Fixed Point Theorems for Dissipative Mappings in Complete Probabilistic Metric Spaces.

Mohamed A. Khamsi,
Department of Mathematical Sciences,
Vladik Ya. Kreinovich,
Computer Science department,
The University of Texas at El Paso.

Abstract

the physical notion of dissipative process gave rise to a mathematical notion of dissipative mapping of metric spaces, that has many applications to mathematical physics and nonlinear analysis.
We describe a physical motivated way to generalize this notion to probabilistic metric spaces(also called Menger spaces) and prove that well-known fixed point theorems for dissipative mappings can be generalized to Menger spaces.

1980 Mathematics Subject Classification: Primary 54H25; Secondary 47H10.
Key words and phrases: Menger spaces, porbabilistic metric spaces, distribution functions, fixed points, contraction mappings.

Mailing address:
Mohammed A. Khamsi,
Department of Mathematical Sciences,
The University of Texas at El Paso,
El Paso, Texas 79968-0514.

1 Introduction and Preliminaries.

The notion of a probabilistic metric space (Menger space) corresponds to the situations when we do not know the distance between the points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.
For a detailed discussion of Menger spaces and their applications, we refer to [5]. Let us give a formal definition

Definition 0. A map $F : R \to R$ is called a distribution function if it is nondecreasing, right-continuous with $\inf F(x) = 0$ and $\sup F(x) = 1$. We will denote by Δ the set of all distribution functions and by Δ^+ the set of distributions F such that $F(0) = 0$.
A commutative, associative and nondecreasing mapping $t : [0,1] \times [0,1] \to [0,1]$ is called a T-norm if and only if

(i) $t(a, 1) = a$ for all $a \in [0,1]$,

(ii) $t(0,0) = 0$.

One can easily check that $t(a, b) = Min(a, b)$ is a T-norm, and that any T-norm satisfies the inequality

$$t(a, b) \leq Min(a, b).$$

Definition 1. A Menger space is a triple (X, F, t), where X is a set, $F : X \times X \to \Delta^+$ and t is a T-norm. By $F_{x,y}$ we mean $F(x, y)$. The map F must satisfy the following three conditions.

(1) $F_{x,y} = F_{y,x}$ for all $x, y \in X$,

(2) $F_{x,y}(\epsilon) = 1$ if and only if $x = y$,

(3) for all $x, y, z \in X$ and $\epsilon, \delta > 0$, we have

$$F_{x,y}(\epsilon + \delta) \geq t(F_{x,z}(\epsilon), F_{z,y}(\delta)).$$

Let us recall the notion of a dissipative mapping that is physically well motivated and has many applications to nonlinear analysis and to mathematical physics. First a physical motivation. Suppose we consider a particle (or, more general, a physical system) that can be located in different points of a space X (X can also be phase space,i.e., represent different values of some additional parameters). By $E(x)$, for $x \in X$, let us denote the potential energy of a particle located in the point x. Consider $e(x, y)$ to be the minimal energy that is necessary to use in order to move the particle from the position x to the position y. Then evidently we have :

(1) $e(x, x) = 0$ for every $x \in X$.

(2) $e(x, y) \leq e(x, z) + e(z, y)$ for every $x, y, z \in X$.

We assume that the function $e(.,.)$ is symmetric. Then one can easily check that $d(x,y) = e(x,y)$ defines a pseudometric on X, i.e. in other words the relation $d(x,x) = 0$ is an equivalence relation and d is a metric on the factor-space with respect to this relation.
If a function T describes the evolution of this system (or particle) (in the sense that if it was initially in x then in the next moment of the time it will be in Tx), then evidently

$$d(x, Tx) \leq E(x) - E(Tx),$$

because the transformation T is just one of the ways to achieve $y = Tx$ from x and $d(x, Tx)$ is by definition the minimal possible energy loss. This inequality is used as a definition of a E-dissipative mapping.
The correspondent physical situation corresponds to the irreversible loss (dissipation) of energy. However, from physical viewpoint this picture is only approximately true. When we say that the evolution is irreversible in the sense that starting from a chaotic state (e.g. all molecules of a gas uniformly distributed) we cannot come spontaneously to a more organized state (all molecules are in one half of the vessel) really we understand that in principle such an evolution is possible, but the probability of such a "fluctuation" is small. This is due to the fact that when we describe the state x by the values of the corresponding macro-observables we mix together several microstates. The transition is possible without the energy loss even when $d(x,y) > 0$. In other words, in this case the minimal energy loss $d(x,y)$ from state x to a state y depends on the microstate and is a random variable. We will describe these random variables by their distributions

$$F_{x,y}(a) = P(d(x,y) < a).$$

What is the analog of the triangle inequality here? If $d(x,y) < a$ and $d(y,z) < b$ then $d(x,z) < a+b$. If we consider all the pairs as independent we conclude that the probability that $d(x,z) < a+b$ is greater or equal than the product of the probabilities that $d(x,y) < a$ and $d(y,z) < b$. In general we can make other assumptions and therefore come with different functions instead of a product that correspond to "and". The natural demands that $A\&B$ means the same as $B\&A$, that $(A\&B)\&C$ and $A\&(B\&C)$ mean the same, lead naturally to the definition of a T-norm t, and so we come to the inequality

$$F_{x,z}(a+b) \geq t(F_{x,y}(a), F_{y,z}(b)).$$

Moreover it is natural to assume that if we change the probability of A or B a little bit then the probability of $A\&B$ should also change slightly, so we will restrict ourselves to continuous T-norms. So we conclude that X is a stochastic metric space.
Let us now denote by $p(E)$ the probability of a "normal" evolution (i.e. the evolution that obeys the statistical laws) in case the energy changes by E. If we do not have any change at all then $p(0) = 1$. If we make a transition with the energy difference $E + E'$, it can be in principle represented as first diminishing the energy by E and then by E'. Then in order that this transition be normal it is necessary (but may be not sufficient) that the transitions on E and E' are normal, so

$$p(E + E') \leq t(p(E), p(E')).$$

In these terms our definition of E(x) leads to the conclusion that, unless the fluctuation occurs, $E(Tx) \leq E(x) - d(x, Tx)$, i.e. $d(x, Tx) \leq E(x) - E(Tx)$. Therefore the probability ***********************************

$$F_{x,Tx}(E(x) - E(Tx)) \geq p(E(x) - E(Tx)).$$

Definition 2. Let (X, F, t) be a Menger space, where t is a continuous T-norm. Let $\lambda : X \to R_+$ be a lower semi-continuous function. We will say that $T : X \to X$ is λ-dissipative if and only if there exists $h : R \to (0, 1)$ for which $\lim\limits_{\epsilon \to 0} h(\epsilon) = 1$ and $h(a + b) \leq t(h(a), h(b))$ such that

$$F_{x,Tx}(\lambda(x) - \lambda(Tx)) \geq h(\lambda(x) - \lambda(Tx)),$$

for every $x \in X$.

Remark. The inequality $h(a + b) \leq t(h(a), h(b))$ implies easily that h is nonincreasing. And if $t(.,.) = Min(.,.)$, then this inequality is indeed equivalent to the fact that h is non-increasing.

Recall that (X, F, t) is a Hausdorff topological space where the topology is defined by the following family of neighborhoods

$$\{N_x(a, b); x \in X, \ \ a > 0 \ \ and \ \ b \in (0, 1)\},$$

where $N_x(a, b) = \{y \in X; \ \ F_{x,y}(a) > 1 - b\}$. In [5], the proof is given of the fact that this topology is metrizable.
For the seek of completeness, we give the proof of the following technical lemma.

Lemma. Let (X, F, t) be a Menger space with continuous t. If $\lim_n x_n = x$ and $\lim_n y_n = y$, where (x_n) and (y_n) are some sequences from X and $x, y \in X$, then $weak - \lim_n F_{x_n,y_n} = F_{x,y}$, i.e. $F_{x_n,y_n}(\epsilon) \to F_{x,y}(\epsilon)$ for every ϵ which is a point of continuity of $F_{x,y}$.

Proof. Since $x_n \to x$ and $y_n \to y$ we conclude that for every $a > 0$ and $b \in (0, 1)$, there exists an $n_0 \geq 1$ such that for every $n \geq n_0$ we get $F_{x,x_n}(a) > 1 - b$ and $F_{y,y_n}(a) > 1 - b$. Therefore applying the triangle inequality we conclude that

$$F_{x_n,y_n}(a') > t(F_{x,x_n}(a), F_{x,y}(a' - 2a), F_{y,y_n}(a)) \geq t(1 - b, 1 - b, F_{x,y}(a' - 2a)),$$

where a' is a point of continuity of $F_{x,y}$. Therefore for arbitrary small $\epsilon > 0$ and for small b, we have

$$F_{x,y}(a' - 2a) > F_{x,y}(a') - \epsilon.$$

The function t is continuous so

$$t(1 - b, 1 - b, F_{x,y}(a') - \epsilon) > t(1, 1, F_{x,y}(a') - \epsilon) - \epsilon = (F_{x,y}(a') - \epsilon) - \epsilon.$$

Therefore for every $n \geq n_0$, we have

$$F_{x_n,y_n}(a') > F_{x,y}(a') - 2\epsilon.$$

Let us complete the proof by showing that

$$F_{x_n,y_n}(a') < F_{x,y}(a') + 2\epsilon.$$

Indeed, using the same triangle inequality we conclude that

$$F_{x,y}(a' + 2a) \geq t(F_{x,x_n}(a), F_{x_n,y_n}(a'), F_{y,y_n}(a)) \geq t(1 - a, 1 - a, F_{x_n,y_n}(a')).$$

The function t is continuous on a square (that is compact), so it is uniformly continuous. Therefore for sfficently small b we have

$$t(1 - b, 1 - b, z) > t(1, 1, z) - \epsilon = z - \epsilon.$$

So for such b we conclude that $F_{x_n,y_n}(a') < F_{x,y}(a' + 2a) + \epsilon$. But a' is a point of continuity of $F_{x,y}$, therefore for sufficiently small b we get

$$F_{x,y}(a' + 2a) < F_{x,y}(a') + \epsilon.$$

The proof is therefore complete.

Example. Let (M, d) be a metric space. Define $F : M \times M \to \Delta^+$ by

$$F_{x,y}(\epsilon) = H(\epsilon - d(x, y)),$$

where we denote by H the distribution function defined by

$$H(\epsilon) = 0 \;\; if \;\; \epsilon < 0, \;\; and \;\; 1 \;\; otherwise.$$

Then (M, F, Min) is a Menger space, and

$$N_x(\epsilon, b) = \{y \in M; \;\; d(x, y) \leq \epsilon\}.$$

So (M, F, Min) is a complete Menger space if and only if (M, d) is a complete metric space. For the definition of complete Menger spaces, one can consult [5].

2 Main results.

Before stating any fixed point result, we introduce the analog partial order that was used by Bronsted [1] (see also [4]).

Theorem 1. Let (X, F, t) be a complete Menger space. Let $\lambda : X \to [0, \infty)$ be a lower semi-continuous function and $h : R \to (0, 1)$ for which $\lim_{\epsilon \to 0} h(\epsilon) = 1$ and $h(a + b) \leq t(h(a), h(b))$. Define the partial order $\prec$ by

$$x \prec y \;\; iff \;\; F_{x,y}(\lambda(y) - \lambda(x)) \geq h(\lambda(y) - \lambda(x)).$$

Then, any decreasing chain in $(X, \prec)$ has a lower bound. In particular X contains minimal elements with respect to $\prec$.

<u>Proof.</u> First, it is elementary that $\prec$ defines indeed a partial order on X. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a decreasing chain, where Γ is a directed set. Then, since every $F_{x,y}$ belongs to Δ^+, we obtain that $(\lambda(x_\alpha))_{\alpha\in\Gamma}$ is a decreasing net of positive numbers. let

$$\lambda_0 = \inf\{\lambda(x_\alpha);\ \ \alpha \in \Gamma\}.$$

So, for every $n \geq 1$, there exists $\alpha_n \in \Gamma$ such that

$$\lambda(x_{\alpha_n}) \leq \lambda_0 + \frac{1}{n}.$$

One can assume that (α_n) is increasing. Let $a > 0$ and $b \in (0, 1)$, then one can find $n_0 \geq 1$ such that

$$\lambda(x_{\alpha_n}) - \lambda(x_{\alpha_m}) \leq a, \ \ and \ \ h(\lambda(x_{\alpha_n}) - \lambda(x_{\alpha_m})) \geq 1 - b,$$

for every $m \geq n \geq n_0$. Therefore, we have

$$x_{\alpha_m} \in N_{x_{\alpha_n}}(a, b),$$

for every $m \geq n \geq n_0$.
This means that (x_{α_n}) is a Cauchy sequence in (X, F, t). Let $\omega \in X$ be the limit of (x_{α_n}), which exists since (X, F, t) is complete. We show that ω is in fact a lower bound for $(x_\alpha)_{\alpha\in\Gamma}$.
First notice that $\omega \prec x_{\alpha_n}$ for all $n \geq 1$. Indeed, using the lower semi-continuity of λ, one gets

$$\lambda(\omega) \leq \lambda_0.$$

For every $m \geq n \geq 1$, we have

$$F_{x_{\alpha_m}, x_{\alpha_n}}(\lambda(x_{\alpha_n}) - \lambda(\omega)) \geq h(\lambda(x_{\alpha_n}) - \lambda(\omega))$$

since $F_{x,y}$ are nondecreasing for every $x, y \in X$ and h is nonincreasing. Let $a > \lambda(x_{\alpha_n}) - \lambda(\omega)$ be a point of continuity of $F_{x_{\alpha_n},\omega}$. Then using the technical lemma, we get

$$F_{x_{\alpha_n},\omega}(a) = \lim_{m\to\infty} F_{x_{\alpha_n}, x_{\alpha_m}}(a) \geq h(\lambda(x_{\alpha_n}) - \lambda(\omega)).$$

Since the points of discontinuity of $F_{x_{\alpha_n},\omega}$ are countable and using the right continuity of $F_{x_{\alpha_n},\omega}$, we have

$$F_{x_{\alpha_n},\omega}(\lambda(x_{\alpha_n}) - \lambda(\omega)) \geq h(\lambda(x_{\alpha_n}) - \lambda(\omega)),$$

i.e. $\omega \prec x_{\alpha_n}$ for every $n \geq 1$.
Now, we complete the proof of our claim. Indeed, let $\alpha \in \Gamma$. If $\alpha \leq \alpha_n$ for some $n \geq 1$, then

$$\omega \prec x_{\alpha_n} \prec x_\alpha.$$

Otherwise, assume that $\alpha_n \leq \alpha$ for all $n \geq 1$. Then $\lambda(x_\alpha) \leq \lambda(x_{\alpha_n})$, which clearly implies that

$$\lim_{n\to\infty} \lambda(x_{\alpha_n}) = \lambda_0 \geq \lambda(x_\alpha)$$

since λ is lower semi-continuous. It is therefore obvious that we have $\lambda(x_\alpha) = \lambda_0$. This will imply easily that

$$\lim_{n\to\infty} x_{\alpha_n} = x_\alpha.$$

Uniqueness of the limit will oblige x_α to be equal to ω. Therefore, we get

$$\omega \prec x_\alpha$$

for all $\alpha \in \Gamma$. In order to complete the proof of Theorem 1, one can use Zorn's lemma.

The next theorem can be seen as an analoguous to Caristi' s result [2] (see also [1,4]).

<u>**Theorem 2.**</u> Let (X, F, t) be a complete Menger space and $T : X \to X$ be a λ-dissipative map.
Then T has a fixed point, i.e. there exists $x \in X$ such that $T(x) = x$.

<u>Proof.</u> Using the order described in Theorem 1, we get

$$T(x) \prec x$$

for every $x \in X$. Let ω be a minimal element in X. Then clearly we have $T(\omega) = \omega$, which finishes the proof of Theorem 2.

In [5], the concept of contraction maps in Menger spaces is defined. let us recall this definition.

<u>**Definition 4.**</u> Let (X, F, t) be a Menger space and $T : X \to X$ be a map. We will say tha T is a contraction if there exists $k \in (0, 1)$ such that

$$F_{Tx,Ty}(\epsilon) \geq F_{x,y}(\frac{\epsilon}{k})$$

for every $x, y \in X$ and $\epsilon > 0$.

It is not hard to prove that there exists $\lambda : X \times (0, 1] \to R_+$ such that

$$(**) \qquad F_{x,Tx}(\lambda(x, \epsilon) - \lambda(Tx, \epsilon)) \geq \epsilon$$

for every $\epsilon \in (0, 1]$ and $x \in X$. Moreover, the function λ is weak-lower semi-continuous, i.e. the maps $\lambda_\epsilon : X \to R_+$ defined by

$$\lambda_\epsilon(x) = \lambda(x, \epsilon)$$

are lower semi-continuous.

In the next theorem, we discuss a fixed point result for mappings satifying $(**)$.

Theorem 3. Let (X, F, t) be a complete Menger space, $\lambda : X \times (0, 1] \to R_+$ be weak-lower semi-continuous and $h : (0, 1] \to (0, 1]$ be a function satifying $\sup h(\epsilon) = 1$. Define the partial order $\prec$ on X by

$$x \prec y \quad iff \quad F_{x,y}(\lambda(y, \epsilon) - \lambda(x, \epsilon)) \geq h(\epsilon).$$

Then, any decreasing chain in X has a lower bound. In particular X contains minimal elements with respect to $\prec$.

Proof. It is easy to check that $\prec$ defines a partial order on X. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a decreasing chain, where Γ is a directed set. Then again since $F_{x,y} \in \Delta^+$, we deduce that $(\lambda(x_\alpha, \epsilon))$ is a decreasing net of positive numbers, for every $\epsilon \in (0, 1]$. For every $a > 0$ and $b \in (0, 1)$, there exists $\alpha \in \Gamma$ such that

$$x_\beta \in N_{x_\alpha}(a, b)$$

for any $\beta \geq \alpha$. Indeed, let $\epsilon \in (0, 1]$ such that $h(\epsilon) > 1 - b$. Then, one can find $\alpha_0 \in \Gamma$ such that

$$\lambda(x_{\alpha_0}, \epsilon) \leq \inf_{\alpha\in\Gamma} \lambda(x_\alpha, \epsilon) + a.$$

Let $\beta \geq \alpha_0$, then

$$\inf_{\alpha\in\Gamma} \lambda(x_\alpha, \epsilon) \leq \lambda(x_\beta, \epsilon) \leq \lambda(x_{\alpha_0}, \epsilon) \leq \inf_{\alpha\in\Gamma} \lambda(x_\alpha, \epsilon) + a.$$

Then,

$$F_{x_\beta,x_{\alpha_0}}(a) \geq F_{x_\beta,x_{\alpha_0}}(\lambda(x_{\alpha_0}, \epsilon) - \lambda(x_\beta, \epsilon)) \geq h(\epsilon) \geq 1 - b,$$

$$i.e. \quad x_\beta \in N_{x_{\alpha_0}}(a, b).$$

Let (a_n, b_n) such that $a_n \downarrow 0$ and $b_n \uparrow 1$ as $n \to \infty$. Then, one can find $(\alpha_n) \subset \Gamma$ such that for every $\beta \geq \alpha_n$, we have

$$x_\beta \in N_{x_{\alpha_n}}(a_n, b_n)$$

for every $n \geq 1$. Since Γ is a directed set, one can assume that (α_n) is increasing. The choice of (α_n) implies that (x_{α_n}) is a Cauchy sequence. Therefore its limit, say $\omega \in X$, exists since (X, F, t) is complete. Let us show that $\omega \prec x_\alpha$ for any $\alpha \in \Gamma$.
First, let us show that $\omega \prec x_{\alpha_n}$ for all $n \geq 1$. Indeed, for every $m \geq n \geq 1$, we have

$$F_{x_{\alpha_m},x_{\alpha_n}}(\lambda(x_{\alpha_n}, \epsilon) - \lambda(x_{\alpha_m}, \epsilon)) \geq h(\epsilon),$$

for every $\epsilon > 0$. Using the weak-lower semi-continuity of λ, one can get

$$\lambda(\omega, \epsilon) \leq \inf_{n\geq 1} \lambda(x_{\alpha_n}, \epsilon),$$

which implies

$$F_{x_{\alpha_m},x_{\alpha_n}}(\lambda(x_{\alpha_n}, \epsilon) - \lambda(\omega, \epsilon)) \geq h(\epsilon),$$

for every $\epsilon > 0$. Using the same idea, as in the proof of the Theorem 1, one can get

$$F_{\omega,x_{\alpha_n}}(\lambda(x_{\alpha_n}, \epsilon) - \lambda(\omega, \epsilon)) \geq h(\epsilon),$$

for every $n \geq 1$ and $\epsilon > 0$. This clearly implies that for all $n \geq 1$, we have

$$\omega \prec x_{\alpha_n}.$$

Let $\alpha \in \Gamma$, and assume that $\alpha \leq \alpha_n$ for some $n \geq 1$, then

$$\omega \prec x_{\alpha_n} \prec x_\alpha.$$

In order to complete the proof of our statement, let $\alpha \in \Gamma$ be such that $\alpha_n \leq \alpha$ for every $n \geq 1$. By definition of α_n, we have

$$x_\alpha \in N_{x_{\alpha_n}}(a_n, b_n),$$

for any $n \geq 1$, which implies that (x_{α_n}) converges to x_α. The uniqueness of the limit implies that

$$\omega = x_\alpha.$$

This completes the proof of our claim and Zorn's lemma will complete the proof of Theorem 3.

As a direct application of Theorem 3, one can get the following fixed point result.

Theorem 4. Let (X, F, t) be a complete Menger space, $\lambda : X \times (0, 1] \to R_+$ be weak-lower semi-continuous and $h : (0, 1] \to (0, 1]$ be a function satifying $\sup h(\epsilon) = 1$. Let $T : X \to X$ be a map satisfying for every $x \in X$ and $\epsilon \in (0, 1]$,

$$(***) \quad F_{x,Tx}(\lambda(x, \epsilon) - \lambda(Tx, \epsilon)) \geq h(\epsilon).$$

Then, T has a fixed point.

Proof. Let T be as assumed and $\prec$ be the order described in Theorem 3. Then clearly we have for every $x \in X$,

$$T(x) \prec x.$$

Let $x_0 \in X$ be a minimal element with respect to $\prec$. Then, clearly, x_0 is a fixed point for T.

As a corollary, one can get the following result (see [5]).

Corollary 1. Let (X, F, t) be a complete Menger space. Then any contraction on X has a unique fixed point.

Proof. We need just to show that the fixed point, which exists by Theorem 4, is unique. Indeed, let $T : X \to X$ be a contraction. Then, there exists $k \in (0, 1)$ such that for every $x, y \in X$ and $\epsilon > 0$, we have

$$F_{Tx,Ty}(\epsilon) \geq F_{x,y}(\frac{\epsilon}{k}).$$

Let x and y be two fixed points for T. Then,

$$F_{x,y}(\epsilon) \geq F_{x,y}(\frac{\epsilon}{k^n}),$$

for every $\epsilon > 0$ and $n \geq 1$. Using the definition of $F_{x,y}$, we get

$$F_{x,y}(\epsilon) \geq \sup_{\zeta>0} F_{x,y}(\zeta) = 1.$$

Which clearly implies that $x = y$.

Remark. For both Theorem 2 and Theorem 4, one can get a common fixed point result for any family of maps $(T_i)_{i\in I}$, if all the maps T_i are dissipative with respect to the same function λ (with the same function h) in Theorem 2, and satisfying the inequality $(***)$ in Theorem 4 for the same functions λ and h. Indeed, for both theorems, one can check that any minimal element is a common fixed point.

References

[1] A. Bronsted,"Fixed point and partial orders", Proc. Amer. Math. Soc. 60(1976),365-366.

[2] J. Caristi," fixed point theorems for mappings satisfying inwardness conditions", Trans. Amer. Math. Soc. 215(1976),241-251.

[3] J. Caristi and W.A. Kirk," Mapping theorems in metric and Banach spaces", Bull. L'Acad. Polon. Sci. 25(1975),891-894.

[4] I. Ekeland,"Sur les problemes variationnels", Comptes Rendus Acad. Sci. Paris,275(1972),1057-1059.

[5] B. Schweizer and A. Sklar,"Probabilistic metric spaces", North-Holland Series in Probability and Applied Mathematics 5(1983).