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Some Geometrical Properties and Fixed Point Theorems in Orlicz Spaces

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Let (G, Σ, μ) be a finite, atomless measure space and let L^ϕ be an Orlicz space of measurable functions on G. We consider some geometrical properties of the functional $\rho(f) = \int_G \phi(f(t))\, d\mu(t)$, called the Orlicz modular. These properties, like strict convexity, uniform convexity or uniform convexity in every direction, can be equivalently expressed in terms of the properties of the corresponding Orlicz function ϕ. We use these properties in order to prove some fixed point results for mappings $T: B \to B$, $B \subset L^\phi$, that are nonexpansive with respect to the Orlicz modular ρ, i.e., $\rho(Tf - Tg) \leqslant \rho(f - g)$ for all f and g in B. We prove also existence and uniqueness in L^ϕ of the best approximant with respect to ρ and some convex subsets of L^ϕ. Our results are valid also in the case when the Orlicz function ϕ does not satisfy the Δ_2-condition. This demonstrates the advantage of our method because, in the latter case, both Luxemburg's and Orlicz's norms cannot possess suitable convexity properties. © 1991 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider classical Orlicz function spaces L^ϕ, but at the same time we introduce some nonstandard geometrical properties and apply them to obtain fixed point theorems for mappings acting within L^ϕ.

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The advantage of our approach consists in: (1) we are able to study the structure of an Orlicz space L^ϕ even if the function ϕ does not satisfy the Δ_2-condition (in the latter case, L^ϕ is a very bad space from the point of view of geometry of Banach spaces); (2) our conditions can be verified much easier because they do not involve the norm $\|\cdot\|_\phi$ which is indirectly defined, but instead they employ the Orlicz modular which is a simple integral functional.

Let us start with a brief review of some basic concepts and facts of the theory of Orlicz and modular spaces.

1.1. DEFINITION. A convex, continuous, even function $\phi: \mathbb{R} \to \mathbb{R}^+$ is called an N-function if and only if the following conditions are satisfied:

(1) $\phi(0) = 0$,

(2) ϕ is strictly increasing in $[0, \infty)$,

(3) $\lim_{u \to 0} \phi(u)/u = 0$ and $\lim_{u \to \infty} \phi(u)/u = \infty$.

Let ϕ be an N-function and let (G, Σ, μ) be a measure space, μ being finite and atomless. Let us consider the space $L^0(G)$ consisting of all measurable real-valued functions on G, and define for every $f \in L^0(G)$ the Orlicz modular $\rho(f)$ by the formula

$$\rho(f) = \int_G \phi(f(x))\, d\mu(x).$$

The Orlicz space L^ϕ is then defined as follows:

$$L^\phi = \{f \in L^0(G); \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\},$$

or equivalently as

$$L^\phi = \{f \in L^0(G); \rho(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

The vector space L^ϕ can be equipped with Luxemburg's norm defined by

$$\|f\|_\phi = \inf\{\lambda > 0; \rho(f/\lambda) \leqslant 1\}.$$

It is well known that $(L^\phi, \|\cdot\|_\phi)$ is a Banach space. For the review of the theory of Orlicz spaces see, e.g., [15, 17, 21]. The functional ρ satisfies:

(a) $\rho(f) = 0$ if and only if $f = 0$ μ—a.e.,

(b) $\rho(-f) = \rho(f)$ for any $f \in L^\phi$,

(c) $\rho(\alpha f + \beta g) \leqslant \alpha\rho(f) + \beta\rho(g)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $f \in L^\phi$.

In this paper, ρ will be called an Orlicz modular induced by ϕ (for the review of modular spaces see [17]); Orlicz modular is a special case of a function modular (cf. [13, 14]). We have two structures in L^ϕ: one is that of a Banach space induced by the norm $\|\cdot\|_\phi$, while the other is a structure of a modular space introduced by the Orlicz modular ρ. The basic fact that relates one structure to the other is that $\|f_n\|_\phi \to 0$ if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$. The geometry of the space $(L^\phi, \|\cdot\|_\phi)$ is relatively well known (cf. [6–9, 18–20]). It turns out, however, that some basic geometrical properties of $(L^\phi, \|\cdot\|_\phi)$ like reflexivity, strict convexity, and uniform convexity can hold only if ϕ satisfies the Δ_2-condition, i.e., there exist $u_0 > 0$, $k > 0$ such that $\phi(2u) \leqslant k\phi(u)$ for all $u \geqslant u_0$. If ϕ fails to satisfy Δ_2, we cannot even determine the dual space $(L^\phi)^*$, while in the Δ_2 case $(L^\phi)^*$ is an Orlicz space L^{ϕ^*}, where ϕ^* is complementary to ϕ in the sense of Young. It is worth recalling that Δ_2 is equivalent to the fact that $\rho(f) < \infty$ for any $f \in L^\phi$. Also, ϕ satisfies Δ_2 if and only if from $\rho(f_n) \to 0$ it follows that $\|f_n\|_\phi \to 0$.

In Section 2 we introduce some geometrical properties of Orlicz modulars such as strict and uniform convexity. We characterize them in terms of simple properties of an N-function ϕ. In Theorem 2.11 we state that under some assumptions there exists a unique best approximant, i.e., such a function $g_0 \in C$ that $\inf\{\rho(f-g);\ g \in C\} = \rho(f-g_0)$, where $f \in L^\phi$ and C is a (ρ)-closed, convex subset of L^ϕ. This interesting approximation result is then used to prove Theorem 2.12 in which we state that L^ϕ has a property that resembles reflexivity, provided the Orlicz modular is uniformly convex. This and the other results are then applied in Section 3 to prove a fix point theorem (Theorem 3.10) for ρ-nonexpansive mappings acting within L^ϕ, i.e., for $T: B \to B$, $B \subset L^\phi$ such that $\rho(Tf - Tg) \leqslant \rho(f-g)$ for all f, g in B. To do this we introduce a notion of a modular normal structure. Fixed point theorems for ρ-nonexpansive maps in Musielak–Orlicz spaces were considered in [16]. In [10, 14] the ρ-nonexpansive mappings in modular function spaces were studied. All above mentioned results used some compactness arguments while in this paper we do not need to assume any compactness at all.

2. Geometrical Properties of Orlicz Modulars

To the end of this paper we assume that μ is finite and atomless. However, by simple modification of proofs all our results can be obtained in the σ-finite case. Let us introduce some properties of N-functions and of Orlicz modulars generated by them.

2.1. Definition. An n-function ϕ is said to be strictly convex (SC) if and only if for every $u \neq v$ there holds

$$\phi\left(\frac{u+v}{2}\right) < \frac{\phi(u)+\phi(v)}{2}.$$

2.2. Definition. An Orlicz modular ρ is called strictly convex (SC) if for every $f, g \in L^\phi$ such that $\rho(f) = \rho(g)$ and

$$\rho\left(\frac{f+g}{2}\right) = \frac{\rho(f) + \rho(g)}{2},$$

there holds $f = g$.

Following [10] let us recall the following definition.

2.3. Definition. (a) For any nonzero $h \in L^\phi$ and $r > 0$ the ρ-modulus of uniform convexity in the direction of h is defined by

$$\delta_\phi(r, h) = \inf\{1 - \tfrac{1}{2}\rho(f + \tfrac{1}{2}h)\},$$

where the infimum is taken over all $f \in L^\phi$ such that $\rho(f) \leqslant r$ and $\rho(f+h) \leqslant r$.

(b) We say that ρ is uniformly convex in every direction (UCED) if

$$\delta_\phi(r, h) > 0 \text{ for every } h \in L^\phi \setminus \{0\} \text{ and } r > 0.$$

(c) We say that ρ is uniformly convex (UC) if for any $\varepsilon > 0$ and any $r > 0$, the ρ-modulus of uniform convexity, defined as

$$\delta_\phi(r, \varepsilon) = \inf\{\delta_\phi(r, h); h \in L^\phi, \rho(h/2) \geqslant r\varepsilon\}$$

is strictly positive.

Observe that

$$\delta_\phi(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right); \rho(f) \leqslant r, \rho(g) \leqslant r, \rho(g) \leqslant r, \rho\left(\frac{f-g}{2}\right) \geqslant r\varepsilon\right\}.$$

Remark. Let us observe that $\delta_\phi(r, \varepsilon)$ is an increasing function of ε for every fixed r. Moreover, for $r_1 < r_2$ there holds

$$1 - \frac{r_2}{r_1}\left(1 - \delta_\phi\left(r_2, \varepsilon\frac{r_1}{r_2}\right)\right) \leqslant \delta_\phi(r_1, \varepsilon).$$

Let us also mention that, since ρ does not have to be homogeneous, $\delta_\phi(r, h)$ depends on h, not only on the direction of h. Similarly, we have to consider the dependence of $\delta_\phi(r, h)$ upon r. We decided however to preserve the "norm space" terminology because UCED plays a similar role in our theory. It was proved by Kaminska [9] that for the norm $\|\cdot\|_\phi$ the following assertions are equivalent: (a) $\|\cdot\|_\phi$ is UCED, (b) $\|\cdot\|_\phi$ is SC, (c) ϕ is SC and ϕ satisfies Δ_2. Our next theorem gives the modular version of that result. Note, however, that replacing $\|\cdot\|_\phi$ by ρ, we are able to eliminate the Δ_2-assumption.

2.5. THEOREM. *The following three conditions are equivalent*:

(i) ϕ *is SC*;

(ii) ρ *is UCED*;

(iii) ρ *is SC*.

Proof. (i)$\Rightarrow$(ii) Let $r>0$ and $h\in L^\phi\backslash\{0\}$ be arbitrary. We want to show that $\delta_\phi(r,h)$ is strictly positive. In order to do this, take any $f\in L^\phi$ such that $\rho(f)\leqslant r$ and $\rho(f+h)\leqslant r$. Let us choose $0<\alpha<\beta$ for which the set $G_1=\{t\in G;\ \alpha\leqslant|h(t)|\leqslant\beta\}$ has positive measure. Write $M=\phi^{-1}(2r/\mu(G_1))$. Since ϕ is SC on $[-(M+\beta), M+\beta]$, it follows by the compactness argument that there exists a $\delta>0$ such that

$$\phi\left(\frac{u+v}{2}\right)\leqslant(1-\delta)\frac{\phi(u)+\phi(v)}{2}. \qquad (*)$$

for $u,v\in[-(M+\beta),\ M+\beta]$ with $|u-v|\geqslant\alpha$. Denote $G_0=\{t\in G;$ $|f(t)|\geqslant M\}$, and observe that

$$r\geqslant\rho(f)\geqslant\int_{G_0}\phi(f(t))\,d\mu(t)\geqslant\phi(M)\,\mu(G_0).$$

Hence,

$$\mu(G_0)\leqslant\frac{r}{\phi(M)}=\frac{\mu(G_1)}{2}$$

and

$$\mu(G_1\backslash G_0)\geqslant\mu(G_1)-\mu(G_0)\geqslant\frac{\mu(G_1)}{2}.$$

Let us note that for $t\in G_1\backslash G_0$, we have $f(t)\in[-(M+\beta),\ M+\beta]$ and

$$(h(t)+f(t))\in[-(M+\beta), M+\beta].$$

By $(*)$ (with $u=h+f$, $v=f$) we obtain then that

$$\int_{G_1\backslash G_0}\phi\left(f(t)+\frac{h(t)}{2}\right)d\mu(t)$$
$$\leqslant(1-\delta)\int_{G_1\backslash G_0}\frac{1}{2}\left(\phi(f(t))+\phi(h(t)+f(t))\right)d\mu(t). \qquad (**)$$

Since,

$$\rho\left(f+\frac{h}{2}\right)=\int_{G_1\setminus G_0}\phi\left(\left(f+\frac{h}{2}\right)(t)\right)d\mu(t)+\int_{G_3}\phi\left(\left(h+\frac{h}{2}\right)(t)\right)d\mu(t),$$

where $G_3=G\setminus(G_1\setminus G_0)$, by $(**)$ and convexity of ϕ we conclude then that

$$\begin{aligned}\rho\left(f+\frac{h}{2}\right)&\leqslant(1-\delta)\int_{G_1\setminus G_0}\frac{1}{2}(\phi(f(t))+\phi(h(t)+f(t)))\,d\mu(t)\\&\quad+\int_{G_3}\frac{1}{2}(\phi(f(t))+\phi(h(t)+f(t)))\,d\mu(t)\\&=\int_G(\phi(f(t))+\phi(h(t)+f(t)))\,d\mu(t)\\&\quad-\delta\int_{G_1\setminus G_0}\frac{1}{2}(\phi(f(t))+\phi(h(t)+f(t)))\,d\mu(t).\end{aligned}$$

Since ϕ is convex and even, we obtain that

$$\begin{aligned}\phi(h(t)/2)&\leqslant\frac{1}{2}(\phi(-f(t))+\phi(f(t)+h(t)))\\&=\frac{1}{2}(\phi(f(t))+\phi(f(t)+h(t))).\end{aligned}\qquad(***)$$

For each $t\in G_1\setminus G_0$, there holds $|h(t)|/2\geqslant\alpha/2$ and in view of $(***)$, we have

$$\begin{aligned}&\int_{G_1\setminus G_0}\frac{1}{2}(\phi(f(t))+\phi(h(t)+f(t)))\,d\mu(t)\\&\qquad\geqslant\phi(\alpha/2)\,\mu(G_1\setminus G_0)\geqslant\frac{1}{2}\phi(\alpha/2)\,\mu(G_1).\end{aligned}$$

Hence,

$$\begin{aligned}\rho\left(f+\frac{h}{2}\right)&\leqslant\frac{1}{2}(\rho(f)+\rho(h+f))-\frac{1}{2}\delta\phi(\alpha/2)\,\mu(G_1)\\&\leqslant r-\frac{1}{2}\delta\phi(\alpha/2)\,\mu(G_1).\end{aligned}$$

Finally, we can observe that

$$\delta_\phi(r,h)\geqslant\frac{1}{2r}\delta\phi(\alpha/2)\,\mu(G_1)>0.$$

(ii) ⇒ (iii) Suppose f and g belong to L^ϕ, $f \neq g$ and $\rho(f) = \rho(g) = r$. Put $h = f - g$, then $\rho(f + g) \leqslant r$ and $\rho(g) \leqslant r$. Since $h \neq 0$, then $r > 0$ and by (ii) we obtain $\delta_\phi(r, h) > 0$. Thus,

$$\rho\left(\frac{1}{2}(f+g)\right) = \rho\left(g + \frac{h}{2}\right) \leqslant (1 - \delta_\phi(r, h))\, r,$$

and since $\delta_\phi(r, h) > 0$, then $\rho(\frac{1}{2}(f+g)) < \frac{1}{2}(\rho(f) + \rho(g)) = r$.

(iii) ⇒ (i) Take $a \neq b$. Since μ is atomless, there exist G_1 and G_2 such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = \varnothing$ and $\mu(G_1) = \mu(G_2)$. Let us define the function

$$f = a\chi_{G_1} + b\chi_{G_2} \qquad \text{and} \qquad g = b\chi_{G_1} + a\chi_{G_2}.$$

Then, $\frac{1}{2}(f+g) = \frac{1}{2}(a+b)\,\chi_G$ and $f - g = (a-b)(\chi_{G_1} - \chi_{G_2}) \neq 0$. Thus,

$$\rho(f) = \tfrac{1}{2}(\phi(a) + \phi(b))\,\mu(G) = \rho(g).$$

By strict convexity of ρ we have $\rho(\frac{1}{2}(f+g)) < \frac{1}{2}(\rho(f) + \rho(g))$ which implies that

$$\phi(\tfrac{1}{2}(a+b)) < \tfrac{1}{2}(\phi(a) + \phi(b)).$$

The proof of Theorem 2.5 is complete.

2.6. Definition. A function ϕ is said to be very convex (VC) if and only if for any $\varepsilon > 0$ and any $u_0 > 0$ there exists $\delta > 0$ such that

$$\phi\left(\frac{1}{2}(u - v)\right) \geqslant \frac{\varepsilon}{2}(\phi(u) + \phi(v)) \geqslant \varepsilon\phi(u_0) \tag{$*$}$$

implies

$$\phi\left(\frac{1}{2}(u + v)\right) \leqslant \frac{1}{2}(1 - \delta)(\phi(u) + \phi(v)). \tag{$**$}$$

It is easy to see that ϕ is VC if ϕ is uniformly convex (for the study of uniform convexity of ϕ see [8]). For our next result we need the following technical lemma.

2.7. Lemma. *Let ϕ be very convex. To every $\varepsilon > 0$, $s > 0$ there exists $\eta > 0$ which depends only on ε and s such that $\delta_\phi(r, \varepsilon) > \eta$ for $r > s/3$.*

Proof. Let us fix $\varepsilon > 0$, $s > 0$ and choose $u_0 > 0$ such that $\phi(u_0)\,\mu(G) \leqslant s\varepsilon/3$. Since ϕ is VC we can pick up a $\delta > 0$ such that

$$\phi\left(\frac{1}{2}(u - v)\right) \geqslant \frac{\varepsilon}{6}\left(\frac{1}{2}(\phi(u) + \phi(v))\right) \geqslant \frac{\varepsilon}{6}\phi(u_0)$$

implies

$$\phi\left(\frac{1}{2}(u+v)\right) \leqslant \frac{1}{2}(1-\delta)(\phi(u)+\phi(v)).$$

Let us fix any $r > s/3$, take any functions $f, h \in L^{\phi}$ with $\rho(f) \leqslant r$, $\rho(f+h) \leqslant r$ and $\rho(h/2) \geqslant r\varepsilon$. Let

$$G_0 = \{t \in G; \phi(f(t)) + \phi(f(t)+h(t)) \geqslant \phi(u_0)\},$$

$$G_1 = \left\{t \in G_0; \phi\left(\frac{h(t)}{2}\right) \geqslant \frac{\varepsilon}{12}(\phi(f(t)) + \phi(f(t)+h(t))\right\},$$

and

$$G_2 = G_0 \backslash G_1.$$

Then,

$$\begin{aligned}
\rho\left(f+\frac{h}{2}\right) &= \int_{G\backslash G_0} \phi\left(\left(f+\frac{h}{2}\right)(t)\right) d\mu(t) + \int_{G_1} \phi\left(\left(f+\frac{h}{2}\right)(t)\right) d\mu(t) \\
&\quad + \int_{G_2} \phi\left(\left(f+\frac{h}{2}\right)(t)\right) d\mu(t) \\
&\leqslant \int_{(G\backslash G_0)\cup G_2} \frac{1}{2}(\phi(f(t)) + \phi(h(t)+f(t)))\, d\mu(t) \\
&\quad + (1-\delta)\int_{G_1} \frac{1}{2}(\phi(f(t)) + \phi(h(t)+f(t)))\, d\mu(t) \\
&= \int_G \frac{1}{2}(\phi(f(t)) + \phi(h(t)+f(t)))\, d\mu(t) \\
&\quad - \delta\int_{G_1} \frac{1}{2}(\phi(f(t)) + \phi(h(t)+f(t)))\, d\mu(t).
\end{aligned}$$

As in the proof of Theorem 2.5, we obtain

$$\rho\left(f+\frac{h}{2}\right) \leqslant r - \delta\int_{G_1} \phi\left(\frac{h(t)}{2}\right) d\mu(t).$$

Let us observe that by convexity of ϕ and by definition of G_0, there holds

$$\begin{aligned}
\int_{G\backslash G_0} \phi\left(\frac{h(t)}{2}\right) d\mu(t) &\leqslant \int_{G\backslash G_0} \frac{1}{2}(\phi(f(t)) + \phi(h(t)+f(t)))\, d\mu(t) \\
&\leqslant \frac{1}{2}\phi(u_0)\,\mu(G\backslash G_0) \leqslant \frac{1}{2}\phi(u_0)\,\mu(G) \leqslant \frac{\varepsilon s}{6}.
\end{aligned}$$

On the other hand, by definition of G_2, we have

$$\int_{G_2} \phi\left(\frac{h(t)}{2}\right) d\mu(t) \leqslant \frac{\varepsilon}{6} \int_{G_2} \frac{1}{2} (\phi(f(t)) + \phi(h(t) + f(t)))\, d\mu(t) \leqslant \frac{\varepsilon}{6} r.$$

We have therefore,

$$\begin{aligned} \int_{G_1} \phi\left(\frac{h(t)}{2}\right) d\mu(t) &= \int_{G} \phi\left(\frac{h(t)}{2}\right) d\mu(t) \\ &\quad - \int_{G\setminus G_0} \phi\left(\frac{h(t)}{2}\right) d\mu(t) - \int_{G_2} \phi\left(\frac{h(t)}{2}\right) d\mu(t) \\ &\geqslant r\varepsilon - \frac{\varepsilon s}{6} - \frac{\varepsilon r}{3} = \varepsilon\left(\frac{2}{3} r - \frac{s}{6}\right). \end{aligned}$$

Hence,

$$\rho\left(f + \frac{h}{2}\right) \leqslant r - \delta \int_{G_1} \phi\left(\frac{h(t)}{2}\right) d\mu(t) \leqslant r - \frac{\delta\varepsilon}{3}\left(2r - \frac{s}{2}\right),$$

thus

$$\delta_\phi(r, h) \geqslant \frac{\delta\varepsilon}{3}\left(2 - \frac{s}{2r}\right) > \frac{\delta\varepsilon}{6} > 0$$

and consequently

$$\delta_\phi(r, \varepsilon) \geqslant \frac{\delta\varepsilon}{6} = \eta.$$

The proof of Lemma 2.7 is complete.

In [8] it was proved that the norm $\|\cdot\|_\phi$ is uniformly convex if and only if ϕ is uniformly convex and ϕ satisfies the condition Δ_2. We will now prove the analogous characterization of uniformly convex Orlicz modulars.

2.8. Theorem. *The following conditions are equivalent*:

(i) *ϕ is VC*;

(ii) *ρ is UC.*

Proof. (i) $\Rightarrow$ (ii) Follows immediately from Lemma 2.7.

(ii) $\Rightarrow$ (i) Suppose ϕ is not very convex. Then, there exist $\varepsilon_0 > 0$, $u_0 > 0$ and u_n, $v_n \geqslant 0$ such that

$$\phi\left(\frac{1}{2}(u_n - v_n)\right) \geqslant \frac{\varepsilon_0}{2} (\phi(u_n) + \phi(v_n)) \geqslant \varepsilon_0 \phi(u_0),$$

but

$$\phi\left(\frac{1}{2}(u_n+v_n)\right) > \frac{1}{2}\left(1-\frac{1}{n}\right)(\phi(u_n)+\phi(v_n)).$$

Choose $G_n \subset G$ such that $(\phi(u_n)+\phi(v_n))\,\mu(G_n)=\phi(u_0)\,\mu(G)$ and $E_n \subset G_n$ with $\mu(E_n)=\frac{1}{2}\mu(G_n)$, $F_n = G_n \backslash E_n$. Let us define functions f_n and h_n by $f_n = u_n\chi_{E_n} + v_n\chi_{F_n}$ and $h_n = (v_n - u_n)(\chi_{E_n} - \chi_{F_n})$. Thus,

$$\rho(f_n)=\tfrac{1}{2}(\phi(u_n)+\phi(v_n))\,\mu(G_n)$$

and

$$\rho(f_n+h_n)=\rho(v_n\chi_{E_n}+u_n\chi_{F_n})=\rho(f_n).$$

Hence,

$$\begin{aligned}\rho(\tfrac{1}{2}h_n) &= \phi(\tfrac{1}{2}(v_n-u_n))\,\mu(G_n)\\ &\geqslant \tfrac{1}{2}\varepsilon_0(\phi(u_n)+\phi(v_n))\,\mu(G_n)=\varepsilon_0\phi(u_0)\,\mu(G).\end{aligned}$$

Compute

$$\begin{aligned}\rho\left(f_n+\frac{1}{2}h_n\right) &= \phi\left(\frac{1}{2}(u_n+v_n)\right)\mu(G_n)\\ &> \frac{1}{2}\left(1-\frac{1}{n}\right)(\phi(u_n)+\phi(v_n))\,\mu(G_n)\\ &= \frac{1}{2}\left(1-\frac{1}{n}\right)\phi(u_0)\,\mu(G).\end{aligned}$$

Let $\varepsilon = 2\varepsilon_0$ and $r=\frac{1}{2}\phi(u_0)\,\mu(G)$ and note that

$$\delta_\phi(r,\varepsilon)=\inf\left\{\delta_\phi(r,h);\,\rho\left(\frac{1}{2}h\right)\geqslant r\varepsilon\right\}\leqslant\frac{1}{n}.$$

This implies that $\delta_\phi(r,\varepsilon)=0$ because n is arbitrary.

The following functions: $\phi_1(t)=e^{|t|}-|t|-1$, $\phi_2(t)=e^{t^2}-1$ may serve as examples of very convex (and hence strictly convex) N-functions that do not satisfy Δ_2-condition (cf. [15, 19]). Nevertheless, by Theorems 2.5 and 2.8, we can obtain some information about the geometrical properties of Orlicz modulars and about fixed points of ρ-nonexpansive mappings (Theorem 3.11, later in this paper).

Beside norm convergence, there are some other types of convergence in Orlicz spaces. Frequently, convergence in measure and convergence

μ—almost everywhere are used. Modular convergence of (f_n) to 0 denotes that there exists $\alpha > 0$ such that $\rho(\alpha f_n) \to 0$. Certainly, modular convergence is weaker than the norm convergence (they are equivalent if and only if ϕ satisfies Δ_2). We will use another convergence which is situated between norm and modular convergence.

2.9. DEFINITION. (a) We say that (f_n) is (ρ)-convergent to f and write $f_n \to f(\rho)$, if and only if $\rho(f_n - f) \to 0$.

(b) A sequence of functions (f_n), with $f_n \in L^\phi$, is called (ρ)-Cauchy if $\rho(f_m - f_n) \to 0$ as $n, m \to 0$.

(c) An Orlicz modular ρ is said to be complete if and only if any (ρ)-Cauchy sequence is (ρ)-convergent to an element of L^ϕ.

(d) A set $B \subset L^\phi$ is called (ρ)-closed if for any sequence of $f_n \in B$ the convergence $f_n \to f(\rho)$ implies that f belongs to B.

Let us note that (ρ)-convergence does not necessarily imply (ρ)-Cauchy condition.

2.10. THEOREM. *L^ϕ is (ρ)-complete.*

Proof. Let $\varepsilon > 0$ be given and let (f_n) be a (ρ)-Cauchy sequence of functions from L^ϕ. There then exists a subsequence (f_{n_k}) and a measurable function f such that $f_{n_k} \to f$ μ—a.e. By the Fatou Lemma, for m sufficiently large there holds

$$\rho(f - f_m) \leqslant \liminf_{k \to \infty} \rho(f_{m_k} - f_n) \leqslant \varepsilon.$$

Hence, $f_m \to f(\rho)$. Since $\rho(f - f_m) < \infty$, $f - f_m \in L^\phi$ because $f_m \in L^\phi$.

2.11. THEOREM. *Assume that ϕ is VC, C is a (ρ)-closed, convex subset of L^ϕ. Let $f \in L^\phi$ be such that the (ρ)-distance between f and C, defined by*

$$d_\rho(f, C) = \inf\{\rho(f - g); g \in C\},$$

is finite. There exists then a unique $g_0 \in C$ such that $\rho(f - g_0) = d_\rho(f, C)$.

Proof. Let us denote $d = d_\rho(f, C)$. We know that $d < \infty$. We can assume that $d > 0$ (otherwise, $f \in C$ because C is (ρ)-closed). By the definition of d, there exists a sequence (f_n) such that $f_n \in C$ and $\rho(f - f_n) \leqslant (1 + 1/n)\, d$. We claim that $(\frac{1}{2} f_n)$ is a (ρ)-Cauchy sequence. Indeed, suppose this is not the case. There then exists $\varepsilon_0 > 0$ and a subsequence (f_{n_k}) such that

$$\varepsilon_0 \leqslant \rho(\tfrac{1}{2}(f_{n_k} - f_{n_p}))$$

for $n_k \neq n_p$. By Theorem 2.8, ρ is UC and therefore,

$$\rho(f - \tfrac{1}{2}(f_{n_k} + f_{n_p})) \leqslant (1 - \delta_\phi(\varepsilon_0/d(k,p), d(k,p))\, d(k,p), \qquad (*)$$

where $d(k,p) = \max\{(1 + 1/n_k)\, d,\ (1 + 1/n_p)\, d\}$. For k and p sufficiently large, $d(k,p) \leqslant 2d$. Thus, by Remark 2.4 we obtain

$$\delta_\phi(\varepsilon_0/d(k,p), d(k,p)) \geqslant \delta_\phi(\varepsilon_0/2d, d(k,p)).$$

By Lemma 2.7 we can find $\eta > 0$ such that $\delta_\phi(\varepsilon_0/2d, r) > \eta$ for $r > d/3$. Since $d(k,p) \geqslant d \geqslant d/3$, it follows that $\delta_\phi(\varepsilon_0/2d,\ d(k,p)) > \eta$. In view of $(*)$ then

$$\rho(f - \tfrac{1}{2}(f_{n_k} + f_{n_p})) \leqslant (1 - \eta)\, d(k,p).$$

Since C is convex, we conclude that $\frac{1}{2}(f_{n_k} + f_{n_p}) \in C$, and therefore

$$d \leqslant \rho(f - \tfrac{1}{2}(f_{n_k} + f_{n_p})) \leqslant (1 - \eta)\, d(k,p).$$

Hence, for any natural k and p, $d \leqslant (1 - \eta)\, d(k,p)$ and, since $d(k,p) \to d$ as $k, p \to \infty$, we have $d \leqslant (1 - \eta)\, d$, which is impossible. Thus, $(\frac{1}{2} f_n)$ is (ρ)-Cauchy. Since L^ϕ is (ρ)-complete (Theorem 2.10), there exists a function $g \in L^\phi$ such that $\frac{1}{2} f_n \to g(\rho)$. Surprisingly, $2g$ belongs to C. Indeed,

$$\rho(\tfrac{1}{2}(f_n + f_m) - (\tfrac{1}{2} f_n + g)) \to 0 \qquad \text{as} \quad m \to \infty.$$

Therefore, $\frac{1}{2} f_n + g$ belongs to C by convexity and (ρ)-closedness of C. Letting n tend to infinity, we obtain that $2g \in C$. Observe that

$$\rho(f - \tfrac{1}{2}(f_n + f_m)) \leqslant \tfrac{1}{2}(\rho(f - f_m) + \rho(f - f_m)).$$

For fixed n let $m \to \infty$. By the Fatou Lemma

$$\rho(f - \tfrac{1}{2} f_n - g) \leqslant \tfrac{1}{2}(\rho(f - f_n) + d).$$

By Fatou's Lemma again (as $n \to \infty$) we obtain $\rho(f - 2g) \leqslant d$. Since $2g \in C$ then $\rho(f - 2g) \geqslant d$. Therefore, $\rho(f - 2g) = d$. Put $g_0 = 2g$. We have just found our best approximant. It remains to prove that g_0 is the unique best approximant. Let h_0 be another such element of C. Hence, $\rho(f - g_0) = d$ and $\rho(f - h_0) = d$. By convexity of ρ we obtain

$$\rho(f - \tfrac{1}{2}(g_0 + h_0)) \leqslant d.$$

Since C is convex, $\frac{1}{2}(g_0 + h_0) \in C$ and consequently

$$d \leqslant \rho(f - \tfrac{1}{2}(g_0 + h_0)),$$

which implies that $\rho(f - \frac{1}{2}(g_0 + h_0)) = d$. Since ρ is UC, it follows that ρ is SC and therefore $g_0 = h_0$.

It is well known that an Orlicz space $(L^\phi, \|\cdot\|_\phi)$ cannot be reflexive if ϕ fails to satisfy Δ_2. In Banach spaces reflexivity is equivalent to the property that every sequence of closed convex and bounded sets has nonempty intersection provided that each finite intersection is nonempty. Our next result states that even if ϕ does not satisfy the Δ_2-condition, Orlicz space L^ϕ has a property of that kind provided ϕ is VC.

2.12. THEOREM. *Let ϕ be VC and (C_n) be a decreasing sequence of (ρ)-closed, convex subsets of L^ϕ. Assume that there exists $f \in L^\phi$ such that*

$$\sup_n d_\rho(f, C_n) < \infty. \tag{$*$}$$

Then,

$$\bigcap_{n \in \mathbb{N}} C_n \neq \varnothing.$$

Proof. Since (C_n) is decreasing, the sequence $d_\rho(f, C_n)$ is increasing and by $(*)$ is bounded. Hence, there exists $d = \lim_{n \to \infty} d_\rho(f, C_n) < \infty$. By Theorem 2.11, for every $n \in \mathbb{N}$ there exists a unique $g_n \in C_n$ such that

$$\rho(f - g_n) = d_\rho(f, C_n).$$

If $d = 0$ then $\rho(f - g_n) \to 0$ and consequently $f \in C_n$ for each n in $\mathbb{N}$ because (C_n) is decreasing. Assume now that $d > 0$. Repeating the argument from the proof of Theorem 2.11, we can prove that (g_n) is a (ρ)-Cauchy sequence. Since L^ϕ is (ρ)-complete, then there exists $g \in L^\phi$ such that $g_n \to g(\rho)$. Since C_n are (ρ)-closed and decreasing, it follows then that $g \in C_n$ for any natural n.

3. NONEXPANSIVE MAPPINGS WITH RESPECT TO ORLICZ MODULARS

In this section we will study (ρ)-nonexpansive mappings in Orlicz spaces and relevant fixed point problems (Theorems 3.10 and 3.12). Let us start with the definition of (ρ)-nonexpansiveness.

3.1. DEFINITION. Let B be a subset of L^ϕ and let $T: B \to B$ be an arbitrary mapping. We say that T is (ρ)-nonexpansive if and only if

$$\rho(Tf - Tg) \leqslant \rho(f - g) \qquad \text{for any} \quad f, g \text{ in } B.$$

One of the reasons of our interest in (ρ)-nonexpansive mappings is that the ρ-nonexpansiveness can be easily verified while norm-nonexpansiveness is quite hard to verify because of the indirect definition of the norm. It can be also shown (cf. [10]) that we have to assume much more in order to assure norm-nonexpansiveness of a mapping. Namely, T is $\|\cdot\|_\phi$-nonexpansive if

$$\rho(\gamma(Tf-Tg)) \leqslant \rho(\gamma(f-g)) \qquad \text{for every} \quad \gamma > 0.$$

3.2.a. Definition. A function $\tau: L^\phi \to [0, \infty]$ is called a (ρ)-type if and only if there exists a sequence (f_n) of elements from L^ϕ such that

$$\tau(g) = \limsup_{n \to \infty} \rho(f_n - g) \qquad \text{for any} \quad g \in L^\phi.$$

3.2.b. Definition. A function $\lambda: L^\phi \to [0, \infty]$ is called (ρ)-lower semicontinuous if and only if for any $\alpha > 0$, the set $C_\alpha = \{f \in L^\phi; \lambda(f) \leqslant \alpha\}$ is (ρ)-closed.

It can be proved that (ρ)-lower semicontinuity is equivalent to the condition

$$\lambda(f) \leqslant \liminf_{n \to \infty} \lambda(f_n) \qquad \text{if} \quad f_n \to f(\rho); f, f_n \in L^\phi.$$

3.3. Proposition. *The following two conditions are equivalent*:

(i) *ϕ satisfies Δ_2*,

(ii) *Every (ρ)-type is (ρ)-lower semicontinuous.*

Proof. (i) $\Rightarrow$ (ii) It was proved by Kaminska [8] that if ϕ satisfies Δ_2, then the Orlicz modular is uniformly continuous, i.e., for any $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h+g)| \leqslant \varepsilon \qquad \text{if} \quad \rho(h) \leqslant \delta \text{ and } \rho(g) \leqslant L.$$

Let τ be an arbitrary (ρ)-type. Let us fix $\alpha > 0$; we want to prove that C_α is (ρ)-closed. Without any loss of generality we can assume that C_α is nonempty. Let (f_n) be a sequence of elements from C_α and let $f_n \to f(\rho)$. We have to prove that $\tau(f) \leqslant \alpha$. Since ϕ satisfies Δ_2, and τ is finite for f_n, then τ is finite everywhere. In particular, $\tau(f) < \infty$. Consequently, $L = \sup_{m \in \mathbb{N}} \rho(g_m - f) < \infty$, where (g_m) defines τ, i.e., $\tau(g) = \limsup_{m \to \infty} \rho(g - g_m)$ for any $g \in L^\phi$. Let us fix an arbitrary $\varepsilon > 0$. There exists then a $\delta > 0$ such that

$$|\rho(g) - \rho(h+g)| \leqslant \varepsilon \qquad \text{if} \quad \rho(h) \leqslant \delta \text{ and } \rho(g) \leqslant L.$$

Since $\rho(f - f_n) \to 0$, there exists $n' \in \mathbb{N}$ with $\rho(f - f_n) \leqslant \delta$ for $n \geqslant n'$. Put $h = f - f_{n'}$, and $g = g_m - f$. Therefore, $|\rho(g_m - f) - \rho(g_m - f_{n'})| \leqslant \varepsilon$ for

$n \in \mathbb{N}$. By the definition of τ, we obtain $|\tau(f) - \tau(f_{n'})| \leqslant \varepsilon$. This implies that $\tau(f) \leqslant \alpha + \varepsilon$ because $f_{n'} \in C_\alpha$. By arbitrariness of ε, $\tau(f)$ is less than or equal to α, which gives (ii).

(ii) $\Rightarrow$ (i) Suppose ϕ does not satisfy Δ_2. There then exists f in L^ϕ such that $\rho(f) < \infty$ while $\rho(\lambda f) = \infty$ for any $\lambda > 1$. Define

$$f_m(x) = \begin{cases} f(x), & |f(x)| \leqslant m \\ 0, & |f(x)| > m \end{cases}$$

and $g_m = f_m - f$. Let the (ρ)-type τ be defined by

$$\tau(g) = \limsup_{m \to \infty} \rho(g - g_m).$$

It is easy to check that $|f_n - g_m| \leqslant |f|$ for $m \geqslant n$ and therefore

$$\tau(f_n) = \limsup_{m \to \infty} \rho(f_n - g_m) \leqslant \rho(f) < \infty.$$

Denoting $\alpha = \rho(f)$, we have $f_n \in C_\alpha$ for any $n \in \mathbb{N}$. Observe that $f_n \to f$ pointwise, $|f_n - f| \leqslant |f|$ and then by Lebesgue's Theorem, $\rho(f_n - f) \to 0$. Let us compute

$$\tau(f) = \limsup_{m \to \infty} \rho(f - g_m) = \limsup_{m \to \infty} \rho(2f - f_m),$$

and observe that $\rho(2f - f_m) = \infty$ since $\rho(2f) = \infty$, $\rho(f) < \infty$ and $\mu(G) < \infty$. Finally, $\tau(f) = \infty$, that is, f cannot be in C_α and this would imply that C_α is not (ρ)-closed. Contradiction completes the proof.

The importance of the above result consists in the fact that, in order to obtain a fixed point theorem for (ρ)-nonexpansive mappings in L^ϕ (ρ being UC) we cannot simply mimic the constructive proofs from [2, 4, 5] because they are based on the lower semicontinuity of types. By Proposition 3.3 this would be possible only if ϕ satisfied the Δ_2-condition. We have, therefore, to introduce a notion of a (ρ)-normal structure (for the definition of normal structure in normed linear spaces see, e.g., [1]), and to prove a modular analog of Kirk's fixed point theorem [11].

3.4. Definition. Let $f \in L^\phi$ and let A be a nonempty subset of L^ϕ. Let us define the following quantities:

(a) $r_\rho(f, A) = \sup\{\rho(f - g); g \in A\}$,

(b) $R_\rho(A) = \inf\{r_\rho(h, A); h \in A\}$,

(c) $\delta_\rho(A) = \sup\{\rho(h - g);\ h \in A, g \in A\}$,

(d) $\mathscr{C}_\rho(A) = \{h \in A;\ r_\rho(h, A) = R_\rho A\}$.

We say that $f \in A$ is a (ρ)-diametral point if $r_\rho(f, A) = \delta_\rho(A)$. We say that A is a (ρ)-diametral set if and only if each f in A is a (ρ)-diametral point. A set A is said to be (ρ)-bounded if $\delta_\rho(A) < \infty$.

3.5. DEFINITION. We say that L^ϕ has (ρ)-normal structure if any (ρ)-bounded, (ρ)-closed, convex, not reduced to a single point, subset C of L^ϕ is not a (ρ)-diametral set, or equivalently, if $\mathscr{C}_\rho(C) \neq C$.

We are now going to prove an analog of Garkavi's characterization of UCED in normed linear spaces [3].

3.6. THEOREM. *The following assertions are equivalent*:

(a) *ρ is UCED,*

(b) *If C is a nonempty, (ρ)-bounded, (ρ)-closed, convex subset of L^ϕ then $\mathscr{C}_\rho(C)$ has at most one point.*

Proof. (a) $\Rightarrow$ (b) Let $C \subset L^\phi$ be nonempty, (ρ)-bounded, (ρ)-closed and convex. Assume to the contrary that there exist $f \neq g$ in $\mathscr{C}_\rho(C)$. For every $h \in C$ there holds $\rho(f-h) \leqslant R_\rho(C)$ and $\rho(g-h) \leqslant R_\rho(C)$. Since ρ is UCED there holds then

$$\rho(\tfrac{1}{2}(f+g)-h) \leqslant (1-\delta)\, R_\rho(C),$$

where $\delta = \delta_\rho(R_\rho(C),\ f-g)$. Let us note that $\delta > 0$ since $f \neq g$ and $R_\rho(C) > 0$ because $\rho(f-g) \leqslant R_\rho(C)$. Thus,

$$r_\rho(\tfrac{1}{2}(f+g), C) = \sup_{h \in C} \rho\,\tfrac{1}{2}(f+g)-h) \leqslant (1-\delta)\, R_\rho(C).$$

This implies that $R_\rho(C) \leqslant r_\rho(\tfrac{1}{2}(f+g), C) \leqslant (1-\delta)\, R_\rho(C)$, which contradicts the fact that $R_\rho(C) > 0$.

(b) $\Rightarrow$ (a) Assume to the contrary that ρ is not UCED. There then exists a function $f \in L^\phi \backslash \{0\}$ and $r > 0$ such that $\delta_\rho(r, f) = 0$. By the definition of δ_ρ we can find a sequence (g_n) in L^ϕ such that $\rho(g_n) \leqslant r$, $\rho(g_n + f) \leqslant r$ and

$$\rho(g_n + \tfrac{1}{2}f) \to r \qquad \text{as} \quad n \to \infty.$$

Let $h_n = g_n + \frac{1}{2}f$ and $C = \text{conv}\{h_n,\ -h_n,\ (\alpha/2)f\}$, where $\alpha < 1$ is such that $\rho(\alpha f) < r$. Let us prove that $R_\rho(C) = r$. Observe that, by convexity of ρ,

$$\rho(-h_n) = \rho(h_n) \leqslant r$$

and

$$\rho\left(\frac{\alpha}{2}f\right) \leqslant \frac{1}{2}\,\rho(\alpha f) < r.$$

Thus, $\rho(g) \leqslant r$ for any $g \in C$. Since 0 belongs to C, it follows that $r_\rho(0, C) \leqslant r$, and consequently $R_\rho(C) \leqslant r$. Take any $g \in C$. Since $h_n = \frac{1}{2}(h_n - g) + \frac{1}{2}(h_n + g)$, then

$$\rho(h_n) \leqslant \tfrac{1}{2}\rho(h_n - g) + \tfrac{1}{2}\rho(h_n + g) \leqslant \tfrac{1}{2}r_\rho(g, C) + \tfrac{1}{2}r_\rho(g, C) = t_\rho(g, C).$$

Since $\lim_{n \to \infty} \rho(h_n) = r$, it follows that $r_\rho(g, C) \geqslant r$. Hence, $R_\rho(C) \geqslant r$ and then $R_\rho(C) = r$. Since $r_\rho(0, C) \leqslant r$, then it actually equals zero, which means that 0 belongs to $\mathscr{C}_\rho(C)$. Let us note that $(\alpha/2) f \neq 0$ and belongs to $\mathscr{C}_\rho(C)$ as well. Indeed, we have

$$h_n - \frac{\alpha}{2} f = (1 - \beta)\, g_n + \beta(g_n + f) \qquad \text{with} \quad \beta = \frac{1 - \alpha}{2},$$

and

$$h_n + \frac{\alpha}{2} f = (1 - \beta')\, g_n + \beta'(g_n + f) \qquad \text{with} \quad \beta' = \frac{1 + \alpha}{2}.$$

Thus, $\rho(h_n - (\alpha/2) f) \leqslant r$ and $\rho(h_n + (\alpha/2) f) \leqslant r$. It follows from the definition of C that for any $g \in C$ there holds $\rho(g - (\alpha/2) f) \leqslant r$ and consequently $(\alpha/2) f \in \mathscr{C}_\rho(C)$. Finally, we have found two different elements in $\mathscr{C}_\rho(C)$. Contradiction completes the proof.

The next result is an immediate consequence of Theorem 3.6.

3.7. THEOREM. *If ρ is UCED then L^ϕ has (ρ)-normal structure.*

Before we state our fixed point theorem we give the following definition.

3.8. DEFINITION. We say that L^ϕ has property (R) if and only if every decreasing sequence (C_n) of (ρ)-bounded, (ρ)-closed, convex subsets of L^ϕ has nonempty intersection provided $C_n \neq \varnothing$ for any $n \in \mathbb{N}$.

3.9. *Remark.* In view of Theorem 2.12, it is clear that L^ϕ has property (R) if ϕ is VC.

3.10. THEOREM (Fixed Point Theorem). *Assume that L^ϕ has property (R) and (ρ)-normal structure. If $C \subset L^\phi$ is a nonempty, (ρ)-closed, (ρ)-bounded, convex set and T: $C \to C$ is (ρ)-nonexpansive, then the mapping T has a fixed point in C.*

Before we give the proof of Theorem 3.10, we have to prove the following lemma (a similar result for metric cases can be found in [12]). Recall that $K \subset L^\phi$ is T-invariant if $T(K) \subset K$.

3.11. LEMMA. *Let C and T be as in Theorem 3.10. There exists $K \subset C$, which is nonempty, (ρ)-closed, convex, T-invariant and such that*

$$\delta_\rho(K) \leqslant \tfrac{1}{2}\delta_\rho(C) + \tfrac{1}{2}R_\rho(C).$$

Proof of Lemma 3.11. If $R_\rho(C) = \delta_\rho(C)$ then we can take $K = C$. Assume therefore that $R_\rho(C) < \delta_\rho(C)$. Let $r = \frac{1}{2}\delta_\rho(C) + \frac{1}{2}R_\rho(C)$. Since $R_\rho(C) < r$, there exists then $f \in C$ such that $r_\rho(f, C) \leqslant r$. Define the family

$$\mathscr{F} = \{H \subset C;\ T(H) \subset H, f \in H,\ H \text{ is } (\rho)\text{-closed and convex}\}$$

and observe that $\mathscr{F} \neq \varnothing$ because $C \in \mathscr{F}$. Let $K = \bigcap_{H \in \mathscr{F}} H$ and note that $K \subset C$, $f \in K$, K is (ρ)-closed, convex and that $T(K) \subset K$. Let us define (ρ)-balls, i.e., sets of the form $B_\rho(g, \alpha) = \{h \in L^\phi;\ \rho(g-h) \leqslant \alpha\}$, and observe that by Fatou's lemma $B_\rho(g, \alpha)$ are (ρ)-closed. We introduce a family $\mathscr{J}$ of all (ρ)-balls that contain $T(K) \cup \{f\}$. Consider the set

$$\operatorname{conv}(T(K) \cup \{f\} = \bigcap_{B \in \mathscr{J}} B \cap K,$$

and observe that convex hull of $T(K) \cup \{f\}$ is (ρ)-closed, convex and contains f. Let us prove that it is also T-invariant. Denote $D = \operatorname{conv}(T(K) \cup \{f\})$ and observe that $D \subset K$. Then,

$$T(D) \subset T(K) \subset D.$$

Now define $K_r = \{h \in K;\ r_\rho(h, K) \leqslant r\}$. Clearly, f belongs to K_r. Let us note that

$$K_r = \bigcap_{g \in K} B_\rho(g, r) \cap K.$$

Hence, K_r is convex and (ρ)-closed. Let us prove that K_r is T-invariant. Take any $h \in K_r$, then $h \in K$ and, by definition of K_r, $K \subset B_\rho(h, r)$. Since T is (ρ)-nonexpansive, we obtain $T(K) \subset B_\rho(T(h), r)$. On the other hand, f belongs to $B_\rho(T(h), r)$ because $r_\rho(f, C) \leqslant r$ and $T(h) \in K$. Thus, $K = D \subset B_\rho(T(h), r)$, which implies $r_\rho((T(h), K) \leqslant r$, that is, $T(h)$ belongs to K_r. Consequently, $K_r \in \mathscr{F}$ and by definition of K, we obtain that $K \subset K_r$. Finally, $K = K_r$ and then $\delta_\rho(K) \leqslant r$. The proof of Lemma 3.11 is complete.

Proof of Theorem 3.10. Consider the family $\mathscr{D}$ of all nonempty, (ρ)-closed, convex and T-invariant subsets of C. Let us define $\delta_0 : \mathscr{D} \to \mathbb{R}^+$ by

$$\delta_0(D) = \inf\{\delta_\rho(F);\ F \in \mathscr{D},\ F \subset D\}.$$

Let $\{\varepsilon_n\}$ be an arbitrary sequence tending to zero. By induction, we can construct a decreasing sequence (D_n) such that $D_n \in \mathscr{D}$ and $\delta_\rho(D_{n+1}) \leqslant \delta_0(D_n) + \varepsilon_n$. By the property (R), $D = \bigcap_{n \in \mathbb{N}} D_n \neq \emptyset$. It is clear that $D \in \mathscr{D}$. Let us prove that D is reduced to a single point. Indeed, by Lemma 3.11 there exists $D^* \in \mathscr{D}$ with $D^* \subset D$ and such that

$$\delta_\rho(D^*) \leqslant \tfrac{1}{2}(\delta_\rho(D) + R_\rho(D)).$$

Since $D^* \subset D_n$ for every $n \in \mathbb{N}$, it follows that $\delta_0(D_n) \leqslant \delta_\rho(D^*)$. Hence,

$$\delta_\rho(D^*) \leqslant \delta_\rho(D) \leqslant \delta_\rho(D_{n+1}) \leqslant \delta_0(D_n) + \varepsilon_n \leqslant \delta_\rho(D^*) + \varepsilon_n,$$

which implies that

$$\delta_\rho(D^*) \leqslant \delta_\rho(D) \leqslant \delta_\rho(D^*) + \varepsilon_n$$

and passing with n to infinity, we have $\delta_\rho(D^*) = \delta_\rho(D)$, this implies that

$$\delta_\rho(D) \leqslant \tfrac{1}{2}(\delta_\rho(D) + R_\rho(D))$$

and, since there always holds $\delta_\rho(D) \geqslant R_\rho(D)$, we obtain that $\delta_\rho(D) = R_\rho(D)$. Since L^ϕ has (ρ)-normal structure, we deduce that D is reduced to a single point. Since D is T-invariant, it follows that this point is a fixed point for T. This completes the proof.

Let ϕ be VC. By Theorem 2.8 ρ is UC and consequently ρ is UCED, which, in view of Theorem 3.7, implies that L^ϕ has (ρ)-normal structure. On the other hand, the property (R) follows from Theorem 2.12. Hence, the following result is true.

3.12. THEOREM. *Let ϕ be very convex and let C and T be as in Theorem* 3.10. *Then T has a fixed point in C.*

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