Chapter 1

ULTRA-METHODS IN METRIC FIXED POINT THEORY

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1. Preface

Over the last two decades ultrapower techniques have become major tools for the development and understanding of metric fixed point theory. In this short chapter we develop the Banach space ultrapower and initiate its use in studying the weak fixed point property for nonexpansive mappings. For a more extensive and detailed treatment than is given here the reader is referred to [1] and [21].

2. Introduction to Ultrapowers of Banach spaces

Throughout the chapter I will denote an index set, usually the natural numbers $\mathbf{N}$ for most situations in metric fixed point theory.

Definition 2.1 *A filter on* I *is a nonempty family of subsets* $\mathcal{F} \subseteq 2^I$ satisfying

(i) $\mathcal{F}$ is closed under taking supersets. That is, $A \in \mathcal{F}$ and $A \subseteq B \subseteq I \implies B \in \mathcal{F}$.

(ii) $\mathcal{F}$ is closed under finite intersections: $A,\, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$.

Examples.

(1) The power set of I, 2^I, defines a filter.

(2) The *Fréchet filter* $\{A \subseteq I : I \backslash A$ is finite $\}$

(3) For $i_0 \in I$, $\mathcal{F}_{i_0} := \{A \subseteq I : i_0 \in A\}$. Filters of the form $\mathcal{F}_{i_0}$ for some $i_0 \in I$ are termed *trivial*, or *non-free* filters.

(4) If $(I, \preceq)$ is a lattice, then the family of supersets of sets of the form $\mathcal{M}_{i_0} = \{i : i \succeq i_0\}$, for $i_0 \in I$, is a filter. To see this, note that $\mathcal{M}_{i_0} \cap \mathcal{M}_{j_0} = \mathcal{M}_{i_0 \vee j_0}$.

A filter $\mathcal{F}$ is *proper* if it is not equal to 2^I, the power set of I. Equivalent conditions are: $\emptyset \notin \mathcal{F}$, or $\mathcal{F}$ has the finite intersection property; that is, all finite intersections of filter elements are nonempty.

Throughout this chapter, we will take filter to mean proper filter.

Definition 2.2 An *ultrafilter* $\mathcal{U}$ on I is a filter on I which is maximal with respect to ordering of filters on I by inclusion: that is, if $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{F}$ is a filter on I, then $\mathcal{F} = \mathcal{U}$. Zorn's lemma ensures that every filter has an extension to an ultrafilter.

Lemma 2.3 *A filter $\mathcal{U} \subset 2^I$ is an ultrafilter on I if and only if for every $A \subseteq I$ precisely one of A or $I \backslash A$ is in $\mathcal{U}$.*

Proof. $(\Rightarrow)$ We show that if $I \setminus A \notin \mathcal{U}$, then $A \in \mathcal{U}$. If $I \setminus A \notin \mathcal{U}$, then $I \setminus A$ has no subset which is an element of $\mathcal{U}$; hence every element of $\mathcal{U}$ meets A. The family $\mathcal{B} = \{A \cap U : U \in \mathcal{U}\}$ therefore has the finite intersection property and so its supersets form a filter $\mathcal{F}_{\mathcal{B}}$. But $\mathcal{U} \subseteq \mathcal{F}_{\mathcal{B}}$, because $U \supseteq U \cap A \in \mathcal{F}_{\mathcal{B}}$, and so by the maximality $\mathcal{F}_{\mathcal{B}} = \mathcal{U}$. Also, $A = A \cap I \in \mathcal{F}_{\mathcal{B}}$ by $\mathbf{F}$(i), and so $A \in \mathcal{U}$.

$(\Leftarrow)$ Note: the condition automatically ensures $\mathcal{U}$ is proper because $I \in \mathcal{U}$ and so $\emptyset = I \setminus I \notin \mathcal{U}$. Now, let $\mathcal{F}$ be a filter on I with $\mathcal{U} \subseteq \mathcal{F}$, we show $\mathcal{F} = \mathcal{U}$. Assume not, then there exists $A \in \mathcal{F}$ with $A \notin \mathcal{U}$. However, we then have $I \setminus A \in \mathcal{U} \subseteq \mathcal{F}$. So both A and $I \setminus A$ belong to $\mathcal{F}$ which , by $\mathbf{F}$(i), implies that $\emptyset = A \cap (I \setminus A) \in \mathcal{F}$, contradicting $\mathcal{F}$ proper. ■

As a consequence of this lemma: For an ultrafilter $\mathcal{U}$ on I if $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{U}$ then at least one of the sets $A_1, A_2, \cdots, A_n$ is in $\mathcal{U}$, and an ultrafilter is nontrivial (free) if and only if it contains no finite subsets .

It will henceforth be a standing assumption that all the filters and ultrafilters with which we deal are nontrivial.

We say $\mathcal{U}$ is *countably complete* if it is closed under countable intersections. Ultrafilters which are not countably complete are particularly useful for some purposes. It is readily seen that an ultrafilter $\mathcal{U}$ is countably incomplete if and only if there exist elements $A_0, A_1, \cdots, A_n, \cdots$ in $\mathcal{U}$ with

$$I = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \quad \text{and} \quad \bigcap_{n=0}^{\infty} A_n \;=\; \emptyset.$$

We shall see that this structure allows us to readily extended inductive and diagonal type arguments into ultrapowers. Every ultrafilter $\mathcal{U}$ over $\mathbf{N}$ is necessarily countably incomplete (consider the countable family of nested sets $A_n := \{n, n+1, n+2, \cdots\} \in \mathcal{U}$).

One of the most exiting result about ultrafilters deal with compactness. Before we state this result, we will need to link ultrafilters with the concept of convergence in topological spaces.

Definition 2.4 For a Hausdorff topological space $(\Omega, \mathcal{T})$, an ultrafilter $\mathcal{U}$ on I, and $(x_i)_{i \in I} \subseteq \Omega$ we say

$$\lim_{\mathcal{U}} x_i \;\left(\equiv \mathcal{T} - \lim_{\mathcal{U}} x_i\right) \;=\; x_0$$

if for every neighbourhood N of x_0 we have $\{i \in I : x_i \in N\} \;\in\; \mathcal{U}$.

Limits along $\mathcal{U}$ are unique and if $\mathcal{U}$ is on $\mathbf{N}$ and (x_n) is a bounded sequence in $\mathbf{R}$ then

$$\liminf_n x_n \leq \lim_{\mathcal{U}} x_n \leq \limsup_n x_n.$$

Moreover, if C is a closed subset of Ω and $(x_i)_{i\in I} \subseteq C$, then $\lim_{\mathcal{U}} x_i$ belongs to C whenever it exists.

Remark 2.5 *Let X be a metric space. If $\mathcal{U}$ is an ultrafilter and $\lim_{\mathcal{U}} x_n = x$, with $(x_n) \subset X$, then there exists a subsequence of (x_n) which converges to x.*

The next theorem is fundamental since it characterizes compactness by use of ultrafilters.

Theorem 2.6 *Let K be a Hausdorff topological space. K is compact if and only if $\lim_{\mathcal{U}} x_i$ exists for all $(x_i)_{i\in I} \subset K$ and any ultrafilter $\mathcal{U}$ over I.*

When the space in question is a linear topological vector space, convergence over an ultrafilter has similar behaviour to traditional convergence. In particular, we have:

Proposition 2.7 *Let X be a linear topological vector space, and $\mathcal{U}$ an ultrafilter over an index set I.*

(i) *Suppose that $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are two subsets of X such that $\lim_{\mathcal{U}} x_i$ and $\lim_{\mathcal{U}} y_i$ exist. Then*

$$\lim_{\mathcal{U}} \Big(x_i + y_i \Big) = \lim_{\mathcal{U}} x_i + \lim_{\mathcal{U}} y_i \quad \textit{and} \quad \lim_{\mathcal{U}} \alpha\, x_i = \alpha \, \lim_{\mathcal{U}} x_i,$$

for any scalar $\alpha \in \mathbf{R}$.

(ii) *If X is a Banach lattice and $(x_i)_{i\in I}$ is a subset of positive elements of X, i.e. $x_i \geq 0$, then $\lim_{\mathcal{U}} x_i$ is also positive.*

Now we are ready to define the ultrapower of a Banach space. Let X be a Banach space and $\mathcal{U}$ an ultrafilter over an index set I. We can form the substitution space

$$\ell_\infty(X) \;:=\; \{(x_i)_{i\in I} : \|(x_i)\|_\infty := \sup_{i\in I} \|x_i\| < \infty\}.$$

Then,

$$N_{\mathcal{U}}(X) \;:=\; \{(x_i)_{i\in I} \in \ell_\infty(X) : \lim_{\mathcal{U}} \|x_i\| = 0\}$$

is a closed linear subspace of $\ell_\infty(X)$.

Definition 2.8 The *Banach space ultrapower* of X over $\mathcal{U}$ is defined to be the Banach space quotient

$$(X)_{\mathcal{U}} \;:=\; \ell_\infty(X)/N_{\mathcal{U}}(X),$$

with elements denoted by $[x_i]_{\mathcal{U}}$, where (x_i) is a representative of the equivalence class. The quotient norm is canonically given by

$$\|[x_i]_{\mathcal{U}}\| \;=\; \lim_{\mathcal{U}} \|x_i\|.$$

Remark 2.9 The mapping $\mathcal{J} : X \longrightarrow (X)_{\mathcal{U}}$ defined by

$$\mathcal{J}(x) := [x] := [x_i]_{\mathcal{U}}, \quad \text{where } x_i = x, \quad \text{for all } i \in I$$

is an isometric embedding of X into $(X)_{\mathcal{U}}$. Using the map $\mathcal{J}$, one may identify X with $\mathcal{J}(X)$ seen as a subspace of $(X)_{\mathcal{U}}$. When it is clear we will omit mention of the map J and simply regard X as a subspace of $(X)_{\mathcal{U}}$.

In what follows, we describe some of the fundamental results related to ultrapowers. We will not be exhaustive and leave it to the interested reader to pursue the subject further by consulting [21], for example.

Proposition 2.10 *Let $(X)_{\mathcal{U}}$ be an ultrapower of a Banach space X. Then for any $\varepsilon > 0$ and any finite dimensional subspace Y_0 of $(X)_{\mathcal{U}}$, there exists a subspace X_0 of X with the same dimension and a linear map $T : X_0 \to Y_0$ such that*

$$(1-\varepsilon)\|x\| \leq \|T(x)\| \leq (1+\varepsilon)\|x\|$$

holds for all $x \in X_0$.

Proof. Let $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ be a unit basis for M and choose representatives

$$\left(x_i^{(k)}\right) \text{ of } x^{(k)} \text{ such that } \left\|x_i^{(k)}\right\| \leq 2,\ (k = 1, 2, \ldots, n).$$

Consider the vector space

$$M_i = \left\langle x_i^{(k)} \right\rangle_{k=1}^{n}$$

and define $T_i : M \to M_i$ by its action on the basis;

$$T_i\left(x^{(k)}\right) = x_i^{(k)} \quad (k = 1, 2, \ldots, n).$$

Then, T_i is linear with $\|T_i\| \leq 2K$, where

$$K = \max\left\{\sum_{k=1}^{n} |\lambda_k| : \ \left\|\sum_{k=1}^{n} \lambda_k \, x^{(k)}\right\| = 1\right\}.$$

For any $x = \sum_{k=1}^{n} \lambda_k \, x^{(k)} \in M$, we have

$$\begin{aligned}
\|x\| = \left\|\sum_{k=1}^{n} \lambda_k \, x^{(k)}\right\| &= \left\|\sum_{k=1}^{n} \lambda_k \left(x_i^{(k)}\right)_{\mathcal{U}}\right\| \\
&= \lim_{\mathcal{U}} \left\|\sum_{k=1}^{n} \lambda_k \, x^{(k)}\right\| \\
&= \lim_{\mathcal{U}} \|T_i x\|
\end{aligned}$$

Thus,

$$I_x = \left\{i \in I : \ \big|\|T_i x\| - \|x\|\big| \leq \frac{\varepsilon}{2}\|x\|\right\} \in \mathcal{U}.$$

Now, let δ be a positive number (to be chosen later) and let $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ be a finite δ-net in the unit sphere of M and set

$$I_0 = \bigcap_{k=1}^{m} I_y(k),$$

then for $i \in I_0$ and $x \in M$ with $\|x\| = 1$ we have

$$\begin{aligned}
\big|\|T_i x\| - \|x\|\big| &\leq \min_{k=1,2,\ldots,m} \left(\left\|T_i\Big(x - y^{(k)}\Big)\right\| + \left\|x - y^{(k)}\right\| + \left|\left\|T_i\Big(y^{(k)}\Big)\right\| - \left\|y^{(k)}\right\|\right|\right) \\
&\leq (2K+1)\delta + \frac{\varepsilon}{2}.
\end{aligned}$$

The conclusion now follows by taking $\delta = \dfrac{\varepsilon}{2(2K+1)}$. ■

Any Banach space which enjoys a similar property as the one described above for $(X)_{\mathcal{U}}$ is called *finitely representable* in X. Therefore, any ultrapower of X is finitely representable in X. Note that we may avoid using the map T by introducing the so-called Banach-Mazur distance between normed spaces.

Definition 2.11 Let X and Y be Banach spaces. The *Banach-Mazur distance* between X and Y is

$$d(X,Y) = \inf\{\|T\|\,\|T^{-1}\|;\ \text{where } T \text{ is an isomorphism from } X \text{ onto } Y\}\ .$$

When X and Y are not isomorphic we simply set $d(X,Y) = \infty$.

Therefore, X is finitely representable in Y if and only if for any $\varepsilon > 0$ and any finite dimensional subspace X_0 of X, there exists a subspace Y_0 of Y with the same dimension such that $d(X_0, Y_0) < 1+\varepsilon$. It is a stunningly useful fact that an ultrapower of a Banach space X can capture isometrically all the spaces finitely represented in X. Indeed, we have

Theorem 2.12 *Let Y be a separable Banach space which is finitely represented in X. Then there is an isometric embedding of Y into the ultrapower $(X)_{\mathcal{U}}$ for each countably incomplete ultrafilter $\mathcal{U}$.*

Proof. Let $\mathcal{U}$ be a countably incomplete ultrafilter on I; that is, there is a countable chain $I_1 \supseteq I_2 \supseteq \ldots$ with $I_n \in \mathcal{U}$ and

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

By the separability of Y we can find a linearly independent sequence $(x(n))_{n=1}^{\infty}$ such that $Y = \overline{\langle \{x(n)\}_{n=1}^{\infty} \rangle}$. Since Y is finitely represented in X, for each N in $\mathbf{N}$, there exists a $1/N$-isometry

$$T_N : X_N \equiv \left\langle \{x(n)\}_{n=1}^{N} \right\rangle \to X.$$

Now define $J : Y \to (X)_{\mathcal{U}}$ by its action on the $x(m)$,

$$J(x(m)) = (x_i(m))\ ,$$

where

$$x_i(m) = \begin{cases} 0 & \text{if } i \in I \setminus I_m, \\ T_n\Big(x(m)\Big) & \text{if } i \in I_m, \text{ where } n \geq m \text{ and } T_n(x(m)) \\ & \text{is the unique number such that } i \in I_n \setminus I_{n+1}. \end{cases}$$

Note that since $\bigcap_{n=1}^{\infty} = \emptyset$, x_i is defined for each $i \in I$. To see that J is an isometry observe that; for

$$x = \sum_{k=1}^{K} \lambda_k\, x(m_k) \quad (\text{such } x \text{ are dense in } Y)$$

we have

$$\begin{aligned}\|Jx\| &= \left\| \sum_{k=1}^{K} \lambda_k \, (x_i(m_k))_{\mathcal{U}} \right\| \\ &= \lim_{\mathcal{U}} \left\| \sum_{k=1}^{K} \lambda_k \, x_i(m_k) \right\| \\ &= \|x\|.\end{aligned}$$

To see this, given $\varepsilon > 0$, choose $N > (1/\varepsilon, \max_k m_k)$, then we have $x \in X_N$ and for $i \in I_N \in \mathcal{U}$,

$$\begin{aligned}\left| \left\| \sum_{k=1}^{K} \lambda_k \, x_i(m_k) \right\| - \|x\| \right| &= \left| \left\| \sum_{k=1}^{K} \lambda_k \, T_n x(m_k) \right\| - \|x\| \right| \quad \text{(for some } n \geq N) \\ &= \Big| \|T_n x\| - \|x\| \Big| \\ &\leq \varepsilon \|x\|.\end{aligned}$$

■

Example 2.13 Ultrapowers of a Hilbert space It is known that a Banach space X is a Hilbert space if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Let $(X)_{\mathcal{U}}$ be an ultrapower of X and let $[(x_i)]$ and $[(y_i)]$ be two elements in $(X)_{\mathcal{U}}$, then we have

$$\Big\| [(x_i)] + [(y_i)] \Big\|^2 = \Big\| [(x_i + y_i)] \Big\|^2 = \lim_{\mathcal{U}} \|x_i + y_i\|^2$$

and

$$\Big\| [(x_i)] - [(y_i)] \Big\|^2 = \Big\| [(x_i - y_i)] \Big\|^2 = \lim_{\mathcal{U}} \|x_i - y_i\|^2 \; .$$

Since

$$\lim_{\mathcal{U}} \|x_i + y_i\|^2 + \lim_{\mathcal{U}} \|x_i - y_i\|^2 = \lim_{\mathcal{U}} \left(\|x_i + y_i\|^2 + \|x_i - y_i\|^2 \right) \; ,$$

and using the Hilbert structure of X, we get

$$\lim_{\mathcal{U}} \|x_i + y_i\|^2 + \lim_{\mathcal{U}} \|x_i - y_i\|^2 = \lim_{\mathcal{U}} \left(2\|x_i\|^2 + 2\|y_i\|^2 \right) \; .$$

Whence,

$$\Big\| [(x_i)] + [(y_i)] \Big\|^2 + \Big\| [(x_i)] - [(y_i)] \Big\|^2 = 2 \Big\| [(x_i)] \Big\|^2 + 2 \Big\| [(y_i)] \Big\|^2$$

which implies $(X)_{\mathcal{U}}$ is a Hilbert space.

This example, though easy to prove, is extremely rich in many ways. Indeed, what the reader should learn from it is that the ultrapower catches any finitely determined property satisfied by the Banach space. Maybe one of the most useful instances of this concerns lattice structure. If X is a lattice Banach space, then any ultrapower $(X)_{\mathcal{U}}$ is also a Banach lattice when the order is defined by taking $\tilde{x} \in (X)_{\mathcal{U}}$ to be positive if and only if one can find a representative, (x_i), of $\tilde{x}$ all of whose elements are positive in X. In this case, $(X)_{\mathcal{U}}$ enjoys most of the important lattice properties satisfied by X.

Also from the above example, we see that a nonreflexive Banach space can not be finitely represented in a Hilbert space. In other words, only reflexive Banach spaces may be finitely represented in a Hilbert space. This leads to the concept of a *super-property.*

Definition 2.14 Let $\mathcal{P}$ be a property defined on a Banach space X. We say that X has the property "*super-*$\mathcal{P}$" if every Banach space that is finitely representable in X has $\mathcal{P}$.

The following result is an immediate consequence of Proposition 2.10 and Theorem 2.12.

Theorem 2.15 *If $\mathcal{P}$ is a separably determined Banach space property that is inherited by subspaces, then a Banach space X has super-$\mathcal{P}$ if and only if some ultrapower of X over a countably incomplete ultrafilter has $\mathcal{P}$ (and hence every ultrapower of X has $\mathcal{P}$).*

Remark 2.16 Reflexivity satisfies the requirements of the above theorem. Thus, a Banach space X is superreflexive if and only if some (and hence every) countably incomplete ultrapower of X is reflexive.

We also note that Theorem 2.12 remains valid if we replace 'every countably incomplete ultrafilter' by 'there exists an ultrafilter', without the assumption that the property be separably determined. Thus, we always have:

If $\mathcal{P}$ is a Banach space property that is inherited by subspaces, then a Banach space X has super-$\mathcal{P}$ if and only every ultrapower of X has $\mathcal{P}$.

In particular Hilbert spaces are superreflexive. One may think that these are the only examples of superreflexive Banach spaces. In the following example, we show that this is far from the case, indeed the family of superreflexive Banach spaces is quite a rich one.

Example 2.17 Let X be a Banach space. For any $\varepsilon > 0$, define

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\|; x, y \in X \ \text{ and } \ \|x\| \le 1\ ,\ \|x\| \le 1\right\}\ .$$

The function $\delta_X(\varepsilon)$ is called the *modulus of uniform convexity* of X. The *characteristic of uniform convexity* of X is defined by

$$\varepsilon_0(X) = \sup\{\varepsilon; \delta_X(\varepsilon) = 0\}\ .$$

A Banach space X is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for any $\varepsilon > 0$. A space is said to be *uniformly nonsquare*, or *inquadrate* if and only if $\varepsilon_0(X) < 2$. We next discuss the link between these concepts and ultrapowers.

Let $(X)_{\mathcal{U}}$ be an ultrapower of X. Then, for any $\varepsilon > 0$, we have

$$\delta_X(\varepsilon) = \delta_{(X)_{\mathcal{U}}}(\varepsilon)\ .$$

Consequently, we also have $\varepsilon_0(X) = \varepsilon_0((X)_{\mathcal{U}})$. In particular, a Banach space is uniformly convex (uniformly nonsquare) if and only if some, and hence every, ultrapower is uniformly convex (uniformly nonsquare). It is also worth mentioning that an ultrapower is uniformly convex if and only if it is strictly convex. Since uniformly nonsquare spaces are reflexive, we deduce that uniformly nonsquare Banach spaces are also superreflexive. In fact, Enflo [8] (see also Pisier [18]) has shown that X is superreflexive if and only if there exists an equivalent norm (on X) which is uniformly convex. More on this may be found in, for example, [3], or [21].

Next we discuss ultraproducts of maps. Let X and Y be two Banach spaces and let $(X)_{\mathcal{U}}$ and $(Y)_{\mathcal{U}}$ be their associated ultrapowers with respect to a given ultrafilter $\mathcal{U}$ on I. Let $T_i : D \subset X \to Y$ be a family of maps indexed by I. Consider

$$[D]_{\mathcal{U}} := \{\tilde{x} \in (X)_{\mathcal{U}} \ : \ \text{there exists a representative } (d_i) \text{ of } \tilde{x} \text{ with } d_i \in D\} \ .$$

Define $[(T_i)]_{\mathcal{U}} : [D]_{\mathcal{U}} \to (Y)_{\mathcal{U}}$ by

$$[(T_i)]_{\mathcal{U}}\Big([(d_i)]_{\mathcal{U}}\Big) = [(T_i(d_i))]_{\mathcal{U}} \ .$$

We of course have to ensure that the T_i satisfy suitable conditions for $[(T_i)]_{\mathcal{U}}$ to be well defined. When this is the case we have, in particular, the following.

Proposition 2.18 *Using the above notations, we have*

(i) $[D]_{\mathcal{U}}$ *is convex if* D *is convex;*

(ii) $[D]_{\mathcal{U}}$ *is closed if* D *is closed;*

(iii) $[D]_{\mathcal{U}}$ *is bounded if* D *is bounded;*

(iv) $[(T_i)]_{\mathcal{U}}$ *is Lipschitzian provided the* T_i *are Lipschitzian mappings whose Lipschitz constants* λ_i *are uniformly bounded, in which case the Lipschitz constant of* $[(T_i)]_{\mathcal{U}}$ *is equal to* $\lim_{\mathcal{U}} \lambda_i$*;*

(v) $[(T_i)]_{\mathcal{U}}$ *is a bounded linear operator provided the* T_i *are linear operators which are uniformly bounded; that is,* $\sup_{i\in I} \|T_i\| < \infty$*, and then* $\|[(T_i)]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_i\|$*.*

Proof. Most of these results follow directly from the relevant definitions. Consequently, we restrict ourselves to proving (ii) in the case of particular interest when $\mathcal{U}$ is an ultrafilter over $\mathbf{N}$. Thus, let $\mathbf{N} = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, be a nested sequence of sets with each $A_n \in \mathcal{U}$, and $\bigcap_{n\geq 1} A_n = \emptyset$.

Suppose $[t_i^1]_{\mathcal{U}}$, $[t_i^2]_{\mathcal{U}}$, $\cdots$ is a sequence of points in $[D]_{\mathcal{U}}$, with each $t_i^j \in D$, which converges to $[x_i]_{\mathcal{U}} \in (X)_{\mathcal{U}}$. By passing to a subsequence if necessary we may without loss of generality assume that

$$\|[t_i^m] - [x_i]\| \ = \ \lim_{\mathcal{U}} \|t_i^m - x_i\| \ < \ \frac{1}{m}.$$

For each $m \in \mathbf{N}$ let

$$B_m \ := \ \left\{ i \in \mathbf{N} : \|t_i^m - x_i\| < \frac{2}{m} \right\} \cap A_m \ \in \ \mathcal{U}.$$

and put $B_0 := \mathbf{N}$ and $t_i^0 := 0$, then

$$\mathbf{N} = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m \supset \cdots , \quad \text{and } \cap_{m=0}^{\infty} B_m = \emptyset.$$

From this it follows that for each $i \in \mathbf{N}$ there is a unique m such that $i \in B_m \backslash B_{m+1}$. Define $y_i := t_i^m$, for this m, in particular then $y_i \in D$.

Now, given any $m \in \mathbf{N}$, for each $i \in B_m$ there is a unique $p \geq m$ with $i \in B_p \backslash B_{p+1}$. thus,

$$\|y_i - x_i\| \ = \ \|t_i^p - x_i\| \ < \ \frac{2}{p} \ \leq \ \frac{2}{m},$$

and so

$$\left\{ i \in \mathbf{N} : \|y_i - x_i\| < \frac{2}{m} \right\} \supseteq B_m \in \mathcal{U}.$$

We therefore have that $\mathcal{U} - \lim \|y_i - x_i\| = 0$, which yields the desired conclusion that $[x_i]_{\mathcal{U}} \in [D]_{\mathcal{U}}$. ■

For the above theorem, recall that a map $T : D \subset X \to Y$ is said to be a *Lipschitz mapping* with Lipschitz constant λ if

$$\|T(x) - T(y)\| \leq \lambda \|x - y\|$$

for all $x, y \in D$.

The above results are useful for studying the dual space of an ultrapower. Indeed, let (x_i^*) be a uniformly bounded family of linear functionals defined on X (i.e. elements of the dual space X^*). Then from the above results, we can generate a bounded linear functional $[(x_i^*)]_{\mathcal{U}}$. This linear functional belongs to the dual of the ultrapower; that is, $\Big((X)_{\mathcal{U}}\Big)^*$. One may then ask whether this construction yields all the elements of $\Big((X)_{\mathcal{U}}\Big)^*$. An answer is provided by the following theorem.

Theorem 2.19 *Let X be a Banach space. Then*

$$\Big((X)_{\mathcal{U}}\Big)^* = (X^*)_{\mathcal{U}}$$

if and only if X is superreflexive.

More on this and similar results may be found in [21].

We will close this section with an important example.

Example 2.20 In this example, we discuss ultrapowers of the L_p-spaces, $1 \leq p < \infty$.

Let (Ω, Σ, μ) be a σ-additive measure space and for $1 \leq p < \infty$, let $L_p(\mu)$ denote the real space $L_p(\Omega, \Sigma, \mu)$. Then, we will show that if $\mathcal{U}$ is an ultrafilter on I then there exists a measure space (Ω, Σ, ν) with $(L_p(\mu))_{\mathcal{U}}$ lattice isometric to $L_p(\nu)$.

First note that under a 'component-wise' definition of order, $(L_p(\mu))_{\mathcal{U}}$ is a Banach lattice. Thus, by the classical theorem of Bohnenblust and Nakano, see for example [14], it is sufficient to prove that the norm in $(L_p(\mu))_{\mathcal{U}}$ is p-additive; that is, whenever $x \wedge y = 0$ we have that

$$\|x + y\|^p = \|x\|^p + \|y\|^p.$$

To this end, let $(x_i)_{\mathcal{U}}$ and $(y_i)_{\mathcal{U}}$ be elements of $(L_p(\mu_i))_{\mathcal{U}}$ such that $(x_i)_{\mathcal{U}} \wedge (y_i)_{\mathcal{U}} = 0$. Let $z_i = x_i \wedge y_i$, then $(x_i - z_i) \wedge (y_i - z_i) = 0$ and so

$$\|x_i - z_i\|^p + \|y_i - z_i\|^p = \|x_i + y_i - 2z_i\|^p \quad (i \in I).$$

On the other hand, we have $0 = (x_i \wedge y_i)_{\mathcal{U}} = (z_i)_{\mathcal{U}}$, so $\lim_{\mathcal{U}} \|z_i\| = 0$, but then, $(x_i) \overset{\equiv}{_{\mathcal{U}}} (x_i - z_i), (y_i) \overset{\equiv}{_{\mathcal{U}}} (y_i - z_i)$ and $(x_i + y_i) \overset{\equiv}{_{\mathcal{U}}} (x_i + y_i - 2z_i)$ and so

$$\|(x_i)_{\mathcal{U}}\|^p + \|(y_i)_{\mathcal{U}}\|^p = \|(x_i)_{\mathcal{U}} + (y_i)_{\mathcal{U}}\|^p,$$

as required.

This argument does not provide us with any information on the structure of the measure space (Ω, Σ, ν), for information on this and related questions see, for example, [21].

3. Fixed Point Theory

Definition 3.1 A Banach space X is said to have the *weak fixed point property* (w-fpp) if for every nonempty weakly compact convex subset C of X and every nonexpansive mapping $T : C \to C$ we have $\mathrm{Fix}(T)$, the fixed point set of T, is nonempty. Recall that $x \in \mathrm{Fix}(T)$ if and only if $T(x) = x$.

To establish the w-fpp for a Banach space X we work toward a contradiction. Thus, assume that X fails to have the w-fpp then there exists a weakly compact convex subset C of X and a nonexpansive mapping $T : C \to C$ with $\mathrm{Fix}(T) = \emptyset$.

Let $\mathcal{F}$ denote the family of all nonempty closed convex subsets K of C that are invariant under T (that is, $T(K) \subset K$). Clearly, $\mathcal{F}$ is not empty, since $C \in \mathcal{F}$. The weak-compactness of C ensures that $\mathcal{F}$ satisfies the assumptions for Zorn's lemma. Therefore $\mathcal{F}$ has minimal elements.

Definition 3.2 A convex set K is said to be a *minimal invariant set* for T if K is a minimal element of $\mathcal{F}$.

Clearly any set K which is a minimal invariant set for T contains more than one point; that is,

$$\mathrm{diam}(K) = \sup\{\|x - y\| : x, y \in K\} > 0 ,$$

otherwise T would have a fixed point.

We proceed to investigate the properties of minimal invariant sets.

Proposition 3.3 *Let K be a minimal invariant set for T. Then*

$$\overline{\mathrm{conv}}\Big(T(K)\Big) = K .$$

The next result gives an interesting property satisfied by minimal invariant sets.

Lemma 3.4 *Let K be a minimal set for T, and let $\alpha : K \to \mathbf{R}_+$ be a lower semi-continuous convex function such that*

$$\alpha\Big(T(x)\Big) \leq \alpha(x) , \quad \textit{for all } x \in K.$$

Then α is a constant function.

Taking $\alpha(x) := \sup\{\|x - y\| : y \in K\}$ and using proposition 3.3 to replace the supremum over K with a supremum over $T(K)$ we see that the above lemma applies and readily yields:

Proposition 3.5 *Any minimal invariant set K for T is a diametral set; that is,* $\mathrm{diam}(K) > 0$ *and*

$$\sup\{\|x - y\| : y \in K\} = \mathrm{diam}(K)$$

for all $x \in K$.

Spaces which contain no weakly compact convex diametral sets are said to have *weak normal structure*, clearly such spaces have the w-fpp.

The property of normal structure (the absence of diametral closed bounded convex subsets) was introduced by W. A. Kirk in 1965 when he showed that reflexive spaces with the property had the fixed point property. It was quickly realized that this result subsumed most of the then known existence results for fixed points of nonexpansive mappings by F. Browder, D. Gohde, M. Edelstein and others. The main thrust of metric fixed point theory during the late 1960's and throughout the 1970's was the quest for natural, and easily verified, conditions on a Banach space that are sufficient for weak normal structure coupled with an exploration of other consequences of normal structure and related properties such as asymptotic normal structure. Details of this work, together with relevant references, may be found in the chapter on the *Classical theory of nonexpansive mappings*.

Initially, it was unknown whether all reflexive spaces necessarily had normal structure, or if weak normal structure and the weak fixed point property were equivalent. Then, in 1975 and 1976, the two questions were settled in the negative by R. C. James and L. Karlovitz respectively.

Example 3.6 For $\beta > 1$ let X_β denote the Hilbert space l_2 equipped with the equivalent norm

$$\|(x_n)\|_\beta = \max\{\|(x_n)\|_{l_2}, \beta\|(x_n)\|_{l_\infty}\}.$$

James observed that these spaces are all superreflexive, but that X_2 fails to have normal structure. Indeed, it is quite easy to verify that X_β fails to have normal structure for $\beta \geq \sqrt{2}$. On the other hand, Karlovitz showed that $X_{\sqrt{2}}$ has the fixed point property for nonexpansive mappings. Subsequently, this family of spaces has been the subject of considerable investigation. For example, in 1981 Baillon and Schöneberg [2] observed that, for $\beta < 2$, X_β has asymptotic normal structure; a geometric property which they showed implies the fixed point property. For larger values of β the situation remained unclear, though Baillon managed to give some highly technical demonstrations of the fixed point property for certain values of β, until finally, in 1984, it was shown [4] that X_β has the fixed point property for all values of β, see also [15].

Normal structure precludes the presence of diametral sets and as such only involves the mapping T in so far as minimal invariant sets of fixed point free nonexpansive maps provide instances of such diametral sets. To establish the weak fixed point property in the absence of weak normal structure requires properties of minimal invariant sets that involve the mapping T in a more explicit way. One such property was used by Karlovitz to establish the fixed point property for the space $X_{\sqrt{2}}$. The property was independently discovered by K. Goebel and the result has subsequently become known as the Goebel-Karlovitz lemma. Before presenting it we need some more facts about nonexpansive mappings.

Let K be a nonempty, bounded, closed, convex subset of a Banach space X, and $T : K \to K$ be nonexpansive. Fix $\varepsilon \in (0,1)$ and $x_0 \in K$, and consider the map $T_\varepsilon : K \to K$ defined by

$$T_\varepsilon(x) = \varepsilon x_0 + (1-\varepsilon)T(x)$$

for all $x \in K$. T_ε is clearly a contraction mapping. Hence it has a unique fixed point $x_\varepsilon \in K$, i.e. $T_\varepsilon(x_\varepsilon) = x_\varepsilon$. We have

$$\|T(x_\varepsilon) - x_\varepsilon\| \leq \varepsilon \operatorname{diam}(K) \ .$$

In other words, we have

$$\inf\{\|T(x) - x\|;\ x \in K\} = 0 \ .$$

Definition 3.7 A sequence (x_n) satisfying $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, is called an *approximate fixed point sequence.*

The above construction shows that a nonexpansive self mapping of a closed bounded convex set always has an approximate fixed point sequence.

The Goebel-Karlovitz lemma is the following

Lemma 3.8 *Let C be a weakly compact convex set and let K be a minimal invariant set for $T : C \to C$. Then for any approximate fixed point sequence (x_n) of T in K, we have*

$$\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam}(K)$$

for all $x \in K$; that is, (x_n) is a diameterizing sequence for K.

The proof is an easy consequence of lemma 3.4 with $\alpha(x) := \limsup_n \|x - x_n\|$.

In the first instance, one might think that the presence of diameterizing sequences in minimal invariant sets of fixed point free noneexpansive mappings would provide a lever for establishing the w-fpp in the absence of normal structure. Unfortunately this is not the case. A simple construction shows that if a space contains a diametral set D then it also contains a diametral set with a diameterizing sequence. Indeed, one can construct within D a sequence (x_n) with $\operatorname{dist}(x_{n+1}, \operatorname{conv}\{x_1, x_2, \cdots, x_n\}) \to \operatorname{diam}(D)$. Such a sequence is diameterizing for its closed convex hull which is therefore a diametral subset of D with the same diameter as D. To proceed in the absence of weak normal structure, the mapping T must be brought back into play, via the Goebel-Karlovitz lemma, and the fact that the diameterizing sequence is an approximate fixed point sequence for T exploited. Such arguments are necessarily both delicate and subtle. It was B. Maurey [17] who, in a brilliant series of results (see section 4), first demonstrated the usefulness of ultrapowers as a setting for such arguments. His methods brought a new dimension to metric fixed point theory and, together with Alspach's seminal example, began what might be described as the 'non-classical theory'.

We now turn to the basic constructions such methods employ.

Let C be a nonempty bounded convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping with no fixed point. Let $\mathcal{U}$ be an ultrafilter on the set of natural numbers. In $(X)_{\mathcal{U}}$ we may define

$$\tilde{C} \ := \ \{[x_n]_{\mathcal{U}} : \ x_n \in C, \quad \text{for all } n \in \mathbf{N}\}.$$

Then, $\tilde{C}$ is a convex subset, with $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C)$, containing an isometric copy, $\mathcal{J}(C)$, of C and on which $\tilde{T} : \tilde{C} \longrightarrow \tilde{C}$ defined by

$$\tilde{T}\Big([x_n]_{\mathcal{U}}\Big) = [T(x_n)]_{\mathcal{U}},$$

where the representative (x_n) is chosen to be a sequence of points from C, is a well defined nonexpansive mapping [proposition 2.18 (iv)] which leaves $\mathcal{J}(C)$ invariant. We now list a number of basic results for $\tilde{C}$ and $\tilde{T}$ constructed as above. From proposition 2.18 (ii) we have the following.

Proposition 3.9 *The set $\tilde{C}$ in $(X)_{\mathcal{U}}$ is closed. Hence, when X is a superreflexive space $\tilde{C}$ is weakly-compact.*

The next proposition follows directly from the definitions.

Proposition 3.10 *If (x_n) is an approximate fixed point sequence for T, then $[x_n]_{\mathcal{U}}$ is a fixed point of $\tilde{T}$. Consequently, $\tilde{T}$ always has fixed points in $\tilde{C}$.*

Conversely, from a fixed point (indeed an approximate fixed point sequence) for $\tilde{T}$ in $\tilde{C}$ we can readily extract an approximate fixed point sequence for T.

If C is a weakly compact minimal invariant set for T, so that the Goebel-Karlovitz lemma applies, then in the above proposition we also have $\|[x_n]_{\mathcal{U}} - \mathcal{J}x\| = \operatorname{diam}(C)$, for all $x \in C$. Since, in this case we can always assume without loss of generality that

$\operatorname{diam} C = 1$ and that (x_n) converges weakly to 0 (so, $0 \in C$), we may suppose that $\|[x_n]_{\mathcal{U}}\| = \operatorname{dist}([x_n]_{\mathcal{U}}, \mathcal{J}C) = 1$.

The following is a significant observation of B. Maurey [17].

Lemma 3.11 *Given any two fixed points $\tilde{a} = [a_n]_{\mathcal{U}}$ and $\tilde{b} = [b_n]_{\mathcal{U}}$ of $\tilde{T}$ in $\tilde{C}$ there is a fixed point $\tilde{c}$ with*

$$\|\tilde{a} - \tilde{c}\| = \|\tilde{c} - \tilde{b}\| = \frac{1}{2}\|\tilde{a} - \tilde{b}\|.$$

Proof. We may assume that $\lambda := \|\tilde{a} - \tilde{b}\| := \lim_{\mathcal{U}} \|a_n - b_n\| > 0$. For each $m \in \mathbf{N}$ let

$$A_m := \left\{ n \geq m : \|a_n - b_n\| \leq \lambda + \frac{2}{m^2} \text{ and } \|a_n - Ta_n\|,\ \|b_n - Tb_n\| \leq \frac{1}{m^2} \right\},$$

then $A_m \in \mathcal{U}$, $\mathbf{N} =: A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ and $\bigcap_{n \geq 1} A_n = \emptyset$.

For each $n \in \mathbf{N}$ let $C_n := \left\{ c \in C : \|a_n - c\|, \|b_n - c\| \leq \lambda/2 + \frac{1}{m} \right\}$ where m is the unique element of $\mathbf{N}$ for which $n \in A_m \backslash A_{m+1}$. Then C_n is bounded, closed, convex and nonempty since

$$\|a_n - 1/2\,(a_n + b_n)\| = \frac{1}{2}\,\|a_n - b_n\| \leq \frac{\lambda}{2} + \frac{1}{m^2} \leq \frac{\lambda}{2} + \frac{1}{m}$$

and similarly, $\|b_n - 1/2\,(a_n + b_n)\| \leq \lambda/2 + 1/m$, so

$$\frac{1}{2}\Big(a_n + b_n\Big) \in C_n.$$

Now, define a strict contraction, T_n on C_n by,

$$T_n z := (1 - \frac{1}{m})Tz + \frac{1}{2m}(a_n + b_n).$$

To see that C_n is T_n-invariant let $z \in C_n$, then

$$\begin{aligned}
\|a_n - T_n z\| &= \left\| a_n - \left(\Big(1 - \frac{1}{m}\Big)Tz + \frac{1}{2m}(a_n + b_n) \right) \right\| \\
&\leq \Big(1 - \frac{1}{m}\Big)\|a_n - Tz\| + \frac{1}{2m}\|a_n - b_n\| \\
&\leq \Big(1 - \frac{1}{m}\Big)\|a_n - Ta_n\| + \Big(1 - \frac{1}{m}\Big)\|Ta_n - Tz\| + \frac{1}{2m}\|a_n - b_n\| \\
&\leq \Big(1 - \frac{1}{m}\Big)\|a_n - Ta_n\| + \Big(1 - \frac{1}{m}\Big)\|a_n - z\| + \frac{1}{2m}\|a_n - b_n\| \\
&\leq \Big(1 - \frac{1}{m}\Big)\frac{1}{m^2} + \Big(1 - \frac{1}{m}\Big)\Big(\frac{\lambda}{2} + \frac{1}{m}\Big) + \frac{1}{2m}\Big(\lambda + \frac{2}{m^2}\Big) \\
&= \frac{1}{m^2} - \frac{1}{m^3} + \frac{\lambda}{2} + \frac{1}{m} - \frac{1}{2m}\lambda - \frac{1}{m^2} + \frac{1}{2m}\lambda + \frac{1}{m^3} \\
&= \frac{\lambda}{2} + \frac{1}{m}.
\end{aligned}$$

and similarly, $\|b_n - T_n z\| \leq \lambda/2 + 1/m$.

Thus, T_n has a unique fixed point, $c_n \in C_n$. That is,

$$c_n = T_n c_n = (1 - \frac{1}{m})Tc_n + \frac{1}{2m}(a_n + b_n).$$

and we have,

$$\begin{aligned}\|Tc_n - c_n\| &= \frac{1}{m}\|Tc_n - (a_n+b_n)/2\| \\ &\le \frac{1}{2m}(\|Tc_n - a_n\| + \|Tc_n - b_n\|) \\ &\le \frac{1}{m}\Big(\frac{\lambda}{2} + \frac{1}{m}\Big).\end{aligned}$$

It therefore follows from the above construction that for each $m \in \mathbf{N}$ the set of n for which $\|Tc_n - c_n\| \le (1/m)(\lambda/2 + 1/m)$ contains A_m and so is in $\mathcal{U}$. Consequently, for $\tilde{c} := [c_n]_{\mathcal{U}}$ we have

$$\|\tilde{c} - \tilde{T}\tilde{c}\| := \lim_{\mathcal{U}} \|c_n - Tc_n\| = 0$$

and so $\tilde{c}$ is a fixed point of $\tilde{T}$.

Similarly, from $\|a_n - c_n\|, \|b_n - c_n\| \le \lambda/2 + 1/m$ for all $n \in A_m \backslash A_{m+1}$, and consequently for all $n \in A_m$, we have $\|\tilde{a} - \tilde{c}\|$ and $\|\tilde{b} - \tilde{c}\|$ are less than or equal to $\lambda/2$. Since $\lambda = \|\tilde{a} - \tilde{b}\|$, the triangle inequality then ensures that $\|\tilde{a} - \tilde{c}\| = \|\tilde{b} - \tilde{c}\| = \|\tilde{a} - \tilde{b}\|/2$ and the result is established. ■

This Lemma states that the fixed point set of $\tilde{T}$ is metrically convex. An appeal to Menger's theorem then ensures the existence of a continuous path of fixed points joining any two fixed points of $\tilde{T}$ and lying within the metric segment between them.

Remark 3.12 When C is weakly-compact and a minimal invariant set for T it is always possible to find two such fixed points $\tilde{a}$ and $\tilde{b}$ of $\tilde{T}$ with $\|\tilde{a} - \tilde{b}\| = \operatorname{diam} \tilde{C}$. To see this, we may without loss of generality suppose that $\operatorname{diam} C = 1$ and that we have an approximate fixed point sequence (x_n) for T, with (x_n) weakly convergent to 0. Applying the Goebel-Karlovitz lemma we may extract a subsequence (x_{n_i}) such that $\|x_{n_i} - x_{n_{i+1}}\| \longrightarrow \operatorname{diam} C$. Taking $\tilde{a} := [x_{n_{2i}}]$ and $\tilde{b} := [x_{n_{2i-1}}]$ yields two fixed points of $\tilde{T}$ with

$$\|\tilde{a}\| = \|\tilde{b}\| = \|\tilde{a} - \tilde{b}\| = 1.$$

The following generalization of the Goebel-Karlovitz lemma, due to P. K. Lin [15] has proved basic for establishing the fixed point property using ultrapower methods.

Lemma 3.13 *Suppose C is a weakly-compact minimal invariant set for T. If $(\tilde{a}_n)$ is an approximate fixed point sequence for $\tilde{T}$ in $\tilde{C}$ then*

$$\lim_n \|\tilde{a}_n - \mathcal{J}x\| \ = \operatorname{diam}(C), \quad \textit{for all } x \in C.$$

Proof. Suppose this were not the case. Without loss of generality we may take $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C) = 1$, and by passing to a subsequence if necessary assume that $\|\tilde{a}_n - \tilde{T}\tilde{a}_n\| < 1/n$.

Then there are $\varepsilon_0 > 0$, $x_0 \in C$, and $n_0 \in \mathbf{N}$ with

$$\|\tilde{a}_n - \mathcal{J}x_0\| \ < \ 1 - \varepsilon_0, \quad \text{for all } n > n_0.$$

Let $\tilde{a}_n = [a^n_m]_{\mathcal{U}}$, with $a^n_m \in C$, and define

$$A_n := \{m : \|a^n_m - x_0\| < 1 - \varepsilon_0/2\},$$

and

$$B_n := \{m : \|a^n_m - Ta^n_m\| < 2/n\}.$$

Then A_n and B_n are in $\mathcal{U}$.

Put $m_0 = 0$ and for $n \in \mathbf{N}$ inductively choose $m_n \in A_n \cap B_n \cap \{m_{n-1}+1, m_{n-1}+2, \cdots\} \in \mathcal{U}$. Then the sequence $(a^n_{m_n})$ is such that

$$\|a^n_{m_n} \; - \; Ta^n_{m_n}\| \; < \; 2/n.$$

That is, $(a^n_{m_n})$ is an approximate fixed point sequence for T in C. But,

$$\|a^n_{m_n} \; - \; x_0\| \; < \; 1 - \varepsilon_0/2,$$

an observation which is difficult to reconcile with the fact that $(a^n_{m_n})$ is, by the Goebel-Karlovitz lemma, diameterizing for C. ■

Remark 3.14 If W is any nonempty closed convex and $\tilde{T}$-invariant subset of $\tilde{C}$, then, by the standard construction using Banach's contraction mapping principle, W contains an approximate fixed point sequence for $\tilde{T}$. So, by the above lemma, for every $x \in C$ we have $\sup\{\|\tilde{w} - \mathcal{J}x\| : \tilde{w} \in W\} = \operatorname{diam} C$. In particular, if we have 'normalized' so that $\operatorname{diam} C = 1$ and $0 \in C$, then

$$\sup_{\tilde{w} \in W} \|\tilde{w}\| \; = \; 1.$$

This leads to an important strategy for establishing the fixed point property in a class of spaces. Namely, try to construct a nonempty closed convex and $\tilde{T}$-invariant subset W of $\tilde{C}$ in such a way that the hypotheses on the spaces preclude the existence of elements in W with norms arbitrarily close to one; thereby contradicting the above lemma and hence the existence of a fixed point free nonexpansive self mapping of a nonempty weakly compact convex subset in the space.

Indeed, we know of only one proof establishing the fixed point property for a class of spaces via ultraproduct methods that does not use this approach, and that is S. Prus' proof [19, 20] (also see [12]) that uniformly non-creasey spaces have the fixed point property.

We illustrate the strategy outlined in the above remark with just one example, due to Garcia-Falset [9], others may be found scattered throughout this Handbook. See also the Notes and remarks section for references to the literature.

Let $\mathcal{U}$ be a given ultra filter over $\mathbf{N}$ and for each Banach space X define a coefficient $R(X)$ by,

$$R(X) \; := \; \sup\{\lim_{\mathcal{U}} \|x + x_n\| : \|x\| \le 1; \; \|x_n\| \le 1, \text{ for all } n \text{ and } (x_n) \to 0 \text{ weakly}\}.$$

Equivalently, $R(X)$ is the 'smallest' number such that

$$\lim_{\mathcal{U}} \|x + x_n\| \le R(X)\|x\| \vee (\lim_{\mathcal{U}} \|x_n\|),$$

for all $x \in X$ and all weak null sequences (x_n).

For example, $R(c_0) = 1$, while $R(L_1) = 2$, in general $1 \le R(X) \le 2$.

Proposition 3.15 *If X is a Banach space with $R(X) < 2$, then X has the weak fixed point property.*

Proof. Suppose X fails the weak-fixed point property. Then there exists a weakly-compact convex set C with $\operatorname{diam}(C) = 1$ which is a minimal invariant set for some nonexpansive mapping T. Further we may assume that C contains a weakly-null approximate fixed point sequence (a_n) for T. Let $\tilde{C}$ and $\tilde{T}$ be defined as above and define

$$W \ := \ \{[w_n]_{\mathcal{U}} : w_n \in C, \ \text{for } n \in N, \ \|[w_n]_{\mathcal{U}} - [a_n]\| \leq 1/2, \ \text{and } D[w_n] \leq 1/2\},$$

where $D[w_n] := \lim_{\mathcal{U}m,} \lim_{\mathcal{U},n} \|w_m - w_n\|$. Then, W is readily seen to be a $\tilde{T}$-invariant, closed, convex, nonempty (as $(1/2)[a_n] \in W$) subset of $\tilde{C}$. Thus, by the above remark

$$\sup\{\|\tilde{w}\| : \tilde{w} \in W\} = 1.$$

On the other hand, let $\tilde{w} = [w_n]_{\mathcal{U}}$ be any element of W, where without loss of generality $w_n \in C$, for all $n \in \mathbf{N}$, and let w_0 be the weak-limit with respect to $\mathcal{U}$ of (w_n). Then,

$$\begin{aligned} \|\tilde{w}\| &= \lim_{\mathcal{U}} \|w_n\| \\ &= \lim_{\mathcal{U}} \|(w_n - w_0) + w_0\| \\ &\leq R(X)(\lim_{\mathcal{U}} \|w_n - w_0\|) \vee \|w_0\|, \end{aligned}$$

by definition of $R(X)$, as $(w_n - w_0)$ converges weakly to 0, hence

$$\|\tilde{w}\| \leq R(X) \lim_{\mathcal{U},n} \lim_{\mathcal{U},m} \|w_n - w_m\| \vee \|w_n - a_n\|,$$

by lower semi-continuity of the norm, since

$$\lim_{\mathcal{U}} w_m = \lim_{\mathcal{U}} (w_n - a_n) = w_0 \ .$$

Hence

$$\|\tilde{w}\| \leq R(X) \times \frac{1}{2} \vee \frac{1}{2} = R(X)/2 < 1.$$

A contradiction which establishes the result. ■

Our choice of the above result to illustrate the strategy in the previous remark is based on its utility; the parameter involved is readily evaluated for many spaces and the criteria is satisfied in a large class of spaces.

Since nearly uniformly smooth (NUS) Banach spaces are readily seen to have $R(X) < 2$ (see [9]), the result answers in the affirmative the long standing question of whether or not NUS spaces have the weak fixed point property.

In a weakly orthogonal Banach lattice $R(X)$ is less than or equal to the Riesz angle $\alpha(X)$ introduced in [4], thus, this proposition generalizes results of [4], [22] and [23].

The above argument is typical of those for many of the more recent 'non-classical' results in metric fixed point theory, starting with Maurey's 1982 proof of the weak fixed point property for c_0, for which it provcides an alternative proof. Note that, since a numeric contradiction is arrived at, by carefully analyzing the proof, the gap (here between $R(X)/2$ and 1) can be exploited to establish the weak fixed point property for spaces whose Banach-Mazur distance from a space satisfying the assumptions is not too great. This is the basis for many of the results given in the chapter on *Stability of the fixed point property for nonexpansive mappings*, where the reader can find many more existence results, in the more general guise of stability results, together with other applications of the methods outlined here.

4. Maurey's fundamental theorems

Maurey's results were deep and particularly significant comming as they did just after Alspach demonstrated the failure of the weak fixed point property in $L^1[0,1]$. As we have already remarked, his results set the stage for the second major revolution in metric fixed point theory. We will not give the details of the proofs for many of his results, and the interested reader is referred to [17], [7] and [1].

Maurey began by establishing the w-fpp for the space c_0.

Theorem 4.1 *The space c_0 has the weak fixed point property.*

This result had eluded proof for many years. From a geometric point of view the space c_0 is a bad space, exhibiting many of the features found in l_∞. Previously, only partial results related to the fixed point property for special domains in c_0 were known and the arguments employed were often extremely intricate and tedious. We will not give Maurey's original proof, as the result is a special case of proposition 3.15 above. However, his proof was both elegant and open to generalization. It exploited the lattice structure of c_0 induced from the canonical basis. Others (see, for example, [4], [22], [23], [**?**] and the Notes and Remarks section below) quickly refined and generalized these ideas to a large class of Banach lattices.

Perhaps the most important result of Maurey is the following.

Theorem 4.2 *Any reflexive subspace X of $L_1[0,1]$ has the fixed point property; that is, any nonexpansive self mapping of a nonempty bounded closed and convex subset of X has a fixed point.*

The ideas behind the original proof of this result may be generalized to obtain the following.

Theorem 4.3 *Let X be a Banach lattice with a uniformly monotone norm and assume that l_1 is not finitely representable in X. Then X has the fixed point property.*

Recall that a Banach lattice X has a uniformly monotone norm if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| \geq \|y\| + \delta$ whenever $x \geq y \geq 0$ and $\|x - y\| \geq \varepsilon$, with $\|y\| = 1$.

In his investigation of the fixed point property, Maurey discovered many fundamental results which led to new insights and a better understanding of the property. For example, in his proof of the above theorem, Maurey used lemma 3.11 and the lattice structure of $L_1[0,1]$ to show that the ultrapower $(X)_{\mathcal{U}}$ of X would contain isometric copies of l_1^n, for all n, if X failed to have the fixed point property. Since reflexive subspaces of $L_1[0,1]$ are superreflexive, this gave the desired contradiction. Following the appearance of his result there have been many attempts to identify a geometric property enjoyed by the reflexive subspaces of $L_1[0,1]$ which would imply the fixed point property. So far such attempts have been in vain.

In the years prior to the appearance of Alspach's example the w-fpp had been established for many of the classical Banach spaces and it was commonly conjectured that all Banach spaces enjoyed the weak fixed point property. His example therefore came as a surprise to many, and helped redefine the direction of investigation. It cast doubt on the likelihood of positive answers to three of the most basic open questions, which we list in decreasing order of strength:

(1) Do all reflexive Banach spaces have the fixed point property? [And conversely; does having the fixed point property imply reflexivity of the space?]

(2) Do all superreflexive Banach spaces have the fixed point property?

(3) Does the Hilbert space ℓ_2 have the fixed point property in all equivalent norms?

To which we would add,

(4) Does c_0 have the weak fixed point property in all equivalent norms? [If on no other ground than in its natural norm the space is about as bad as it can get.]

Maurey's results, in particular theorem 4.2, offsets Alspach's finding and points in the direction of an affirmative to (2) and hence (3). Lin's recent stability result for ℓ_2 (see section 5 and [16]) also lends support to (3). The recent progress described in the chapter on *Renormings of ℓ_1 and c_0 and fixed point properties* may be seen as support for the converse of (1) further support is provided by a result of van Dulst and Pach [6] which shows that the 'super fixed point property' implies superreflexivity. Maurey was unsuccessful in his attemps to settle (2), however, in the course of his investigations he discovered the following tantalizing result, the proof of which again relies on constructions in an ultrapower of the space.

Theorem 4.4 *Let X be a superreflexive Banach space and let K be a bounded nonempty closed convex subset of X. Then any isometry $T : K \to K$ has a fixed point.*

In other words, superreflexive Banach spaces have the fixed point property for isometries.

Before we close this section, it is worth mentioning that Maurey [17] also proved that the Hardy space H^1 has the fixed point property.

5. Lin's results

We will not attempt to give a detailed list of the results obtained in the two decades following Maurey's discoveries, many of which may be found in the chapter on *Stability of the fixed point property for nonexpansive mappings*. However, some of the most important contributions were due to P-K.Lin [15], and we discuss two of these.

Theorem 5.1 *Let X be a Banach space with a 1-unconditional basis, then X has the weak fixed point property.*

Proof. Assume that there exist a weakly compact convex nonempty subset C of X and $T : C \to C$ a nonexpansive map with no fixed point. Let K be a minimal set for T. Let (x_n) be an approximate fixed point sequence in K. Without loss of generality, we may assume that (x_n) converges weakly to $0 \in K$ and $\operatorname{diam}(K) = 1$. Passing to a subsequence, one can construct a sequence of natural projections (P_n), associated to the Schauder basis of X, such that

$$\begin{aligned} P_n \circ P_m &= 0 \quad \text{if } n \neq m, \\ \lim_{n\to\infty} \|P_n(x_n)\| &= 0 \quad \text{for any } x \in X, \text{ and} \\ \lim_{n\to\infty} \|P_n(x_n) - x_n\| &= 0. \end{aligned}$$

Using the Goebel-Karlovitz lemma, one may assume that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 1.$$

Let $(X)_{\mathcal{U}}$ be an ultrafilter of X. Let $\tilde{K}$ and $\tilde{T}$ be associated to K and T. Set

$$\tilde{x} = [(x_n)] \text{ and } \tilde{y} = [(x_{n+1})] \text{ in } \tilde{K}.$$

Both $\tilde{x}$ and $\tilde{y}$ are fixed points for $\tilde{T}$. Consider the projections (on $(X)_{\mathcal{U}}$)

$$\tilde{P} = [(P_n)] \text{ and } \tilde{Q} = [(P_{n+1})].$$

Hence

$$\tilde{P}(\tilde{x}) = \tilde{x}; \;\; \tilde{Q}(\tilde{y}) = \tilde{y}; \;\; \tilde{P}(\tilde{y}) = \tilde{Q}(\tilde{x}) = \tilde{P}(x) = \tilde{P}(x) = 0$$

for any $x \in X$. Since the constant of unconditionality of the basis is 1, then we have

$$\tilde{P} \circ \tilde{Q} = 0; \;\; \|\tilde{I} - \tilde{P}\| \leq 1; \;\; \|\tilde{I} - \tilde{Q}\| \leq 1; \;\; \|\tilde{P} + \tilde{Q}\| \leq 1$$

where $\tilde{I}$ is the identity operator of $(X)_{\mathcal{U}}$. Now set

$$\tilde{W} = \left\{ \tilde{w} \in \tilde{K};\text{ there exists } x \in K;\, \|\tilde{w} - x\| \leq \frac{1}{2} \text{ and } \|\tilde{w} - \tilde{x}\| \leq \frac{1}{2};\, \|\tilde{w} - \tilde{y}\| \leq \frac{1}{2} \right\}\cdot$$

Since

$$\|\tilde{x} + \tilde{y}\| = \|\tilde{P}(\tilde{x}) + \tilde{Q}(\tilde{y})\| \leq \|\tilde{P}(\tilde{x}) - \tilde{Q}(\tilde{y})\| = \|\tilde{x} - \tilde{y}\| = 1,$$

then $\tilde{x} + \tilde{y} \in \tilde{W}$, in other words, $\tilde{W}$ is not empty. It is easy to check that $\tilde{W}$ is $\tilde{T}$-invariant, i.e. $\tilde{T}(\tilde{W}) \subset \tilde{W}$. Let $\tilde{w} \in \tilde{W}$ and $x \in K$ such that $\|\tilde{w} - x\| \leq 1/2$. Hence

$$\begin{aligned} 2\tilde{w} &= (\tilde{P} + \tilde{Q})(\tilde{w}) + (\tilde{I} - \tilde{P})(\tilde{w}) + (\tilde{I} - \tilde{Q})(\tilde{w}) \\ &= (\tilde{P} + \tilde{Q})(\tilde{w} - x) + (\tilde{I} - \tilde{P})(\tilde{w} - \tilde{x}) + (\tilde{I} - \tilde{Q})(\tilde{w} - \tilde{y})\cdot \end{aligned}$$

So

$$\begin{aligned} 2\|\tilde{w}\| &\leq \left\|(\tilde{P} + \tilde{Q})(\tilde{w})\right\| + \left\|(\tilde{I} - \tilde{P})(\tilde{w})\right\| + \left\|(\tilde{I} - \tilde{Q})(\tilde{w})\right\| \\ &\leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}\cdot \end{aligned}$$

which implies $\|\tilde{w}\| \leq 3/4$. This is in contradiction with remark 3.14. ■

This result was quickly extended, see for example [22], [23], [13] and [5]. In fact more was proved. For example, by exploiting the gap between 3/4 and 1, Lin obtained the the conclusion for any Banach space X with an unconditional basis provided that the constant of unconditionality λ is less than $(\sqrt{33} - 3)/2$. This conclusion brings to the surface the problem of whether or not the above result is valid for all Banach spaces with an unconditional basis. This problem is still open and clearly related to the stability problem: Does there exists a Banach space for which the above conclusion is true for any equivalent norm? In the particular case of Hilbert space, Lin [16] improved on all previously known results by establishing the following stability bound.

Theorem 5.2 *Let $(H, \|\cdot\|)$ be a real Hilbert space. Let $|\cdot|$ be an equivalent norm such that*

$$\|x\| \leq |x| \leq \beta\|x\| \text{ for all } x \in H\cdot$$

Then $(H, |\cdot|)$ has the fixed point property provided

$$\beta < \sqrt{\frac{5 + \sqrt{13}}{2}} \doteq 2.07.$$

The proof uses all of the ingredients developed in this chapter and may be found in the chapter on *Stability of the fixed point property for nonexpansive mappings*.

6. Notes and Remarks

The results developed in this chapter have been used to establish a variety of results in metric fixed point theory, in particular the weak fixed point property for a large variety of Banach spaces.

The notion of an ultrafilter dates back to work of Tarski in 1930 and that of an ultrapower to Skolem in 1934. A rich theory for ultrapowers and ultraproducts (of sets and models) has been built up by a succession of logicians: Los, Frayne, Morel, Scott, Tarski, Hanf, Chang, Keisler, Robinson, Luxemburg and Shelah to mention only a few.

Banach space ultrapowers and ultraproducts were formally introduced by Dacunha-Castelle and Krivine in 1972 and were subsequently developed and applied by Stern, Heinrich and many others. They are now an important and widely used tool for probing the geometry and structure of Banach spaces. They have been particularly important in the study of local theory, superproperties, operator ideals and the isomorphic classification of Banach spaces.

Ultrapower methods were first introduced into metric fixed point theory by B. Maurey [17] in 1982 when he used this technique to provide a positive resolution to the long standing question of whether or not c_0 had the weakly-fixed point property. He took the W of remark 3.14 to be the metric midpoint set for two fixed points of $\tilde{T}$ constructed as in remark 3.12. This was generalized in [4] to obtain the weak fixed point property for Banach lattices with a Riesz angle

$$\alpha(X) := \sup\{\| \, |x| \vee |y| \, \| : x, \, y \in B_X\} \ < \ 2$$

and for which

$$\liminf_m \liminf_n \| \, |x_n| \wedge |x_m| \, \| \ = \ 0, \quad \text{whenever } (x_n) \text{ converges weakly to } 0.$$

Lattices with this last property were referred to as *weak orthogonal* Banach lattices. A stronger variant of weak orthogonality, namely:

$$\liminf_n \| \, |x_n| \wedge |x| \, \| \ = \ 0, \quad \text{whenever } (x_n) \text{ converges weakly to } 0 \text{ and } x \in X,$$

was shown to imply the weakly-fixed point property by Sims [22, 23]. The proof employed the W first defined by P. K. Lin in 1983 and used to establish the weak fixed point property for Banach spaces with a 1-unconditional basis [15]. The set W used in these proofs consisted of those points in Maurey's W whose distance from $\mathcal{J}C$ is less than or equal to a half, where in addition the points $\tilde{a}$ and $\tilde{b}$ were chosen to be 'orthogonal' to one another so that $\|\tilde{a} + \tilde{b}\| = \|\tilde{a} - \tilde{b}\| = 1$. A class of spaces in which such a choice is always possible was considered in [22]. Such spaces were said to have property WORTH. It remains an open question whether or not all spaces with WORTH have the weakly-fixed point property.

Several more 'geometric' variants of these conditions have been introduced. For instance A. Jiménez-Melado and E. Lloréns-Fuster [11] considered the property of orthogonal convexity, gave examples of orthogonally convex spaces, and showed that it entails the weakly-fixed point property. A Banach space X is *orthogonally convex* if for every weak-null sequence (x_n) with

$$D(x_n) := \limsup_m \limsup_n \|x_m - x_n\| \ > \ 0$$

there exists $\beta > 1/2$ such that

$$\limsup_m \limsup_n \sup\{\|z\| : \|z - x_m\|, \, \|z - x_n\| \leq \beta\|x_m - x_n\|\} \ < \ D(x_n).$$

The characteristic of a sequence, $D(w_n)$, first introduced in the above context and used in the proof of proposition 3.15, has played a crucial role in many of the more recent results. For example, in the proof that spaces with Kalton's property (M) have the weak fixed point property [10] where it was found necessary to employ a W similar to that used in the proof of proposition 3.15, but defined by the asymmetric constraints $\|[w_n]_{\mathcal{U}} - [a_n]\| \leq 1/2 - \varepsilon$ and $D[w_n] \leq 1/2 + \varepsilon$, with $\varepsilon > 0$. Recall that X has property (M)if weak null types are constant on spheres about 0. That is, $\lim_{\mathcal{U}} \|x - x_n\| = \lim_{\mathcal{U}} \|y - x_n\|$ whenever $\|x\| = \|y\|$ and (x_n) weakly converges to 0. Starting with the proof of the Goebel-Karlovitz lemma, weak null types are seen to play an essential role in many aspects of metric fixed point theory. Indeed, understanding the behaviour of weak null types in a space is often the key to its fixed point properties.

P. K. Lin used a W defined by a combination of all the constraints discussed above to establish what is currently the best known bound for the stability of the fpp in ℓ_2 discussed in section 5.

For many of the results discussed in this chapter, and in many applications, a Banach space ultrapower $(X)_{\mathcal{U}}$ over $\mathbf{N}$ can be replaced by the space

$$\ell_\infty(X)/c_0(X),$$

where the quotient norm is canonically given by $\|[x_n]\| = \limsup_n \|x_n\|$, see for example: [4, 9, 10]. However, calculations in this space usually entail an infestation of subsequence taking. In many instances it is possible to avoid the use of these larger ambient spaces altogether; for example, see [7] where an ultrapower free proof of Maurey's result on the reflexive subspaces of $\mathcal{L}_1$ may be found. However, such proof often obscure the essential argument in a veritable plague of epsilons and deltas. None-the-less, the disadvantages and advantages are largely cosmetic and it is up to the individual to choose which approach is most to their taste.

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