luxemburg norm. It is a basic fact that for every modular ρ, the convergence $\|x_n\|_\rho \to 0$ is equivalent to $\rho(\alpha x_n) \to 0$ for all $\alpha > 0$. One can easily observe that for every fixed $x \in \mathfrak{X}_\rho$ the function $\mathbb{R} \ni \alpha \to \rho(\alpha x)$ is nondecreasing. It is also clear that any F-norm $\|\cdot\|$ can be regarded as a modular provided $\alpha \to \|\alpha x\|$ is increasing for every x in $\mathfrak{X}$. Since for every F-norm $\|\cdot\|$ there exists an equivalent F-norm with this property [23], we can say that up to equivalence, every F-norm is a modular. The converse is certainly not true; as a classical example we may give the Orlicz modular defined for every measurable real function f by the formula:

$$\rho_\varphi(f) = \int_R \varphi(f(t))\, dm(t)$$

where m denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi: \mathbb{R} \to [0, +\infty)$ is even, continuous, $\varphi(0) = 0$ and $\varphi(u) \to +\infty$ as $u \to +\infty$.

The modular space induced by Orlicz's modular ρ_φ is called the Orlicz space L^φ. Orlicz spaces, their generalizations and many other spaces of measurable functions belong to a large class of function spaces called modular function spaces. We want now to present some basic concepts and facts about these spaces. For an exposition of this theory the reader is referred to Kozlowski [13, 14, 15].

Let X be a nonempty set and Σ be a nontrivial σ-algebra of subsets of X. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us also assume that there exists a nondecreasing sequence of sets $X_i \in \mathcal{P}$, such that $X = \bigcup X_i$. Roughly speaking, $\mathcal{P}$ plays the role of the δ-ring of subsets of finite measure (in many examples this is indeed the case). By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathfrak{M}$ we shall denote the space of all measurable functions, i.e. all functions $f: X \to \mathbb{R}$ such that there exists a sequence of $g_n \in \mathcal{E}$, $|g_n| \le |f|$ and $g_n(x) \to f(x)$ for all x in X. The symbol 1_A denotes the characteristic function of the set A.

Let us recall that a set function $\mu: \Sigma \to [0, +\infty]$ is called a σ-subadditive measure if $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ for any $A \subset B$ and $\mu(\bigcup A_n) \le \sum \mu(A_n)$ for any sequence of sets $A_n \in \Sigma$.

Definition 1.4. (c.f. [13, 14, 15]). A functional $\rho: \mathcal{E} \times \Sigma \to [0, +\infty]$ is called a function modular if

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$.

(P_2) $\rho(f, E) \le \rho(g, E)$ whenever $|f(x)| \le |g(x)|$ for any $x \in X, f, g \in \mathcal{E}$ and $E \in \Sigma$.

(P_3) $\rho(f, \cdot): \Sigma \to [0, +\infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$.

(P_4) $\rho(\alpha, A) \to 0$ as $\alpha \downarrow 0$ for every $A \in \mathcal{P}$, where for the sake of simplicity we denote $\rho(\alpha, A) = \rho(\alpha 1_A, A)$.

(P_5) If $\exists\, \alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$, $\forall\, \beta > 0$.

(P_6) $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$ (for any fixed $\alpha > 0$), i.e. $\rho(\alpha, A_k) \to 0$ if $A_k \in \mathcal{P}$ and $A_k \downarrow \emptyset$.

The definition of ρ is then extended to all $f \in \mathfrak{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E) : g \in \mathcal{E}, |g(x)| \le |f(x)| \text{ for all } x \in E\}.$$

In this sense we will understand the notation $\rho(\alpha, E)$ for sets E not belonging to $\mathcal{P}$; for the sake of simplicity, we write $\rho(f)$ instead of $\rho(f, X)$.

Definition 1.5. A set E is said to be ρ-null if and only if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $\omega(x)$ is said to hold almost everywhere (ρ-a.e.) if the exceptional set $\{x \in X; \omega(x) \text{ does not hold}\}$ is ρ-null. For instance, we will frequently say that $f_n \to f$ ρ-a.e.

Let us observe that a countable union of ρ-null sets is still ρ-null. In view of (P_5), if $\rho(\alpha, E) = 0$ for a positive number α, then E is ρ-null. In the sequel we will identify sets A and B whose symmetric difference $A \,\Delta B$ is ρ-null; similarly, we will identify measurable functions which differ only on a ρ-null set.

We recall now a result (c.f. [13, 14, 15]) that justifies the terminology of definition 1.4.

THEOREM 1.6. The functional $\rho: \mathfrak{M} \to [0, +\infty]$ is a modular in the sense of definition 1.1.

In view of theorem 1.6 the following definition makes sense.

Definition 1.7. A modular space determined by a function modular ρ will be called a modular function space and will be denoted by L_ρ.

The F-norm induced by ρ will be denoted by $\|\cdot\|_\rho$. By definition 1.1(c), we have

$$L_\rho = \{f \in \mathfrak{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

In the following theorem we recall some of the properties of modular function spaces. For proofs and details the reader is referred to [13, 14, 15].

THEOREM 1.8.
(1) $(L_\rho, \|\cdot\|_\rho)$ is a complete space and the F-norm $\|\cdot\|_\rho$ is monotone with respect to the natural order in $\mathfrak{M}$.
(2) If there is a number $\lambda > 0$ such that $\rho(\lambda(f_n - f)) \to 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to f$ ρ-a.e.
(3) (Egoroff's theorem) If $f_n \to f$ ρ-a.e. then there exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow X$ and f_n converges uniformly to f on every H_k.
(4) Defining $L_\rho^\circ = \{f \in \mathfrak{M} : \rho(f, \cdot) \text{ is order continuous}\}$ and $E_\rho = \{f \in \mathfrak{M} : \lambda f \in L_\rho^\circ \text{ for every } \lambda > 0\}$ we have:
(a) $L_\rho \supset L_\rho^\circ \supset E_\rho$,
(b) E_ρ has the Lebesgue property, i.e. $\|f1_{D(k)}\|_\rho \to 0$ if $f \in E_\rho$ and $D(k) \downarrow \emptyset$,
(c) E_ρ is the closure of $\mathcal{E}$ (in the sense of $\|\cdot\|_\rho$).
(5) (Vitali's theorem) If $f_n \in E_\rho$, $f \in L_\rho$ and $f_n \to f$ ρ-a.e., then the following conditions are equivalent:
(i) $f \in E_\rho$ and $\|f_n - f\|_\rho \to 0$,
(ii) for every $\alpha > 0$ the subadditive measures $\rho(\alpha f_n, \cdot)$ are equicontinuous, i.e. $\sup_n \rho(\alpha f_n, E_k) \to 0$ whenever $E_k \in \Sigma$, $E_k \downarrow \emptyset$.
(6) (Lebesgue's theorem) If $f_n, f \in \mathfrak{M}$, $f_n \to f$ ρ-a.e. and there exists a function $g \in E_\rho$ such that $|f_n| \le |g|$ ρ-a.e. for all n, then $\|f_n - f\|_\rho \to 0$.
(7) For $f_n, f \in \mathfrak{M}$ the following are equivalent:
(i) ρ has the Fatou property, i.e. $\rho(f_n) \uparrow \rho(f)$ whenever $|f_n| \uparrow |f|$ ρ-a.e.
(ii) ρ is a left continuous modular, i.e. $\rho(\lambda_n f) \uparrow \rho(f)$ whenever $0 < \lambda_n \uparrow 1$.
(iii) $\rho(f) \le \liminf \rho(f_n)$ whenever $f_n \to f$ ρ-a.e.
(8) A function modular ρ is said to satisfy the Δ_2-condition if $\sup_n \rho(2f_n, E_k) \to 0$ whenever $E_k \downarrow \emptyset$ and $\sup_n \rho(f_n, E_k) \to 0$.

It was proved in [15] that Δ_2 is equivalent to the equality $E_\rho = L_\rho$. The other characterization is as follows: ρ has Δ_2 if and only if F-norm convergence is equivalent to modular convergence. The latter means that in order to obtain $\|f_n - f\|_\rho \to 0$ it suffices to verify that there exists $\lambda > 0$ with $\rho(\lambda f_n) \to 0$. An interesting connection between Δ_2 and the separability of L_ρ, as well as some other conditions of this type can be found in [15].

Definition 1.9. A subset B of L_ρ is called
(a) ρ-bounded if $\sup_{f,g \in B} \rho(f - g) < \infty$;
(b) (ρ-a.e.)-closed if from $f_b \to f$ (ρ-a.e.), $f_n \in B$ it follows that $f \in B$;

(c) (ρ-a.e.)-compact if from every sequence of functions $f_n \in B$ we may extract a subsequence $\{g_n\}$ such that $g_n \to f$ (ρ-a.e.), where f belongs to B.

We use the above mentioned terminology because of its formal similarity to the metric case. Since ρ is usually quite far from being a norm or F-norm, one should be extremely careful when dealing with these notions. For instance, there is no reason for a (ρ-a.e.)-compact set to be ρ-bounded. Nevertheless, one can easily observe that every (ρ-a.e.)-compact set is (ρ-a.e.)-closed. Similarly, we introduce the notion of a ρ-ball B_ρ by the natural formula $B_\rho(f, r) = \{g \in L_\rho : \rho(f - g) \leq r\}$. We should not expect $B_\rho(f, r)$ to be ρ-bounded, because ρ in general need not be subadditive. It is worth mentioning that for many examples (see below) all these notions are quite natural and the conditions from definition 1.9 can be easily verified.

Example 1.10. It is easy to check that Orlicz spaces are modular function spaces. Similarly, Musielak–Orlicz spaces, i.e. spaces determined by a modular of the form

$$\rho(f, E) = \int_E \varphi(t, |f(t)|)\, d\mu(t)$$

are modular function spaces, provided φ belongs to the class Φ. For the precise definitions and properties of Musielak–Orlicz spaces see the book by Musielak [18], where they are called generalized Orlicz spaces.

Example 1.11. Suppose $\mathfrak{M}$ is a family of σ-additive measures on (X, Σ), and $\varphi \in \Phi$. One can prove that

$$\rho(f, E) = \sup_{\mu \in \mathfrak{M}} \int_E \varphi(t, |f(t)|)\, d\mu(t)$$

is a function modular. As an example of function spaces determined by a function modular of this type we can mention Lorentz type L^p-spaces, where

$$\rho(f, E) = \sup_{\tau \in \mathfrak{I}} \int_E |f(t)|^p\, d\mu_\tau.$$

Here μ is a fixed σ-finite measure on X, $\mathfrak{I}$ is any set of measurable, invertible transformations $\tau: X \to X$ and $\mu_\tau(E) = \mu(\tau^{-1}(E))$.

Example 1.12. If $\{\rho_n\}$ is a sequence of function modulars, then one can prove that

$$\rho(f, E) = \sum_{n=1}^{\infty} \frac{2^{-n}\rho_n(f, E)}{1 + \rho_n(f, E)}$$

is a function modular as well, while $\rho^\circ(f, E) = \sup \rho_n(f, E)$ is not a function modular in general. One can ask what should be assumed to guarantee that ρ° satisfies (P_1) through (P_6). The other question is when both ρ and ρ° determine the same modular space. These and similar problems were considered in [15] in relation to the theory of summable functions.

Example 1.13. In [15] the reader can find a construction of the domain of continuity for disjointly additive operators defined on simple functions. It turns out that a disjointly additive operator T induces a function modular ρ such that T can be extended to the whole of E_ρ and the extension is continuous on E_ρ.

SECTION 2

In this section we present some fixed point theorems for mappings that are nonexpansive or contractive in the modular sense. Certainly, one can also consider mappings which are nonexpansive with respect to the F-norm $\|\cdot\|_\rho$ induced by the function modular ρ. We should like to mention that, generally speaking, there is no natural relation between these two kinds of nonexpansiveness. Once again we want to emphasize our philosophy that all results expressed in terms of modulars are more convenient in the sense that their assumptions are much easier to verify.

Definition 2.1. Let B be a subset of a modular space L_ρ and let $T: B \to B$ be an arbitrary mapping.

(a) T is said to satisfy a ρ-Lipschitz condition with constant λ if

$$\rho(T(f) - T(g)) \leq \lambda\rho(f - g) \qquad \text{for all } f, g \in B.$$

(b) T is said to be ρ-nonexpansive if T satisfies a ρ-Lipschitz condition with constant 1.

(c) T is called a strict ρ-contraction if T satisfies a ρ-Lipschitz condition with constant $\lambda < 1$.

Our first fixed point theorem can be named the Banach contraction principle for modular function spaces because of its obvious similarity to this classical result. By $T^n(f)$ we understand the nth iterate of the point f with respect to T. We say that f is a fixed point of T if $T(f) = f$ ρ-a.e.

THEOREM 2.2. Let ρ be a function modular satisfying the Δ_2-condition and let B be a $\|\cdot\|_\rho$-closed subset of L_ρ. If $T: B \to B$ is a strict ρ-contraction and there exists $f_0 \in B$ with $\sup \rho(2T^n(f_0)) < \infty$, then T has a fixed point $f \in B$.

Proof. Consider an $f_0 \in B$ such that $R = \sup \rho(2T^n(f_0)) < \infty$. For all natural numbers n and k we have

$$\rho(T^{n+k}(f_0) - T^n(f_0)) \leq \lambda^n[\rho(2T^k(f_0)) + \rho(2f_0)] \leq \lambda^n 2R.$$

Since $\lambda < 1$ and $R < \infty$, $\{T^n(f_0)\}$ is a Cauchy sequence in the sense of modular convergence. By the Δ_2-condition, $\{T^n(f_0)\}$ is a Cauchy sequence in the sense of the F-norm $\|\cdot\|_\rho$ (c.f. theorem 1.8, part (8)). Since $(L_\rho, \|\cdot\|_\rho)$ is complete, there exists a function f in L_ρ such that $\|f - T^n(f_n)\|_\rho \to 0$. The function f belongs to B, because B is $\|\cdot\|_\rho$-closed. We claim that f is the desired fixed point. Indeed,

$$\rho(T(f) - T^k(f_0)) \leq \lambda\rho(f - T^{k-1}(f_0)) \to 0 \qquad \text{as } k \to \infty,$$

and since $\rho(f_0 - T^n(f_0)) \to 0$ as $n \to \infty$, we have:

$$\rho(\tfrac{1}{2}(f - T(f))) \leq \rho(T(f) - T^n(f_0)) + \rho(T^n(f_0) - f) \to 0 \qquad \text{as } n \to \infty.$$

Thus $\rho(\frac{1}{2}(f - T(f))) = 0$, and in view of the definition of a function modular ρ, $T(f) = f$ ρ-a.e.

In order to obtain the uniqueness of the fixed point, we have to add an additional assumption.

PROPOSITION 2.3. Under the assumptions of theorem 2.2, suppose in addition that $\rho(f - g) < \infty$ for all $f, g \in B$. Then the fixed point f is unique.

Proof. Suppose $T(f_1) = f_1$ and $T(f_2) = f_2$. Then

$$\rho(f_1 - f_2) = \rho(T(f_1) - T(f_2)) \leq \lambda\rho(f_1 - f_2).$$

Since $\lambda < 1$ and $\rho(f_1 - f_2)$ is finite, this is possible only if $f_1 = f_2$ ρ-a.e.

It is worth noting that in some modular function spaces, $\rho(f - g) < \infty$ for all f and g in L_ρ if ρ satisfies the Δ_2-condition. For instance, this is the case when L_ρ is an Orlicz or a Musielak-Orlicz space.

Combining theorem 2.2 and proposition 2.3 we obtain immediately the next result.

THEOREM 2.4. Let ρ satisfy the Δ_2-condition, and let B be a $\|\cdot\|_\rho$-closed subset of L_ρ. If $T: B \to B$ is a strict ρ-contraction and B is ρ-bounded, then T has a unique fixed point.

Definition 2.5. A function modular is said to satisfy the (*)-condition if

$$(*) \quad \rho(f, H) \leq \limsup \rho(f_n, H)$$

for every $H \in \mathcal{P}$ such that (f_n) converges uniformly to f on H, where f_n and F belong to L_ρ°.

Let us note that the (*)-condition is satisfied for many function modulars. This is the case, for example, for all modular function spaces with the Fatou property (see remark 2.6), which is equivalent to the left-continuity of ρ (cf. theorem 1.8, part (7)). In particular, all Musielak-Orlicz modulars are left-continuous and, therefore, satisfy (*).

Remark 2.6.

(a) The (*)-condition is equivalent to:

$$(**) \quad \rho(f, H) \leq \liminf \rho(f_n, H), \text{ where } f, f_n \text{ and } H \text{ are the same as in } (*).$$

(b) If ρ satisfies the Fatou property then ρ satisfies the (*)-condition.

Proof. The (**)-condition is satisfied because (*) must hold for every subsequence of (f_n). Since the other implication is obvious, we have (a). By theorem 1.8, part (7), the Fatou property gives the (**)-condition and by part (a), the (*)-condition is then satisfied.

In view of remark 2.6, the (*)-condition is equivalent to the lower semi continuity of ρ with respect to uniform convergence on sets from the δ-ring $\mathcal{P}$. The next result characterizes function modulars that satisfy the (*)-condition.

PROPOSITION 2.7.

(a) ρ has the Fatou property if and only if $B_\rho(r)$ is (ρ-a.e.)-closed for all $r > 0$, where $B_\rho(r) = \{f \in L_\rho : \rho(f) \leq r\}$.

(b) A function modular ρ satisfies the (*)-condition if and only if $B_\rho(r) \cap L_\rho^\circ$ is (ρ-a.e.)-closed in L_ρ° for all $r > 0$.

Proof. (a) is evident. To prove (b), suppose that ρ satisfies (*). Fix an $r > 0$ and take a sequence (f_n), $f_n \in B_\rho(r) \cap L_\rho^\circ$, such that $f_n \to f$ (ρ-a.e.) with $f \in L_\rho^\circ$. We have to show that $f \in B_\rho(r)$. From Egoroff's theorem it follows that there exists a sequence of sets $H_k \in \mathcal{P}$

such that $H_k \uparrow X$, and (f_n) converges uniformly to f on each (H_k). For any $k \in N$ we have

$$\rho(f, H_k) \le \limsup_{n\to\infty} \rho(f_n, H_k) \le \limsup_{n\to\infty} \rho(f_n) \le r$$

and $\rho(f) \le \rho(f, H_k) + \rho(f, X \backslash H_k)$. Since $f \in L_\rho^\circ$, we have $\rho(f, X\backslash H_k) \to 0$ as $k \to \infty$. Therefore $\rho(f) \le r$ and then $f \in L_\rho^\circ \cap B_\rho(r)$. Assume now that $B_\rho(r)$ is (ρ-a.e.)-closed and that (f_n) converges uniformly to f on a set $H \in \mathcal{P}$ $(f_n, f \in L_\rho^\circ)$. If (*) is not satisfied, then there exists a subsequence (g_n) of (f_n) such that

$$\lim \rho(g_n, H) < \gamma < \rho(f, H)$$

for a positive constant γ. Since g_n converges uniformly to f on H, it follows that $g_n 1_H \to f 1_H$ (ρ-a.e.) and, because $B_\rho(\gamma) \cap L_\rho^\circ$ is (ρ-a.e.)-closed in E_ρ, $f 1_H \in B_\rho(\gamma) \cap L_\rho^\circ$, which yields $\rho(f, H) \le \gamma$, a contradiction.

Our next result will play an important role in the proof of a fixed point theorem for strict ρ-contractions when ρ does not satisfy the Δ_2-condition. If $\|\cdot\|$ is an F-norm then clearly $\|f_n - g\| \to \|f - g\|$ if $\|f_n - f\| \to 0$. Since modulars are not subadditive in general, we cannot expect the same result for function modulars. It turns out, however, that the (*)-condition gives some control, namely we have the following "asymptotic center" [4] result.

LEMMA 2.8. Let ρ satisfy the (*)-condition. Assume that for a sequence $\{f_n\} \subset L_\rho$ there exists a subsequence $\{h_n\}$ of $\{f_n\}$ such that $h_n \to f$ ρ-a.e. Then for all $g \in L_\rho$ such that $f - g \in L_\rho^\circ$, we have

$$\rho(f - g) \le \limsup \rho(f_n - g).$$

Proof. Let $\{h_n\}$ be a subsequence of $\{f_n\}$ with $h_n \to f$ ρ-a.e. By Egoroff's theorem (theorem 1.8, part (3)) there exists then a sequence of sets $H_k \in \mathcal{P}$, such that $H_k \uparrow X$ and h_n converges uniformly to f on every H_k. For any $k \in N$ we have, by the (*)-condition,

$$\rho(f - g, H_k) \le \limsup_{n\to\infty} \rho(h_n - g, H_k) \le \limsup_{n\to\infty} \rho(h_n - g) \le \limsup_{n\to\infty} \rho(f_n - g).$$

Since $f - g$ belongs to L_ρ°, we get $\rho(f - g, X\backslash H_k) \to 0$ as $k \to \infty$, and consequently,

$$\rho(f - g, H_k) \to \rho(f - g)$$

as $k \to \infty$. The latter convergence implies that $\rho(f - g) \le \limsup \rho(f_n - g)$, as claimed.

THEOREM 2.9. Let ρ satisfy the (*)-condition and let B be a (ρ-a.e.)-compact, ρ-bounded subset of L_ρ. Assume that $B - B \subset L_\rho^\circ$ (i.e. $f - g \in L_\rho^\circ$, provided f, g are in B). If $T: B \to B$ is a strict ρ-contraction then it has a unique fixed point.

Proof. Take an arbitrary $f_0 \in B$, and set $f_n = T^n(f_0)$. For $g \in B$, define

$$\varphi(g) = \limsup \rho(f_n - g).$$

Since T is a strict ρ-contraction, we get easily $\varphi(T(g)) \le \lambda\varphi(g)$ for all $g \in B$, where $\lambda \in (0, 1)$ is the Lipschitz constant for T. Therefore, $\inf\{\varphi(g) : g \in B\} = 0$. By the ($\rho$-a.e.)-compactness of B we can choose a subsequence (h_n) of (f_n) which is (ρ-a.e.)-convergent to an $f \in B$.

By lemma 2.8, $\rho(f - g) \leq \varphi(g)$ for every $g \in B$. Hence,

$$\rho(f - T^n(g)) \leq \varphi(T^n(g)) \leq \lambda^n \varphi(g)$$

and $\rho(T(f) - T^n(g)) \leq \lambda\rho(f - T^{n-1}(g)) \leq \lambda\varphi(T^{n-1}(g)) \leq \lambda^n\varphi(g)$. Finally, we have

$$\rho(\tfrac{1}{2}[f - T(f)]) \leq \rho(f - T^n(g)) + \rho(T^n(g) - T(f)) \leq 2\lambda^n\varphi(g) \leq 2\lambda^n R.$$

Where $R = \sup\{\rho(f - g); f, g \in B\}$ is finite in view of the ρ-boundedness of B. Since $2\lambda^n R \to 0$ as $n \to \infty$, we deduce that $T(f) = f$ ρ-a.e.

Remark 2.10. Let us observe that for some important modular function spaces, e.g. Orlicz spaces or Musielak-Orlicz spaces, the fact that $B - B \subset L_\rho^\circ$ follows from

$$\sup\{\rho(f - g) : f, g \in B\} < \infty.$$

Indeed, order continuity of $\rho(f - g, \cdot)$ is in such spaces equivalent to $\rho(f - g) < \infty$.

In order to prove a fixed point theorem for ρ-nonexpansive mappings we have first to introduce a new condition for function modulars.

Definition 2.11. The growth function w_ρ of a function modular ρ is defined as follows:

$$w_\rho(t) = \sup[\rho(tf)/\rho(f) : f \in L_\rho, 0 < \rho(f) < \infty], \qquad t \geq 0.$$

Observe that $w_\rho(t) \leq 1$ for all $t \in [0, 1]$.

Definition 2.12. We say that ρ satisfies the regular growth condition if $w_\rho(t) < 1$ for all $t \in [0, 1)$.

The class of function modulars that satisfy the regular growth condition is quite large. For instance, if ρ is s-convex $(0 < s \leq 1)$, then $\rho(tf) \leq t^s\rho(f)$ for $t \leq 1$, and consequently $w_\rho(t) \leq t^s \leq t < 1$. Thus all s-convex function modulars satisfy the regular growth condition. It is not hard to prove that if ρ is an Orlicz modular then, in the case of finite measure, ρ satisfies the regular growth condition if and only if $\limsup_{s\to\infty}[\varphi(ts)/\varphi(s)] < 1$ for all $t \in [0, 1)$, where φ denotes the Orlicz function associated with ρ. If there exists a constant $K > 0$ such that $\rho(2f) \leq K\rho(f)$ for all $f \in L_\rho$, then w_ρ is submultiplicative and hence (see [19]) there is $p > 1$ such that $w_\rho(t) \leq t^p$ for $t \in [0, 1)$. Consequently, such function modulars also satisfy the regular growth condition.

Recall that a set B is said to be star-shaped if there exists $u \in B$ such that $u + \lambda(g - u)$ belongs to B whenever $\lambda \in [0, 1]$ and $g \in B$. Such a point u is called a center of B.

THEOREM 2.13. Assume that ρ satisfies the (*)-condition and the regular growth condition. Let B be a star-shaped ρ-bounded and (ρ-a.e.)-compact subset of L_ρ such that $B - B \subset L_\rho^\circ$. Assume in addition that for every sequence of functions $f_n \in B$ such that $f_n \to f$ ρ-a.e. with $f \in B$ and for every sequence of sets $G_k \downarrow \emptyset$,

$$(+) \quad \lim_{k\to\infty}\left(\sup_n \rho(f_n - f, G_k)\right) = 0.$$

If $T: B \to B$ is ρ-nonexpansive, then it has a fixed point.

Proof. Let u be a center of the star-shaped set B. For each $\lambda \in (0, 1)$, let us consider the mapping $T_\lambda: B \to B$ defined by $T_\lambda(f) = u + \lambda(T(f) - u)$. Observe that $\rho(T_\lambda(f) - T_\lambda(g)) = \rho(\lambda[T(f) - T(g)]) \le w_\rho(\lambda)\rho(T(f) - T(g)) \le w_\rho(\lambda)\rho(f - g)$. Since ρ satisfies the regular growth condition, $w_\rho(\lambda) < 1$ and, consequently, T_λ is a strict ρ-contraction for every $\lambda \in (0, 1)$. In view of theorem 2.9 T_λ has a unique fixed point f_λ. Take a sequence of positive numbers $\lambda(n) \uparrow 1$. Denoting $T_n = T_{\lambda(n)}$ and $f_n = f_{\lambda(n)}$ and using a compactness argument we can assume (passing to a suitable subsequence if necessary) that there exists a function $f \in B$ such that $f_n \to f$ ρ-a.e. By Egoroff's theorem there exists a sequence $\{H_k\}$ of sets from $\mathcal{P}$ such that $H_k \uparrow X$ and f_n converges uniformly to f on every H_k. Observe that for every $k \in N$

(i) $\rho(\frac{1}{2}[T(f) - T_n(f)], H_k) \le \rho(T(f) - T_n(f)) = \rho([1 - \lambda(n)][T(f) - u]) \to 0$ because $T(f) - u \in L_\rho$ and $1 - \lambda(n) \to 0$ as $n \to \infty$. Moreover, we have

(ii) $\rho(T_n(f_n) - f_n, H_k) \le \rho(T_n(f_n) - f_n) = 0$ because f_n is a fixed point for T_n. Observe now that

(iii) $\rho(T_n(f) - T_n(f_n), H_k) \le \rho(T_n(f) - T_n(f_n)) \le \rho(\lambda(n)(T(f) - T(f_n))) \le \rho(T(f) - T(f_n)) \le \rho(f - f_n) \le \rho(f - f_n, H_m) + \rho(f - f_n, X \backslash H_m)$ holds for all natural numbers n, m, and k. Hence

(iv) $\rho(T_n(f) - T_n(f_n), H_k) \le \rho(f - f_n, H_m) + \sup_n \rho(f - f_n, X \backslash H_m)$.

Since ρ satisfies condition (+), we have:

$$\lim_m \left[\sup_n \rho(f - f_n, X \backslash H_m) \right] = 0.$$

Let ε be an arbitrary positive number. There exists $m' \in N$ such that $\rho(f - f_n, X \backslash H_{m'}) \le \varepsilon$ for every $n \in N$. On the other hand, our assumption on $\{f_n\}$ implies that $\rho(f - f_n, H_{m'}) \le \varepsilon$ for n sufficiently large. We conclude, therefore, that for $k \in N$,

(v) $\lim_{n \to \infty} \rho(T_n(f) - T_n(f_n), H_k) = 0$.

Using (i), (ii) and (v) we have for arbitrary $k \in N$,

(vi) $$\rho(\tfrac{1}{4}[T(f) - f_n], H_k) \le \rho(\tfrac{1}{2}[T(f) - T_n(f)], H_k) + \rho(T_n(f) - T_n(f_n), H_k) + \rho(T_n(f_n) - f_n, H_k) \to 0 \quad \text{as } n \to \infty.$$

Finally,

$$\rho(\tfrac{1}{8}[T(f) - f], H_k) \le \rho(\tfrac{1}{4}[T(f) - f_n], H_k) + \rho(\tfrac{1}{4}[f - f_n], H_k) \to 0$$

which implies that $T(f) = f$ ρ-a.e. in H_k. Since $H_k \uparrow X$, we obtain $T(f) = f$ ρ-a.e. in X. The proof is complete.

Remark. An example of a set B satisfying (+) such that $B - B \subset L_\rho^\circ$ is provided by a set B such that $B - B \subset \{f \in L_\rho : |f(x)| \le |g(x)|\}$ where $g \in L_\rho^\circ$.

As mentioned in the introduction, one of the reasons of our interest in ρ-contractions is that the contraction condition can be easily verified. Our next result shows that we have to assume more than ρ-nonexpansiveness in order to obtain norm nonexpansiveness. An example of a mapping which is ρ-nonexpansive but is not nonexpansive in the norm sense will be discussed later. Recall that a modular ρ is called left continuous if $\rho(\lambda f) \uparrow \rho(f)$ as $\lambda \uparrow 1$. It is known (cf. [18, theorem 1.8]) that for convex, left continuous modulars the inequalities $\|f\|_\rho \le 1$ and $\rho(f) \le 1$ are equivalent.

PROPOSITION 2.14. Let ρ be a convex, left continuous function modular. If $\rho(\gamma[T(f) - T(g)]) \le \rho(\gamma[f - g])$ for every $\gamma > 0$, then $\|T(f) - T(g)\|_\rho \le \|f - g\|_\rho$.

Proof. Suppose that there exist f, g in L_ρ and $\alpha > 0$ such that $\|f - g\|_\rho < \alpha < \|T(f) - T(g)\|_\rho$. Then $\|(f - g)/\alpha\|_\rho < 1$, which implies that $\rho([f - g]/\alpha) < 1$, and on the other hand, $1 < \|(T(f) - T(g))/\alpha\|_\rho$. The left continuity of ρ now yields $1 < \rho([T(f) - T(g)]/\alpha)$. Finally, setting $\gamma = \alpha^{-1}$, we obtain $\rho(\gamma[f - g]) < 1 < \rho(\gamma[T(f) - T(g)])$, a contradiction.

Example 2.15. Let $X = (0, \infty)$, and let Σ be the σ-algebra of all Lebesgue measurable subsets of X. Let $\mathcal{P}$ denote the δ-ring of subsets of finite measure. Define a function modular by

$$\rho(f) = \frac{1}{e^2}\int_0^\infty |f(x)|^{x+1}\, dm(x).$$

Let B be the set of all measurable functions f: $(0, \infty) \to \mathbb{R}$ such that $0 \le f(x) \le \frac{1}{2}$. Define a linear operator T by the formula

$$T(f)(x) = \begin{cases} f(x - 1), & \text{for } x \ge 1 \\ 0, & \text{for } x \in [0, 1]. \end{cases}$$

Clearly, $T(B) \subset B$. We claim that for every fixed $\lambda \le 1$ and for all $f, g \in B$,

(i) $\rho(\lambda[T(f) - T(g)]) \le \lambda\rho(\lambda[f - g])$.

Indeed,

$$\begin{aligned}\rho(\lambda(T(f) - T(g)) &= e^{-2}\int_0^\infty \lambda^{x+1}|T(f)(x) - T(g)(x)|^{x+1}\, dm(x) \\ &= e^{-2}\int_1^\infty \lambda^{x+1}|f(x - 1) - g(x - 1)|^{x+1}\, dm(x) \\ &= \lambda e^{-2}\int_0^\infty \lambda^{x+1}|f(x) - g(x)|^{x+1}|f(x) - g(x)|\, dm(x) \\ &\le \lambda e^{-2}\int_0^\infty \lambda^{x+1}|f(x) - g(x)|^{x+1}\, dm(x) = \lambda\rho(\lambda(f - g)).\end{aligned}$$

In particular, if $\lambda = 1$ then (i) shows that T is nonexpansive. It is also easy to see that B is a (ρ-a.e.)-bounded subset of L_ρ. We observe that T is not $\|\cdot\|_\rho$-nonexpansive. Indeed, put $f = 1_{[0,1]}$ and note that $T(f) = 1_{[1,2]}$. We have

$$\rho(ef) = e^{-2}\int_0^\infty e^{x+1}|f(x)|^{x+1}\, dm(x) = e^{-2}\int_0^1 e^{x+1}\, dm(x) = \frac{1}{e}\int_0^1 e^x\, dm(x) = \frac{e - 1}{e} < 1.$$

Since ρ is left-continuous, the latter inequality implies that $\|ef\|_\rho \le 1$ and therefore $\|2^{-1}f\|_\rho \le \frac{1}{2}e$. On the other hand,

$$\rho(ef) = e^{-2}\int_1^2 e^{x+1}\, dm(x) = e^{-1}\int_1^2 e^x\, dm(x) = \frac{e^2 - e}{e} = e - 1 > 1,$$

and consequently $\|eT(f)\|_\rho > 1$. Thus, $\|T(\frac{1}{2}f)\|_\rho = \|\frac{1}{2}T(f)\|_\rho > \frac{1}{2}e$, so that T is not $\|\cdot\|_\rho$-nonexpansive.

The next example shows that sometimes an operator T can itself determine a function modular ρ_T so that T is ρ_T-nonexpansive.

Example 2.16. Let T be the Urysohn integral operator

$$T(f)(x) = \int_0^1 k[x, y, |f(y)|]dy + f_0(x),$$

where f_0 is a fixed function and f: $[0, 1] \to \mathbb{R}$ is measurable. For the kernel k we assume that

(a) k: $[0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable,
(b) $k(x, y, 0) = 0$,
(c) $k(x, y, \cdot)$ is continuous, convex and increasing to ∞,
(d) $\int_0^1 k(x, y, t)\,dx > 0$ for $t > 0$ and $y \in (0, 1)$.

Set $\rho_T(f, A) = \int_0^1(\int_A k(x, y, |f(y)|)\,dy)\,dx$, where $A \subset [0, 1]$ is measurable. It is easy to show that ρ_T is a function modular.

If the following inequality holds for almost all $t \in [0, 1]$ and all $f, g \in L_\rho$,

(i)
$$\int_0^1 \left\{\int_0^1 k[t, u, |k(u, v, |f(v)|) - k(u, v, |g(v)|)|]\,dv\right\} du \le \int_0^1 k[t, u, |f(u) - g(u)|]\,du,$$

then T is ρ_T-nonexpansive. Indeed, there holds

$$\rho_T(T(f) - T(g)) \le \int_0^1 \left\{\int_0^1 \left[\int_0^1 k[x, u, |k(u, v, |f(v)|) - k(u, v, |g(v)|)|]\,dv\right] du\right\} dx$$
$$\le \int_0^1 \left\{\int_0^1 k[x, u, |f(x) - g(x)|]\,du\right\} dx = \rho_T(f - g).$$

The first inequality was obtained by Jensen's inequality while the other one by (i). As a concrete example of such an operator one can take, for instance,

$$T(f)(x) = \int_0^x xy|f(y)|\,dy + f_0(x).$$

We conclude this part of the paper with the following remark on Alspach's counterexample [1]. Let us define the operator:

$$T(f)(x) = \begin{cases} \min\{2, 2f(x)\} & \text{for } x \in [0, \frac{1}{2}] \\ \max\{0, 2f(2x - 1) - 2\} & \text{for } x \in (\frac{1}{2}, 1] \end{cases}$$

on C, a convex subset of L_1, defined by

$$C = \left\{f \in L_1[0, 1] : 0 \le f(x) \le 2 \text{ a.e. and } \int_0^1 f(x)\,dx = 1\right\}.$$

The operator T is an isometry on C, with an empty fixed point set. It seems that the condition $\int_0^1 f(x)d(x) = 1$ is "responsible" for the existence of nonexpansive, fixed point free self-mappings of C, since for $T: B \to B$, where $B = \{f \in L_1 : 0 \le f(x) \le 2 \text{ a.e.}\}$, we obtain the obvious fixed point $f(x) = 0$. Maurey observed in [17] that all weakly compact convex subsets of reflexive subspaces of L_1 have the fixed point property (that is, any nonexpansive self-mapping has a fixed point). Note that our theorem 2.13 gives "an intrinsic" reason why $T: B \to B$ must have a fixed point while $T: C \to C$ does not have to. Moreover, we do not refer to any geometrical properties of subspaces of L_1 (observe that even the convexity of B is not essential here).

SECTION 3

The concept of normal structure was introduced by Brodskii and Milman [2] for the case of linear normed spaces. It was frequently used to prove existence theorems in fixed point theory. There were also some attempts to generalize the concept of normal structure to metric spaces [9, 20] and more abstract sets [8, 21].

In this section we define normal structure in modular function spaces. According to the philosophy of this paper, we introduce normal structure for function modulars, not for the norms or the F-norms generated by them. We prove then a modular analog of Kirk's fixed point theorem [10], and give some natural examples of modular function spaces with normal structure (with respect to the function modular).

First, we have to introduce some basic notions. Let B be a ρ-bounded subset of L_ρ.

Definition 3.1.

(a) By the ρ-diameter of B, we will understand the number

$$\delta_\rho(B) = \sup\{\rho(f - g) : f, g \in B\}.$$

(b) The quantity $r_\rho(f, B) = \sup\{\rho(f - g) : f, g \in B\}$ will be called the ρ-Chebyshev radius of B with respect to f.

(c) The ρ-Chebyshev radius of B is defined by $R_\rho(B) = \inf\{r_\rho(f, B) : f \in B\}$.

(d) The ρ-Chebyshev center of B is defined as the set

$$\mathcal{C}_\rho(B) = \{f \in B : r_\rho(f, B) = R_\rho(B)\}.$$

Note that $R_\rho(B) \le r_\rho(f, B) \le \delta_\rho(B)$ for all $f \in B$ and observe that there is no reason, in general, for $\mathcal{C}_\rho(B)$ to be nonempty.

Definition 3.2.

(a) We say that g is a ρ-diametral point of B if $r_\rho(g, B) = \delta_\rho(B)$.

(b) The set B is called ρ-diametral if every $g \in B$ is a ρ-diamentral point.

(c) A sequence $\{f_n\}$ of functions from L_ρ is called a ρ-diametral sequence if there exists $c > 0$ such that $\delta_\rho(f_n) \le c$ and

$$\lim \text{dist}_\rho[f_{n+1}, \text{conv}(f_i : 1 \le i \le n)] = c$$

where $\text{dist}_\rho[f, A] = \inf\{\rho(f - g) : g \in A\}$ and

$$\text{conv}\{f_i : 1 \le i \le n\} = \left\{\sum_{i=1}^{n} \alpha_i f_i ; \alpha_i \ge 0 \text{ and } \sum_i \alpha_i = 1\right\}.$$

Let us observe that $\text{dist}_\rho(f_{n+1}, \text{conv}(f_i)) \le nc$, while in the norm case this distance can be estimated by the number c itself.

Before we state our first fixed point theorem, we have to introduce the concept of ρ-admissible sets and ρ-normal structure.

Definition 3.3. Let B be a ρ-bounded subset of L_ρ.

(a) We say that A is an admissible subset of B (cf. [3]) if $A = \bigcap_{i \in I} B_\rho(b_i, r_i) \cap B$, where $b_i \in B$, $r_i > 0$ and I is an arbitrary index set. (Recall that $B_\rho(f, r) = \{g \in L_\rho : \rho(f - g) \leq r\}$).

(b) If C is a subset of B we let $co(C) = \bigcap_{f \in C} B(f, r_\rho(f, C)) \cap B$.

(c) B is said to have ρ-normal structure if each ρ-admissible subset A of B, not reduced to a single point, has a ρ-nondiametral point (i.e. a point which is not ρ-diametral).

By $\mathcal{A}(B)$ we denote the family of all admissible subsets of B. For the proof of the analog of Kirk's theorem, we shall need the following lemma, as in [12].

LEMMA 3.4. Let B be a nonempty, ρ-bounded subset of L_ρ. Let $T: B \to B$ be a ρ-nonexpansive mapping. Assume that B has ρ-normal structure. If $D \in \mathcal{A}(B)$ is a T-invariant set, then there exists a nonempty admissible subset D^* of D, which is T-invariant, and such that

$$\delta_\rho(D^*) \leq \tfrac{1}{2}(\delta_\rho(D) + R_\rho(D)).$$

Proof. Set $r = 2^{-1}(\delta_\rho(D) + R_\rho(D))$. We can assume that $\delta_\rho(D) > 0$, otherwise we can take $D^* = D$. Since B has ρ-normal structure, we have $R_\rho(D) < \delta_\rho(D)$. Therefore, the set $A = \{f \in D : D \subset B_\rho(f, r)\}$ is a nonempty admissible subset of B ($A = \bigcap_{f \in D} B_\rho(f, r) \cap D$). *A priori* there is no reason for A to be T-invariant. Put $\mathfrak{F} = \{M \in \mathcal{A}(B) : A \subset M$ and $T(M) \subset M\}$ and $L = \bigcap_{M \in F} M$. Note that $\mathfrak{F}$ is nonempty, since $B \in \mathfrak{F}$. The set L is an admissible subset of B which contains A. Using the definition of $\mathfrak{F}$, we deduce that L is T-invariant. Consider $C = A \cup T(L)$, and observe that $co(C) = L$. Indeed, since $C \subset T(L) \subset L$ and $L \in \mathcal{A}(B)$, we have $co(C) \subset L$. From this we obtain $T(co(C)) \subset T(L) \subset C$ and $C \subset co(C)$, hence $co(C) \in \mathfrak{F}$, and $L \subset co(C)$. This gives the desired equality. Define $D^* = \{f \in L : L \subset B_\rho(f, r)\}$; we claim that D^* is the desired set. Observe that D^* is nonempty since it contains A (by definition of A). Using the same argument we can prove that D^* is an admissible subset of B. On the other hand, it is clear that $\delta_\rho(D^*) \leq r$. To complete the proof, we have to show that D^* is T-invariant. Let $f \in D^*$. By definition of D^* we have $L \subset B_\rho(f, r)$. Since T is ρ-nonexpansive, we have $T(L) \subset B_\rho(T(f), r)$. For any $g \in A$ there holds $L \subset B_\rho(g, r)$. But $T(f) \in L$, so that $T(f) \in B_\rho(g, r)$, which is equivalent to $g \in B_\rho(T(f), r)$. This implies that $A \subset B_\rho(T(f), r)$. Since $C = A \cup T(L)$, we deduce from what we have proved above that $C \subset B_\rho(T(f), r)$. Therefore, we have $co(C) = L \subset B_\rho(T(f), r)$. By the definition of D^*, it follows that $T(f) \in D^*$. In other words, D^* is T-invariant.

We are now ready to prove the analog of Kirk's theorem in modular function spaces.

THEOREM 3.5. Let ρ have the Fatou property. Suppose that a ρ-bounded, (ρ-a.e.)-compact $B \subset L_\rho$ has ρ-normal structure. If $T: B \to B$ is ρ-nonexpansive, then it has a fixed point.

Proof. Let $\mathfrak{F} = \{D \in \mathcal{A}(B) : D \neq \emptyset$ and $T(D) \subset D\}$. The family $\mathfrak{F}$ is not empty since $B \in \mathfrak{F}$. Since ρ has the Fatou property, we deduce from proposition 2.7(a) that the admissible subsets of B are (ρ-a.e.)-closed, hence (ρ-a.e.)-compact. This implies that any decreasing sequence of nonempty elements of $\mathfrak{F}$ has a nonempty intersection (recall that $\mathfrak{F}$ is stable by intersection).

Define $\delta_0: F \to [0, \infty)$ by $\delta_0(D) = \inf\{\delta_\rho(F) : F \in \mathfrak{F}$ and $F \subset D\}$. Put $D_1 = B$, and define $D_2 \in \mathfrak{F}$ by $\delta_\rho(D_2) \le \delta_0(D_1) + \varepsilon_1$ and $D_2 \subset D_1$, where $\{\varepsilon_n\}$ is a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Assume that D_i have been constructed for $i \le n$, and define $D_{n+1} \in \mathfrak{F}$ by $D_{n+1} \subset D_n$ and $\delta_\rho(D_{n+1}) \le \delta_0(D_n) + \varepsilon_n$. Such sets exist by the definition of δ_0. By our previous remarks on $\mathfrak{F}$, we deduce that the intersection of $\{D_n\}$ is a nonempty element of $\mathfrak{F}$. Denote this intersection by C. Let us assume that C is not reduced to a single point. Since C satisfies all the hypotheses of lemma 3.4, there exists C^* in $\mathfrak{F}$, contained in C, such that

$$(**) \qquad \delta_\rho(C^*) \le 2^{-1}(\delta_\rho(C) + R_\rho(C)).$$

We have $\delta_\rho(C^*) \le \delta_\rho(C) \le \delta_\rho(D_{n+1}) \le \delta_0(D_n) + \varepsilon_n$ for all $n \in N$. By the definition of δ_0 we also have $\delta_0(D_n) \le \delta_\rho(C^*)$. Since n is arbitrary and ε_n tends to 0 as $n \to \infty$, we deduce that $\delta_\rho(C^*) = \delta_\rho(C)$. Then the inequality $(**)$ implies that $\delta_\rho(C) \le R_\rho(C)$, a contradiction. Consequently, C is reduced to a single point which is then a fixed point for T.

Following [15], we say that a positive σ-finite measure μ on Σ (finite on $\mathcal{P}$) is absolutely continuous with respect to the function modular ρ ($\mu \ll \rho$) if $\mu(A) = 0$ for any ρ-null set A (recall that A is ρ-null if $\rho(\alpha 1_A) = 0$ for all $\alpha > 0$). For every mapping $T: B \to B$, $F(T)$ denotes the set of all fixed points of T.

Using a similar technique to that used in [8], we can prove the following result.

Theorem 3.6. Let ρ have the Fatou property and let μ be a σ-finite measure on Σ. Suppose that a ρ-bounded, (ρ-a.e.)-closed $B \subset L_\rho$ is compact in the sense of convergence in measure on sets of finite measure and B has ρ-normal structure. Then any commutative family of ρ-nonexpansive self-mappings of B has a common fixed point.

The rest of the paper is devoted to some special cases and examples that show possible applications of theorems 3.5 and 3.6. We will start with an analog of the Brodskii–Milman theorem. First, we have to prove the following technical result.

Lemma 3.7. Let ρ be a convex function modular and B be a convex, ρ-bounded subset of L_ρ. Assume that B is ρ-diametral and not reduced to a single point. Then B contains a ρ-diametral sequence. More precisely, if B is ρ-diametral, then there exists a sequence $\{f_n\}$ in B such that

$$\left(1 - \frac{1}{n+1}\right)\delta_\rho(B) \le \operatorname{dist}_\rho(f_{n+1}, \operatorname{conv}(f_i : 1 \le i \le n)) \le \delta_\rho(B).$$

Proof. Since B is ρ-diametral, $r_\rho(f, B) = \delta_\rho(B)$ for any $f \in B$. Therefore, for each $\varepsilon > 0$, we can find a function $g_\varepsilon \in B$ such that $(1 - \varepsilon)\delta_\rho(B) \le \rho(f - g_\varepsilon)$. Fix an arbitrary $f_1 \in B$. Then there exists $f_2 \in B$ such that $\rho(f_1 - f_2) \ge (1 - \frac{1}{4})\delta_\rho(B)$. Since B is convex, $\frac{1}{2}(f_1 + f_2)$ belongs to B, and therefore we can find $f_3 \in B$ such that $\rho(\frac{1}{2}(f_1 + f_2) - f_3) \ge (1 - \frac{1}{9})\delta_\rho(B)$. Thus we can inductively construct a sequence $\{f_n\}$ of elements of B such that

$$(\#) \qquad \rho\left(f_{n+1} - \frac{f_1 + f_2 + \cdots + f_n}{n}\right) \ge \left(1 - \frac{1}{(n+1)^2}\right)\delta_\rho(B).$$

Let $g \in \operatorname{conv}(f_i;\ 1 \le i \le n)$. Then $g = \sum_{i \le n} \alpha_i f_i$, with $\alpha_i \ge 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Note that $(1/n)\sum_{i=1}^{n} f_i = (1/n\alpha)g + \sum_{i=1}^{n}((1/n) - (\alpha_i/n\alpha))f_i$, where $\alpha = \max\{\alpha_i\}$. Since $(1/n) - (\alpha_i/n\alpha) \geq 0$ and $\sum_{i=1}^{n}((1/n) - (\alpha_i/n\alpha)) + (1/n\alpha) = 1$, we have

$$\frac{1}{\alpha n}\rho(g - f_{n+1}) + \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right)\rho(f_i - f_{n+1}) \geq \rho\left(\frac{1}{n}\sum_{i=1}^{n} f_i - f_{n+1}\right).$$

By (#) we get

$$\frac{1}{\alpha n}\rho(g - f_{n+1}) + \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right)\delta_\rho(B) \geq \left(1 - \frac{n\alpha}{(n+1)^2}\right)\delta_\rho(B).$$

By simple arithmetic we then obtain $\{1 - [n\alpha/(n+1)^2]\}\delta_\rho(B) \leq \rho(g - f_{n+1})$. Since $\alpha n \leq n + 1$ and $\alpha_1, \ldots, \alpha_n$ are arbitrary, we obtain the desired inequality.

THEOREM 3.8. Let ρ be a convex function modular and let B be convex and compact in the sense of the norm $\|\cdot\|\rho$. If B is ρ-bounded and not reduced to a single point, then B contains a ρ-nondiametral point.

Proof. Assume to the contrary that B is ρ-diametral. By lemma 3.7, B contains a ρ-diametral sequence (f_n). Since B is $\|\cdot\|_\rho$-compact, there exists a subsequence $\{f_{n'}\}$ which is $\|\cdot\|_\rho$-convergent to $f \in B$. We have

$$\rho(f_{n'} - f_{m'}) \leq \rho(2(f_{n'} - f)) + \rho(2(f_{m'} - f)) \to 0 \qquad \text{as } n', m' \to \infty.$$

On the other hand, if $n' < m'$ we have

$$\mathrm{dist}_\rho(f_{m'}, \mathrm{conv}(f_i : i < m')) \leq \rho(f_{m'} - f_{n'}).$$

Since $\mathrm{dist}_\rho(f_{m'}, \mathrm{conv}(f_i : i < m')) \to \delta_\rho(f_i)$ as $m' \to \infty$, this contradicts the fact that $\delta_\rho(f_i) > 0$ (see definition 3.2(c)).

We can obtain more interesting examples of sets with ρ-normal structure if we use a "uniform convexity" concept for function modulars. This is not surprising because of the well known connection between normal structure and uniform convexity in Banach spaces [10, 11, 24].

Some authors generalized the notions of normal structure and uniform convexity to metric spaces [9, 11, 20], but there had been no concrete examples until the paper by Goebel *et al.* [5] (for more information see Goebel and Reich [4]).

We will define uniform convexity of function modulars, will prove that uniform convexity of ρ implies ρ-normal structure and give a concrete example of a uniformly convex function modular.

Definition 3.9.

(a) For any nonzero $u \in L_\rho$ and $r > 0$, we define the r-modulus of uniform convexity of ρ in the direction of u to be:

$$\delta(r, u) = \inf\left\{1 - \frac{1}{r}\rho\left(f + \frac{1}{2}u\right)\right\},$$

where the infimum is taken over all $f \in L_\rho$ such that $\rho(f) \leq r$ and $\rho(f + u) \leq r$.

(b) We say that a convex function modular ρ is uniformly convex in every direction (U.C.E.D.) if $\delta(r, u) > 0$ for every nonzero $u \in L_\rho$ and all $r > 0$.

(c) We say that ρ is uniformly convex (U.C.) if for each $\varepsilon > 0$ and $r > 0$,

$$\inf\{\delta(r, u) : u \in L_\rho \text{ and } \rho(u) \geq r\varepsilon\} > 0.$$

Note the formal similarity of our definition to the relevant concept introduced for metric spaces [4, 5]. Musielak [18, definition 11.5] defined uniform convexity for modulars in exactly the same way as for norms. Such a concept of uniform convexity does not seem to be an appropriate tool for dealing with ρ without the Δ_2-condition, which is of particular interest to us (see example 3.11 below).

PROPOSITION 3.10. Let ρ be a U.C.E.D. function modular, and let $B \subset L_\rho$ be star-shaped, ρ-bounded and not reduced to a single point. Then B has a ρ-nondiametral point.

Proof. Let f be a center of B. Take any $g \in B$, with $g \neq f$ and put $\varepsilon = \delta_\rho(B)^{-1}\rho(\frac{1}{2}(f - g))$. Observe that $0 < \varepsilon < \infty$, because $f \neq g$ and B is ρ-bounded and not reduced to a single point. Let us fix temporarily any $h \in B$ and set $u = f - g$, $w = g - h$ and $r = \delta_\rho(B)$. Then $\rho(w) = \rho(g - h) \leq r$ and $\rho(w + u) = \rho(f - h) \leq r$. In view of the U.C.E.D. of ρ, we have then

$$\rho(w + \tfrac{1}{2}u) \leq r(1 - \delta(r, u)), \qquad \text{i.e. } \rho\left(\frac{f + g}{2} - h\right) \leq r(1 - \delta(r, u)).$$

Hence, $\sup \rho[((f + g)/2) - h] \leq \delta_\rho(B)(1 - \delta(r, u))$ and since $\delta(r, u) > 0$, the function $(f + g)/2$ is not a ρ-diametral point.

Example 3.11.

(i) Let us recall that a nonnegative, real function φ is said to be strictly convex (S.C.) if for $u \neq v$ there holds

$$\varphi\left(\frac{u + v}{2}\right) < \frac{\varphi(u) + \varphi(v)}{2}.$$

(ii) Similarly, we can say that a convex function modular ρ is S.C. if $f = g$ whenever $\rho(f) = \rho(g)$ and

$$\rho\left(\frac{f + q}{2}\right) = \frac{\rho(f) + \rho(g)}{2}.$$

(iii) A convex function φ is called uniformly convex (U.C.) if for every $a \in (0, 1)$, there exists $\delta(a) \in (0, 1)$ such that:

$$\varphi\left(\frac{1 + b}{2}\right) \leq (1 - \delta(a))\frac{\varphi(u) + \varphi(bu)}{2} \qquad \text{for every } u > 0 \text{ and } 0 \leq b \leq a.$$

Let $\varphi: \mathbb{R} \to [0, \infty)$ be an even, continuous and convex function such that $\varphi(0) = 0$ and $\varphi(u)/u \to \infty$ as $u \to \infty$. Let us consider the Orlicz space L^φ, i.e. the modular function space determined by the function modular:

$$\rho_\varphi(f) = \int_{\mathbb{R}} \varphi(f(x))\, dm(x).$$

One can prove (cf. [7]) that ρ_φ is U.C.E.D. if and only if φ is S.C. (as a matter of fact, this is also equivalent to the strict convexity of ρ_φ). We also know that the uniform convexity of the Orlicz modular ρ_φ is equivalent to the very convexity of φ. (Recall that a function φ is said to be very convex (V.C.) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta((u + v)/2) \le (1 - \delta)(\varphi(u) + \varphi(v)/2$ whenever $\varphi((u - v)/2) \ge \varepsilon(\varphi(u) + \varphi(v))/2$.)

Using theorem 3.5 and the above remarks one can easily prove the following result.

PROPOSITION 3.12. Let L^φ be an Orlicz space with φ S.C. Assume that $B \subset L^\varphi$ is star-shaped, ρ-bounded, (ρ-a.e.)-closed and compact in the sense of convergence in measure. If $T: B \in B$ is ρ-nonexpansive, then $F(T) \ne \emptyset$.

It is clear that φ is S.C. if φ is V.C. (the converse is not true). It was also proved that φ is V.C. if and only if ρ is U.C. (cf. [7]). As an example of a very convex Orlicz function one can take $\varphi(u) = e^{|u|} - |u| - 1$. Proposition 3.10 shows an advantage of our theory in comparison with the fixed point results in the norm sense, since it was proved by Kaminska [6], that the Luxemburg norm in L^φ is U.C. if and only if φ is U.C. and verifies the Δ_2-condition. Thus for φ without Δ_2 (e.g. $\varphi(u) = e^{|u|} - |u| - 1$) we cannot use the classical results in order to obtain fixed points of norm nonexpansive mappings. Let us also recall that the Δ_2-condition is also necessary for the strict convexity of the Luxemburg norm [25], and for uniform convexity of L^φ with the Orlicz norm (see e.g. [18]). Using different methods, Lami Dozo and Turpin [16] obtained a similar result to our proposition 3.11 for the case of Musielak–Orlicz spaces. Instead of strict convexity of φ they assumed some growth condition and $B \subset L_0^\varphi$ was assumed to satisfy a stronger kind of ρ-boundedness ($\sup\{\rho(\lambda(f - g)) : f, g \in B\} < \infty$ for some $\lambda > 1$).

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