HOLOMORPHIC RETRACTS IN B_H^∞

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Abstract. In this paper we show that the common fixed point set of a commuting family of holomorphic mappings in B_H^∞ is either empty or a holomorphic retract.

1. Introduction

In the case of reflexive spaces P. Mazet and J.-P. Vigué ([34], [35]) obtained a retraction onto the fixed point set of a holomorphic self-mapping by using standard methods of complex analysis. They also showed that their approach fails in the case of the open ball in l^∞. However, it is known that if B_H^∞ is the open unit ball in a Cartesian product of infinitely many Hilbert spaces furnished with the sup norm and f is a holomorphic ($k_{B_H^\infty}$-nonexpansive) self-mapping of B_H^∞ with a nonempty fixed point set $\mathrm{Fix}(f)$, then this set $\mathrm{Fix}(f)$ is a holomorphic ($k_{B_H^\infty}$-nonexpansive) retract of B_H^∞. More generally, if we have a finite family of commuting ($k_{B_H^\infty}$-nonexpansive) holomorphic self-mappings of B_H^∞ with a nonempty common fixed point set, then this set is also a holomorphic ($k_{B_H^\infty}$-nonexpansive) retract of B_H^∞ ([32], see also [31]). Let us observe that in the case of the open unit ball B_H^n in a finitely many Hilbert spaces furnished with the max-norm the common fixed point set of every commuting family of holomorphic ($k_{B_H^n}$-nonexpansive) mappings in B_H^n is either empty or a holomorphic retract and for each finite family of commuting holomorphic ($k_{B_H^n}$-nonexpansive) self-mappings of B_H^∞ with fixed points their common fixed point set is nonempty.

Recently, the first author and T. Kuczumow showed, that if $\mathcal{F}$ is a countable family of holomorphic ($k_{B_H^\infty}$-nonexpansive) commuting self-mappings of B_H^∞ with a nonempty common fixed point set $\mathrm{Fix}(\mathcal{F})$, then the set $\mathrm{Fix}(\mathcal{F})$ is a $k_{B_H^\infty}$-nonexpansive retract of B_H^∞ [8]. In this paper we present the general result of this type: if $\mathcal{F}$ is a family of holomorphic ($k_{B_H^\infty}$-nonexpansive) commuting self-mappings of B_H^∞ with a nonempty common fixed point set $\mathrm{Fix}(\mathcal{F})$, then the set $\mathrm{Fix}(\mathcal{F})$ is a holomorphic ($k_{B_H^\infty}$-nonexpansive) retract of B_H^∞.

2. Preliminaries

In this paper we consider complex Banach spaces. Let B_H denote the open unit ball of a complex Hilbert space $(H, (\cdot,\cdot))$. This ball is called the Hilbert ball. Let k_{B_H} denote the Kobayashi distance on B_H ([24], [25]). We have the following explicit formula for the Kobayashi distance k_{B_H} on B_H

$$k_{B_H}(x,y) = \arg\tanh\left(1-\sigma(x,y)\right)^{\frac{1}{2}},$$

1991 *Mathematics Subject Classification.* 32A10, 46G20, 47H09, 47H10.

Key words and phrases. Fixed points, holomorphic mappings, k_B^∞-nonexpansive mappings, retracts.

where $x, y \in B_H$ and

$$\sigma(x,y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{|1 - (x,y)|^2}$$

([17], see also [11], [16] and [31]).

The metric space (B_H, k_{B_H}) has the following very useful properties:

(i) The Kobayashi distance k_{B_H} is locally equivalent to the norm $\|\cdot\|$ in H ([11], [14], [16], [18], [20], [31]);

(ii) Each ball in (B_H, k_{B_H}) is convex ([16], [17], [33], [31]);

(ii) The metric space (B_H, k_{B_H}) is locally linearly uniformly convex, i.e., for each $z \in B_H$, $R > 0$ and $0 < \epsilon < 2$ we have

$$\left.\begin{array}{r} k_{B_H}(z,x) \le R \\ k_{B_H}(z,y) \le R \\ k_{B_H}(x,y) \ge \epsilon R \end{array}\right\} \Rightarrow \left(z, \frac{1}{2}x + \frac{1}{2}y\right) \le (1 - \delta_0(z,R,\epsilon)) R$$

and

$$\delta(R_1, R_2, R_3, \epsilon_1, \epsilon_2) = \inf\{\delta_0(z,R,\epsilon) : \epsilon_1 \le \epsilon \le \epsilon_2, \quad \|z\| \le R_1, \quad R_2 \le R \le R_3\} > 0$$

for all $0 < R_1$, $0 < R_2 \le R_3$ and $0 < \epsilon_1 \le \epsilon_2 < 2$ ([28], see also [7]).

(iv) If $\{x_\lambda\}_{\lambda \in I}$ and $\{y_\lambda\}_{\lambda \in I}$ are nets in B_H which are weakly convergent to x and y respectively, $x, y \in B_H$, then

$$k_{B_H}(x,y) \le \liminf_{\lambda} k_{B_H}(x_\lambda, y_\lambda),$$

i.e., the Kobayashi distance is lower semicontinuous with respect to the weak topology in H ([27], see also [21] and [29]).

Now, let J be an infinite set of indices,

$$l^\infty(H) = \left\{ x = \{x_j\}_{j \in J} \in \prod_{j \in J} H : \sup_{j \in J} \|x_j\| < \infty \right\},$$

and B_H^∞ the open unit ball in $l^\infty(H)$ with the supremum norm.

The Kobayashi distance in B_H^∞ is given by

$$k_{B_H^\infty}(x,y) = \sup_{j \in J} k_{B_H}(x_j, y_j)$$

and is locally equivalent to the norm ([32], see also [31]).

Now let us recall that a mapping $f : B_H^\infty \to B_H^\infty$ is $k_{B_H^\infty}$-nonexpansive if

$$k_{B_H^\infty}(f(x), f(y)) \le k_{B_H^\infty}(x,y)$$

for all $x, y \in B_H^\infty$. Each holomorphic self-mapping $f : B_H^\infty \to B_H^\infty$ is $k_{B_H^\infty}$-nonexpansive ([32]).

$\mathrm{Fix}(f)$ denotes the fixed point set of a self-mapping f of B_H^∞ and $\mathrm{Fix}(\mathcal{F})$ denotes the common fixed point set of a family $\mathcal{F}$ of self-mappings of B_H^∞.

We need to recall here a few facts about holomorphic mappings.

Theorem 2.1. *(Generalized Hartogs' Theorem) ([32], see also [2], [3], [10], [12] and [19]). Let X be a Banach space and D a nonempty open subset of X. If $f : D \longrightarrow l^{\infty}(H)$ is locally bounded, then the following statements are equivalent:*
(i) $f = \{f_j\}$ is holomorphic;
(ii) each $f_j : D \longrightarrow H$ is holomorphic.

Theorem 2.2. *([32]). Let $f : B_H^{\infty} \to B_H^{\infty}$ be a holomorphic mapping. Then the following statements are equivalent:*
(i) f has a fixed point;
(ii) there exists a ball $B(x,r)$ in $\left(B_H^{\infty}, k_{B_H^{\infty}}\right)$ which is f-invariant;
(iii) there exists an f-invariant, $k_{B_H^{\infty}}$-bounded product $\prod_{j\in J} C_j$ of closed convex subsets of B_H.

Remark 2.1. ([32]). One can observe that in contrast with the case of the open unit ball B_H, there exists in B_H^{∞} a holomorphic fixed-point-free self-mapping f with a $k_{B_H^{\infty}}$-bounded iteration $\left\{f^k(x)\right\}$ for each x.

Now we quote a result due to T. Kuczumow, S. Reich, A. Stachura ([32]).

Theorem 2.3. *If $f : B_H^{\infty} \to B_H^{\infty}$ is holomorphic ($k_{B_H^{\infty}}$-nonexpansive), then $Fix(f)$ is either empty or a holomorphic ($k_{B_H^{\infty}}$-nonexpansive) retract of B_H^{∞}.*

The following theorem is also known ([32], see also [31]).

Theorem 2.4. *Suppose $f_1, ..., f_m$ are commuting $k_{B_H^{\infty}}$-nonexpansive (holomorphic) self-mappings of B_H^{∞} such that $\bigcap_{j=1}^{m} Fix(f_j) \neq \emptyset$. Then $\bigcap_{j=1}^{m} Fix(f_j)$ is $k_{B_H^{\infty}}$-nonexpansive (holomorphic) retract of B_H^n.*

3. A few facts from the metric fixed point theory

Let (M,d) be a metric space. $B(x,r)$ will stand for the closed ball centered at $x \in M$ with the radius $r \geq 0$. For any nonempty bounded subset $A \subset M$, we set

$$\begin{aligned} r_x(A) &= \sup\{d(x,a) : a \in A\}, \quad x \in M, \\ r(A) &= \inf\{r_a(A) : a \in A\}, \\ \delta(A) &= diam(A) = \sup\{r_a(A) : a \in A\} \\ &= \sup\{d(x,y) : x, y \in A\} \end{aligned}$$

Recall that $r(A)$ is called the Chebyshev radius of A [15]).

For a bounded set A of M, set

$$cov(A) = \bigcap\{B(x,r) : x \in M, A \subset B(x,r)\}.$$

We will say that A is an admissible set if and only if $A = cov(A)$, i.e. A is an intersection of closed balls. The family of all admissible subsets of M will be denoted by $\mathfrak{A}(M)$.

A family $\mathcal{S} \subset 2^M$ is called a convexity structure if
(i) $\emptyset, M \in \mathcal{S}$,
(ii) $\{x\} \in \mathcal{S}$ for each $x \in M$,
(iii) $\mathcal{S}$ contains the closed balls of M,
(iv) $\mathcal{S}$ is closed under arbitrary intersections.

Let us observe that the smallest convexity structure is the family $\mathfrak{A}(M)$ of all admissible subsets of M.

We will say that a convexity structure $\mathcal{S}$ of M is compact if each descending chain of nonempty sets in $\mathcal{S}$ has nonempty intersection.

A convexity structure $\mathcal{S}$ is said to be normal if for each $A \in \mathcal{S}$ we have either $\delta(A) = 0$ or $r(A) < \delta(A)$.

The crucial theorem in our next considerations is the following

Theorem 3.1. [23] *Let (M, d) be a bounded metric space with a convexity structure $\mathfrak{A}(M)$ (i.e. the family of all admissible subsets of M). If $\mathfrak{A}(M)$ is compact and normal, then any commuting family $\mathcal{F}$ of nonexpansive self-mappings of M has a common fixed point.*

4. A common fixed point set of commuting holomorphic mappings in B_H^∞

We begin with the following simple observation.

Lemma 4.1. *Let $G = \prod_{j\in J} G_j$ be a $k_{B_H^\infty}$-bounded product of nonempty closed convex subsets of B_H. Then the family $\mathfrak{A}(G)$ of all admissible sets in a metric space $(G, k_{B_H^\infty})$ is compact and normal.*

Proof. It is sufficient to observe that each nonempty admissible set E in $(G, k_{B_H^\infty})$ is a product of nonempty closed convex subsets of B_H, which are weakly compact and that the metric space (B_H, k_{B_H}) is locally linearly uniformly convex. □

Corollary 4.2. *Let $G = \prod_{j\in J} G_j$ be a $k_{B_H^\infty}$-bounded product of nonempty closed convex subsets of B_H. If $\mathcal{F}$ is a commuting family of $k_{B_H^\infty}$-nonexpansive self-mappings of G, then $\mathcal{F}$ has a common fixed point in G.*

Proof. It is sufficient to apply Theorem 3.1. □

Corollary 4.3. *Let $\mathcal{F}$ be a commuting family of $k_{B_H^\infty}$-nonexpansive self-mappings of B_H^∞ and let $G = \prod_{j\in J} G_j$ be a $k_{B_H^\infty}$-bounded product of nonempty closed convex subsets of B_H which is $\mathcal{F}$-invariant. If $\mathcal{F}$ has a common fixed point in B_H^∞, then $\mathcal{F}$ has a common fixed point in G.*

Proof. Let x be a common fixed point of $\mathcal{F}$ in B_H^∞ and $B(x, r)$ a closed ball in $(B_H^\infty, k_{B_H^\infty})$. For sufficiently large $r > 0$ the set $\tilde{G} = G \cap B(x, r) \subset G$ is a nonempty, $k_{B_H^\infty}$-bounded and $\mathcal{F}$-invariant product of closed convex subsets of B_H. By Corollary 4.2, $\mathcal{F}$ has a common fixed point in $\tilde{G}$. □

Now we are ready to prove the main theorem

Theorem 4.4. *For any family $\mathcal{F}$ of commuting holomorphic ($k_{B_H^\infty}$-nonexpansive) self-mappings of B_H^∞ with the nonempty common fixed point set Fix($\mathcal{F}$), the set Fix($\mathcal{F}$) is a holomorphic ($k_{B_H^\infty}$-nonexpansive) retract of B_H^∞.*

Proof. We will use the Bruck method ([4], [5]).

We prove this result only in the holomorphic case. Let

$$\mathcal{N}_\infty = \{g : g \text{ is a holomorphic self-mapping of } B_H^\infty,\ \ Fix\,(\mathcal{F}) \subset Fix\,(g)\}$$

and let $x_0 = \{x_{0j}\} \in Fix\,(\mathcal{F})$ be fixed. We can observe that

$$\mathcal{N}_\infty \subset \prod_{x\in B_H^\infty}\prod_{j\in J}\{y \in B : k_{B_H}(y, x_{0j}) \le k_{B_H^\infty}(x, x_0)\} = \prod_{x\in B_H^\infty}\prod_{j\in J} C_{xj}.$$

If in each C_{xj} we have the weak topology, then each C_{xj} is weakly compact and therefore, by Tychonoff's Theorem ([13], [22]), the product $\prod_{x\in B_H^\infty}\prod_{j\in J} C_{xj}$ is compact in the product topology. Next, the set $\mathcal{N}_\infty$ is closed in the topology of coordinate pointwise weak convergence.

Now, we preorder $\mathcal{N}_\infty$ by setting $g \le h$ if and only if

$$k_{B_H^\infty}(g(x), w) \le k_{B_H^\infty}(h(x), w)$$

for all $w \in Fix\,(\mathcal{F})$ and $x \in B_H^\infty$ and we choose a descending chain $\{g_\lambda\}_{\lambda\in\Lambda} = \{\{g_{\lambda j}\}_{j\in J}\}_{\lambda\in\Lambda}$ in $(\mathcal{N}_\infty, \le)$. By the compactness of $\prod_{x\in B_H^\infty}\prod_{j\in J} C_{xj}$, this chain $\{g_\lambda\}_{\lambda\in\Lambda}$ has a subnet $\{g_{\lambda'}\}_{\lambda'\in\Lambda'}$ for which Λ' is an ultranet ([1], [13], [22]). Hence we get

$$w\text{-}\lim_{\lambda'} g_{\lambda' j}(x) = g_j(x),\ x \in B_H^\infty \text{ and } j \in J.$$

The mapping $g = \{g_j\}_{j\in J}$ is holomorphic. By the weak lower semicontinuity of k_{B_H} the following inequalities are valid:

$$k_{B_H^\infty}(g(x), w) \le \lim_{\lambda'} k_{B_H^\infty}(g_{\lambda'}(x), w)$$

$$\le k_{B_H^\infty}(g_\lambda(x), w)$$

for each $w \in Fix\,(\mathcal{F})$, $x \in B_H^\infty$ and $\lambda \in \Lambda$. This means that g is a lower bound of the chain $\{g_\lambda\}_{\lambda\in\Lambda}$ and therefore by the Kuratowski-Zorn Lemma, $\mathcal{N}_\infty$ contains a minimal element r. We claim that r is a retraction of B_H^∞ onto $\mathrm{Fix}\,(\mathcal{F})$.

Suppose there exists $y \in B_H^\infty$ such that $r(y) \notin Fix\,(\mathcal{F})$. By minimality of r in $\mathcal{N}_\infty$ and the inequality $r \circ r \le r$ we get

$$k_{B_H^\infty}(r(y_0), w) = k_{B_H^\infty}(r(r(y)), r(r(w))) = k_{B_H^\infty}(r(y), r(w)) = k_{B_H^\infty}(y_0, w) > 0$$

for $y_0 = r(y)$ and all $w \in Fix\,(\mathcal{F})$. Next, since for each $j \in J$, after interchanging j-coordinate functions between two arbitrarily chosen mappings from $\mathcal{N}_\infty$, we also have a mapping from $\mathcal{N}_\infty$, and since $g,h\in\mathcal{N}_\infty$ and $0\le\beta\le 1$ imply that $\beta g+(1-\beta)h\in\mathcal{N}_\infty$ too, the set $\mathcal{N}_\infty$ is equal to $\prod_{j\in J} D_j$, where each D_j is convex and weakly compact. Let

$$C=\{(g\circ r)(y_0): g\in\mathcal{N}_\infty\}.$$

Using the same arguments as above we see that C is $k_{B_H^\infty}$-bounded and $C=\prod_{j\in J} C_j$, where each C_j is convex and weakly compact. Directly from the definitions of $\mathcal{N}_\infty$, C and r we obtain that the set C is $\mathcal{F}$-invariant and hence by Corollary 4.3, $C\cap Fix(\mathcal{F})\neq\emptyset$. We choose an arbitrary point $(g\circ r)(y_0)\in C\cap Fix(\mathcal{F})$. Then we get the contradiction

$$\begin{aligned}0&=k_{B_H^\infty}((g\circ r)(y_0),(g\circ r)(y_0))=k_{B_H^\infty}((g\circ r)(y_0),(g\circ g\circ r)(y_0))\\&=k_{B_H^\infty}(r(y_0),(g\circ r)(y_0))>0.\end{aligned}$$

The proof in the $k_{B_H^\infty}$-nonexpansive case is practically the same. This completes the proof of the theorem. □

Remark 4.1. As the example given in [30] shows the assumption in the above theorem that the common fixed point set Fix($\mathcal{F}$) is nonempty is essential.

Remark 4.2. The following problem is still open. Let f_1 and f_2 be commuting $k_{B_H^\infty}$-nonexpansive (holomorphic) self-mappings of B_H^∞ such that $\mathrm{Fix}(f_j)\neq\emptyset$ for $1\le j\le 2$. Is $\mathrm{Fix}(f_1)\cap Fix(f_2)$ nonempty? It is not clear whether this is true when H is a one dimensional vector space. Let us observe that in the case of finite product B_H^n the answer to this question is positive ([26]).

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