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UNIFORM NONCOMPACT CONVEXITY, FIXED POINT PROPERTY IN MODULAR SPACES.

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ABSTRACT. In this work we define the so-called modulus of noncompact convexity in modular spaces. We extend the results obtained in Banach spaces by Goebel and Sekowski while their methods can not be reproduced as.

1. Introduction and Preliminaries.

As early as 1930, Orlicz and Birnbaum tried to generalize the Lebesgue function spaces L^p. Namely, they considered the function spaces

$$L^\Phi = \{f : R \to R; \exists\lambda > 0 \text{ such that } \int_R \Phi(\lambda|f(x)|)dx < \infty\},$$

where Φ behaves similarly to power function $\Phi(t) = t^p$. Later on, the convexity assumptions on Φ were ommitted. An interesting example is given by

$$\Phi(t) = e^t - 1.$$

Application to differential and Integral equations with Kernels of nonpower types were good reasons for the development of the theory of Orlicz spaces. Recently a new interest in classical Orlicz spaces is emerging in connection with problems of convexity, the Boyd indices and rearrangement invariant function spaces (see [10]). The succesful theory of Orlicz spaces inspired Nakano [13] to develop the theory of Modular spaces in connection with the theory of order spaces. This was redefined and generalized by Orlicz and Musielak. Let us give a brief account of some basic facts on modular spaces.

Definition 1. Let X be a vector space over $K(K = C$ or $R)$. A functional $\rho : X \to [0, \infty]$ is called a modular on X if for arbitrary elements f and g of X, it satisfies the following:

(1) $\rho(f) = 0$ if and only if $f = 0$,

(2) $\rho(\alpha f) = \rho(f)$ for every $\alpha \in K$ with $|\alpha| = 1$,

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If we replace (3) by

(3)' $\rho(\alpha f + \beta g) \leq \alpha\beta(f) + \beta\rho(g)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$,

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then the modular ρ is called convex. For a modular ρ on X one can associate a modular space X_ρ defined as

$$X_\rho = \{f \in X; \lim_{\lambda\to 0} \rho(\lambda f) = 0\}.$$

X_ρ is a linear subspace of X. Using the modular ρ, one can define a F-norm [11] on X_ρ by

$$\|f\|_\rho = \inf\{t > 0; \rho(\frac{f}{t}) \leq t\}.$$

If the modular ρ is convex, then

$$\|f\|'_\rho = \inf\{t > 0; \rho(\frac{f}{t}) \leq 1\}$$

is a norm on X_ρ, frequently called the Luxemburg norm [11]. One can also check that $\|f_n - f\|_\rho \to 0$ is equivalent to $\rho(\alpha(f_n - f)) \to 0$ for all $\alpha > 0$.

Examples.

(Example 1.) The Musielak-Orlicz modular spaces (see. e.g. [11]). Let

$$\rho(f) = \int_\Omega \varphi(\omega, f(\omega))d\mu(\omega),$$

where μ, a σ-finite measure on Ω, and $\varphi : \Omega \times R \to [0, \infty)$ satisfy the following:

(i)$\varphi(\omega, u)$ is a continuous even function of u which is nondecreasing for $u > 0$, such that $\varphi(\omega, 0) = 0, \varphi(\omega, u) > 0$ for $u \neq 0$, and $\varphi(\omega, u) \to \infty$ as $u \to \infty$.

(ii)$\varphi(\omega, u)$ is a measurable function of ω for each $u \in R$.

The corresponding modular space is called a Musielak-Orlicz (or a generalized Orlicz) modular function space, and is denoted by L^φ. If φ does not depend on the first variable, then L^φ is called an Orlicz space. An exemple of functions that satisfy the above conditions, is given by

$$\varphi(u) = |u|^\varphi, \text{for } p > 0.$$

Then L^φ is isomorphic to L^p.

(Example 2.) (See e.g. [4,9]) Let

$$\rho(f) = \sup_{\mu\in\Lambda} \int_\Omega \varphi(\omega, f(\omega))d\mu(\omega),$$

where φ is as in Example 1 and Λ is a set of positive measures such that $\sup_{\mu\in\Lambda} \mu(\Omega) < \infty$.

Then ρ is a function modular.

(Example 3.) Lorentz type L^p-spaces, see [4,9]. Let

$$\rho(f) = \sup_{\tau\in\mathcal{T}} \int_\Omega |f(\omega)|^p d\mu_\tau(\omega),$$

where μ is a σ-finite measure on $\Omega, \mathcal{T}$ is the group of all measure preserving transformations $\tau : \Omega \to \Omega$ and

$$\mu_\tau(E) = \mu(\tau^{-1}(E)).$$

Then ρ is a function modular.

Definition 2. Let X_ρ be a modular space.

(a) We say that a sequence (f_n) is ρ-convergent to f and write $f_n \to f(\rho)$, if and only if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(b) ρ is said to satisfy the Δ_2-condition if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to \infty$.

(c) (Fatou property) We say that ρ has the Fatou property if $\rho(f) \leq \liminf_n \rho(f_n)$ whenever (f_n) ρ-converges to f.

(d) A sequence (f_n) is called ρ-Cauchy whenever $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

(e) The modular ρ is called complete if any ρ-Cauchy sequence is ρ-convergent.

(f) A subset $B \subset X_\rho$ is called ρ-closed if for any sequence $(f_n) \subset B$ ρ-convergent to $f \in X_\rho$, we have $f \in B$.

(g) A ρ-closed subset $B \subset X_\rho$ is called ρ-compact if any sequence $(f_n) \subset B$ has a ρ-convergent subsequence.

(h) A subset $B \subset X_\rho$ is said to be ρ-bounded if

$$\delta_\rho(B) = \sup\{\rho(f - g); f, g \in B\} < \infty.$$

(i)Define the ρ-distance between $f \in X_\rho$ and $B \subset X_\rho$ as

$$d_\rho(f, B) = \inf\{\rho(f - g); g \in B\}.$$

Remarks.

(1) Note that ρ-convergence does not necessarily imply ρ-Cauchy since ρ does not a priori satisfy the triangle inequality. One can easily show that this will happen if and only if ρ satisfies the Δ_2-condition.

(2) Since the intersection of ρ-closed sets is still ρ-closed, one can associate to any subset A of X_ρ a ρ-closed subset, denoted $\bar{A}$, which is minimal in the following sence:

$$if\ A \subset B \text{ and } B \text{ is } \rho\textit{-closed},\ then\ \bar{A} \subset B.$$

We will call $\bar{A}$ the ρ-closure of A.

(3) One can give an easy characterization of Fatou property in terms of ρ-balls. Indeed, one can check that ρ-balls are ρ-closed if and only if ρ has the Fatou property. Recall the definition of the ρ-ball $B_\rho(f, r)$ centered at $f \in X_\rho$ with radius r as

$$B_\rho(f, r) = \{g \in X_\rho; \rho(f - g) \leq r\}.$$

In Banach spaces, when we think of reflexivity automatically the dual space is present in our taught. But in modular spaces, it is very hard to conceive the dual space. To circumvent the problem, we use some characterizations of reflexivity.

Definition 3. Let X_ρ be a modular space.

(a) We will say that X_ρ or ρ satisfy the property (R) if and only if every decreasing sequence of nonempty ρ-closed and ρ-bounded convex subset of X_ρ, has a nonempty intersection.

(b) We will say that X_ρ or ρ satisfy the property (R') if and only if for every ρ-bounded sequence $(f_n) \subset X_\rho$, there exists a subsequence $(f_{n'})$ of (f_n) such that the intersection of $(cl(conv\{f_{i'}; i \geq n\}))$ is nonempty and reduced to one point.

Remarks.

(a) By $cl(conv\{A\})$, we mean the ρ-closure of the smallest convex subset containing A.

(b) Clearly the property (R') implies the property (R) and are equivalent in Banach spaces.

2. Uniform noncompact convexity in Modular spaces.

Goebel and Sekowski [5] used the concept of measure of noncompactness to give a new classification of Banach spaces. In this work, we discuss their result under the setting of modular spaces.

Definition 4. Let X_ρ be a modular space. Define the Hausdorff measure of noncompactness by

$\chi(A) = \inf\{\epsilon > 0; A$ *can be convered with a finite number of* ρ*-balls of radius smaller than* $\epsilon\}$,

and the Kuratowski measure of noncompactness by

$\alpha(A) = \inf\{\epsilon > 0; A$*can be covered with a finite number of sets of diameter smaller than* $\epsilon\}$,

for any subset A of X_ρ. We will make the obvious convention that the inf over empty set is infinite.

One can easily notice that $\chi(A) \leq \alpha(A)$ for every $A \subset X_\rho$.

Throughout this work, we will assume that X_ρ is ρ-complete and ρ satisfies the Fatou property.

Proposition 5. *The following properties hold,*

(1) if $A \subset B$, then $\chi(A) \leq \chi(B)$ and $\alpha(A) \leq \alpha(B)$,

(2) $\chi(\bar{A}) = \chi(A)$ and $\alpha(\bar{A}) = \alpha(A)$,

(3) if $\alpha(A) = 0$, then $\bar{A}$ is ρ-compact,

(4) let (A_n) be a decreasing sequence of nonempty ρ-closed subset of X_ρ. Assume that $\lim_n \alpha(A_n) = 0$ (resp. $\lim_n \chi(A_n) = 0$) then $\cap A_n$ is a nonempty ρ-compact set (resp. $\cap A_n$ is nonempty and $\chi(\cap A_n) = 0$).

Proof. The proof of (1) is obvious.

In order to prove (2), first notice that $\chi(A) \leq \chi(\bar{A})$. Let $\epsilon > 0$, then there exists $B_\rho(x_1, r), \cdot\cdot, B_\rho(x_n, r)$ such that

$$A \subset \bigcup_{1 \leq i \leq n} B_\rho(x_i, r)$$

where $r \leq \chi(A) + \epsilon$. Since ρ satisfies the Fatou property, the ρ-balls are ρ-closed and therefore

$$\bar{A} \subset B_\rho(x_1, r) \cup \cdot\cdot \cup B_\rho(x_n, r),$$

which implies $\chi(\bar{A}) \leq r \leq \chi(A) + \epsilon$. Because ϵ is arbitarary, we have $\chi(\bar{A}) \leq \chi(A)$.

The second claim will easily hold if

$$(*) \;\; diam_\rho(A) = diam_\rho(\bar{A}).$$

In order to prove $(*)$, set $diam_\rho(A) = d$. Then for any $x \in A$ we have $A \subset B_\rho(x, d)$. Since the ρ-balls are ρ-closed, we get $\bar{A} \subset B_\rho(x, d)$. So, for any $y \in \bar{A}$ we have $\rho(x - y) \leq d$. One can clearly deduce that $A \subset B_\rho(y, d)$ for all $y \in \bar{A}$. Again using the ρ-closeness of the ρ-balls, we get $\bar{A} \subset B_\rho(y, d)$. Therefore $diam_\rho(\bar{A}) \leq d = diam_\rho(A)$. And since the other inequality is obvious, we deduce $(*)$.

Let us prove (3). Assume that $\alpha(A) = 0$. Let (f_n)be any sequence of elements in $\bar{A}$. Let $\epsilon > 0$, then by using the definition of α one can find a subsequence $(f_{n'})$of (f_n)such that $\rho(f_{n'} - f_{m'}) \leq \epsilon$ for all $n', m' \in N$. An easy argument will show that (f_n)has a ρ-Cauchy subsequence which is ρ-convergent since X_ρ is ρ-complete.

In order to prove (4), let (A_n) be a decreasing sequence of ρ-closed nonempty subsets of X_ρ such that $\lim_n \alpha(A_n) = 0$. Choose $f_n \in A_n$ and let $\epsilon > 0$. Then, there exists $n_0 \geq 1$ such that $\alpha(A_n) < \epsilon$ for $n \geq n_0$. Therefore, there exist $D_1, \cdot\cdot, D_k$ such that

$$A_{n0} \subset \bigcup_{1 \leq i \leq k} D_i,$$

with $diam_\rho(D_i) \leq \epsilon$, for all $1 \leq i \leq k$. So $f_n \in \bigcup_{1\leq i\leq k} D_i$ for all $n \geq n_0$. It is clear, therefore, that one D_i contains infinitely many f_n. So there exists a subsequence $(f_{n'})$of (f_n)such that $\rho(f_{n'} - f_{m'}) \leq \epsilon$ for all n', m'. A classical argument will imply that (f_n)has a ρ-Cauchy subsequence which is therefore ρ-convergent to $f \in X_\rho$. Using the ρ-closeness of A_n, we deduce that $f \in A_n$ for all $n \geq 1$. Then $\cap A_n \neq \emptyset$ and obviously we have $\alpha(\cap A_n) = 0$. Therefore, $\cap A_n$ is ρ-compact using (3).

In order to complete the proof of (4), let (A_n) be a decreasing sequence of ρ-closed nonempty subsets of X_ρ such that $\lim_n \chi(A_n) = 0$. Let $\epsilon > 0$, then there exists $n_0 \in N$ such that $\chi(A_{n0}) < \epsilon$. Let $fnn \in A_n$ for all n, then there exist infinitely many elements from (f_n)which belong to a certain ρ-ball $B_\rho(f, \epsilon)$. This implies that there exists a subsequence $(f_{n'})$such that

$$\rho(\frac{1}{2}(f_{n'} - f_{m'})) \leq \rho(f_{n'} - f) + \rho(f - f_{m'}) \leq 2\epsilon$$

for every n', m'. So, using the same idea as before, there exists a subsequence $(f_{n'})$of (f_n)such that $(\frac{1}{2}f_{n'})$ is ρ-Cauchy. Let $h \in X_\rho$ be its limit. Fix $n'_0 \in N$, then $(\frac{1}{2}(f_{n'_0} + f_{m'}))_{m'\geq n'_0}$ is ρ-convergent to $\frac{1}{2}f_{n'_0} + h$. Threrfore, because $A_{n'_0}$ is ρ-closed we deduce that $\frac{1}{2}f_{n'_0} + h \in A_{n'_0}$. Since kna is decreasing, we get $\frac{1}{2}f_{n'} + h \in A_m$ for all $n' \geq m$ for a fixed m. Again since $(\frac{1}{2}f_{n'} + h)_{n'\geq m}$ is ρ-convergent to $h + h = 2h$, we obtain that $2h \in A_m$. This clearly implies that $\cap A_n$ is nonempty and using (3) we get $\chi(\cap A_n) = 0$. The proof of Proposition 5 is therefore complete.

Remark. Since in general ρ is not subadditive, there is no reason to have $\alpha(A) = 0$ whenever A is ρ-compact. Let us add that it can be shown that ρ satisfies the Δ_2-condition if and only if $\alpha(A) = 0$ whenever A is ρ-compact.

For more on α and χ, one can consult [1].

As Goebel and Sekowski did in Banach spaces, we give a new scaling for modular spaces using the measures of noncompactness α and χ.

Definition 6. The ρ-modulus of noncompact convexity Δ_χ (resp. Δ_α) is defined as

(**) $\Delta_\chi(r, \epsilon) = \sup\{c > 0;$ for any ρ-bounded convex $A \subset X_\rho$ ρ-bounded and $f \in X_\rho$ such that $A \subset B_\rho(f, r)$ with $\chi(A) \geq r\epsilon$, then $dist_\rho(f, A) \leq r(1 - c)\}$,
for every $r > 0$ and $\epsilon > 0$.

For the characteristic of noncompact convexity by

$$\epsilon_\chi(r, X_\rho) \text{ (resp. } \epsilon_\alpha(r, X_\rho)) = \sup\{\epsilon > 0; \Delta_\chi(r, \epsilon) \text{ (resp. } \Delta_\alpha(r, \epsilon)) = 0\}$$

for every $r > 0$.

Since we have $\chi \leq \alpha$, one can get $\Delta_\chi \leq \Delta_\alpha$ and $\epsilon_\alpha \leq \epsilon_\chi$. In any Banach space X, one can easily prove that $\epsilon_\chi(X) = 0$ if and only if $\epsilon_\alpha(X) = 0$. In modular spaces, it is not the case in general.

Definition 7. The modular space X_ρ is said to be α (resp. χ)-uniformly ρ-noncompact convex if and only if $\epsilon_\alpha(r, X_\rho) = 0$ (resp. $\epsilon_\chi(r, X_\rho) = 0$), for every $r > 0$.

Clearly if X_ρ is χ-uniformly ρ-noncompact convex, X_ρ is α-uniformly ρ-noncompact convex.

Example. Recall that the authors in [2] (see also [6,11]) introduced a concept scaling the modular spaces called ρ-uniform convexity. Let us recall their definition. We will say that X_ρ is ρ-uniformly convex if for every $r > 0$ and $\epsilon > 0$, the ρ-modulus of uniform fonvexity, defined as

$$\delta_\rho(r, \epsilon) = \inf\{1 - \frac{1}{r}\rho(f + \frac{1}{2}h)\},$$

(where the infimum is taken over all $f, h \in X_\rho$ such that $\rho(f) \leq r, \rho(f+h) \leq r$ and $\rho(\frac{h}{2}) \geq r\epsilon$) is strictly positive.
One can also find, in [2], examples of modular spaces which are ρ-uniformly convex. Let us show that ρ-uniform convexity implies α-uniform ρ-noncompact convexity. Indeed, let $A \subset X_\rho$ be nonempty ρ-bounded and convex such that $A \subset B_\rho(f,r)$ with $\alpha(A) \geq r\epsilon$ for some $f \in X_\rho, r > 0$ and $\epsilon > 0$. Let $\zeta < 1$ then, by definition of α, one can find $h_1, h_2 \in A$ such that $\rho(h_1 - h_2) \geq r\zeta\epsilon$. Therefore, we have

$$\rho(\frac{(f-h_1)+(f-h_2)}{2}) \leq r(1-\delta_\rho(r,\zeta\epsilon)).$$

Since $\frac{h_1+h_2}{2} \in A$, we deduce that

$$dist_\rho(f,A) \leq r(1-\delta_\rho(r,\zeta\epsilon)).$$

Which clearly implies that

$$\delta\rho(r,\zeta\epsilon)) \leq \Delta_\alpha(r,\epsilon),$$

for every $r > 0, \epsilon > 0$ and $\zeta < 1$.
This obviously completes the proof of our claim.

3. Main results

Goebel and Sekowski [5] proved that whenever the characteristic of uniform noncompact convexity of any Banach space is less than 1, then the space is reflexive and has the normal structure property. In what follows, we investigate the validity of these results in modular spaces. Let us point out that their proofs are entirely based on the rich structure of the Banach spaces, specially the existence of the dual space.
The first result discuss the link between proximinality and the ρ-moduls of noncompact convexity. More exactly, given a function $f \in X_\rho$, we consider the minimization problem of finding $h \in C$ such that

$$\rho(f-h) = \inf\{\rho(f-g); g \in C\},$$

for a given $C \subset X_\rho$. Such a h is called a best approximant. Problems of finding best approximants are important in approximation theory [12] and probability theory [3].
Theorem 1. *Let X_ρ be a ρ-complete modular space. Assume that ρ is convex, satisfy the Fatou property and X_ρ is α-uniformly ρ-noncompact convex.* Then for any nonempty C ρ-bounded ρ-closed convex subset of X_ρ and $f \in X_\rho$ such that $dist_\rho(f,C) < \infty$, the set

$$P_\rho(f,C) = \{g \in C; dist_\rho(f,C) = \rho(f-g)\}$$

is a nonempty ρ-compact convex subset.

Proof. We assume without any loss of generality that $d = dist_\rho(f,C) > 0$. Consider the sets $C_n = C \cap B_\rho(f, d+\frac{1}{n})$ for $n \geq 1$. Then clearly C_n is a decreasing sequence of ρ-closed nonempty convex subsets of C. Assume that $\inf \alpha(C_n) = \lim_n \alpha(C_n) > 0$. Then since X_ρ is α-uniformly ρ-noncompact convex there exists $\Delta > 0$ such that

$$dist_\rho(f,C_n) \leq (1-\Delta)(d+\frac{1}{n})$$

for every $n \geq 1$. Since $dist_\rho(f,C) \leq dist_\rho(f,C_n)$, we get

$$d \leq (1-\Delta)(d+\frac{1}{n})$$

for every $n \geq 1$. This contradicts the fact that $d > 0$. So $\lim_n \alpha(C_n) = 0$, and by Proposition 5, we deduce that $\cap C_n$ is a nonempty ρ-compact convex subset of C. Obviously we have

$$P_\rho(f, C) = \cap_{n \geq 1} C_n.$$

The proof of Theorem 1 is therefore complete.

Remark. We are unable to prove whether the conclusion of Theorem 1 is true if we juste assume that $\epsilon_\alpha(X_\rho) < 1$.

Theorem 2. *Let X_ρ be a ρ-complete modular space. Assume that ρ is convex, satisfy the Fatou property and X_ρ is α-uniformly ρ-noncompact convex.*
Then X_ρ has the Property (R).

Proof. Let (C_n) be a decreasing sequence of ρ-bounded ρ-closed nonempty convex subsets of X_ρ. Fix $f \in C_0$, then

$$r = \sup_n dist_\rho(f, C_n) \leq diam_\rho(C_0) < \infty.$$

Define $K_n = C_n \cap B_\rho(f, r)$. Clearly (K_n) is a decreasing sequence of nonempty ρ-closed convex subsets of X_ρ. Using the same argument as for the proof of Theorem 1, one can show that $\lim_n \alpha(K_n) = 0$. So, $\cap K_n$ is a nonempty ρ-compact convex of *ekp*. Therefore $\cap C_n$ is nonempty, which completes the proof of Theorem 2.

Remark. One can ask if $\bigcap_{\beta \in \Gamma} A_\beta$ is nonempty under the assumptions of Theorem 2, for any decreasing family $(A_\beta)_{\beta \in \Gamma}$ of ρ-bounded, ρ-closed nonempty convex subsets of X_ρ and any directed set Γ. The answer to this problem is in the affirmative. Indeed, let $(A_\beta)_{\beta \in \Gamma}$ be as described before. We can assume without any loss of generality that Γ has a minimum, say β_0. Let $f \in A_{\beta_0}$, then

$$r = \sup\{dist_\rho(f, A_\beta); \beta \in \Gamma\} \leq diam_\rho(A_{\beta_0}) < \infty.$$

Set $K_\beta = A_\beta \cap B_\rho(f, r)$ for any $\beta \in \Gamma$. Then $(K_\beta)_{\beta \in \Gamma}$ is a decreasing family of nonempty ρ-closed convex subsets of X_ρ. Uning the same argument as before, we get $\inf\{\alpha(K_\beta); \beta \in \Gamma\} = 0$. Therefore for every $n \geq 1$, there exists $\beta_n \in \Gamma$ such that

$$\alpha(K_{\beta_n}) \leq \frac{1}{n}.$$

Clearly one can choose $(\beta_n) \subset \Gamma$ such that (K_{β_n}) is decreasing. Using Proposition 5, we obtain that $K_\infty = \cap K_{\beta_n}$ is a nonempty ρ-compact subset.
We can assume that there exists $\beta' \in \Gamma$ such that $\beta_n \leq \beta'$ for every $n \geq 1$. Otherwise, we would have

$$\bigcap_{n \geq 1} K_{\beta_n} = \bigcap_{\beta \in \Gamma} K_\beta.$$

Since (K_β) is decreasing, we deduce that $K_\beta \subset K_\infty$ for every $\beta \geq \beta'$. Put $\delta = diam_\rho(K_{\beta'})$. Because $\alpha(K_{\beta'}) = 0$, one can find $(A_i)_{1 \leq i \leq k}$ such that

$$K_{\beta'} \subset \bigcup_{1 \leq i \leq k} A_i \text{ with } diam_\rho(A_i) \leq \frac{\delta}{2}.$$

Let $f_\beta \in K_\beta$ for every $\beta \geq \beta'$. It is not hard to show that there exists $i_0 \in \{1, 2, .., k\}$ such that for every $\beta \geq \beta'$ there exists $\gamma \geq \beta$ such that $f_\gamma \in A_{i_0}$. Indeed, assume to the contrary that for every $i \in \{1,..,k\}$ there exists $\beta_i \geq \beta'$ such that f_γ does not belong to A_i for all $\gamma \geq \beta_i$. Since Γ is a directed set, there exists $\beta \in \Gamma$ such that $\beta \geq \beta_i$ for every $i \in \{1,..,k\}$. So f_β does not belong to any A_i. Contradicting the fact that $f_\beta \in K_\beta \subset K'_\beta \subset A_1 \cup \cdot\cdot \cup A_k$. Therefore, there exists such $i_0 \in \{1,..,k\}$. Set

$$C_\beta = \{f_\gamma; \gamma \geq \beta \text{ and } f_\gamma \in C_{i_0}\}.$$

Then for every $\beta \geq \beta', C_\beta$ is nonempty and moreover we have for every $\zeta \geq \beta$, there exists $\gamma \geq \zeta$ such that $f_\gamma \in A_\beta$. Since $C_\beta \subset A_i$, we get $diam_\rho(C_\beta) \leq \frac{\delta}{2}$. Let $L_\beta = cl(conv\{C_\beta\})$ for all $\beta \geq \beta'$. Then $diam_\rho(L_\beta) \leq \frac{\delta}{2}$ and $(L_\beta)_{\beta \geq \beta'}$ is a decreasing family of nonempty ρ-closed convex sets with $L_\beta \subset K_\beta$ for every $\beta \geq \beta'$. Using $\alpha(K_\beta) = 0$ we deduce that $\alpha(L_\beta) = 0$ for every $\beta \geq \beta'$.
One can repeat the same construction to get another decreasing family (L^2_β) of ρ-closed convex subsets with $L^2_\beta \subset L_\beta$ for all $\beta \geq \beta'$ and $diam_\rho(L^2_\beta) \leq \frac{\delta}{4}$, since $\alpha(L_{\beta'}) = 0$.
So by induction, we obtain a decreasing family $(L^n_\beta)_{\beta \geq \beta'}$ of ρ-closed convex nonempty subsets with

$$L^n_\beta \subset L^{n-1}_\beta \text{ and } diam_\rho(L^n_\beta) \leq \frac{\delta}{2^n},$$

for all $\beta \geq \beta'$ and all $n \geq 2$. Since X_ρ is ρ-complete, we get

$$\bigcap_{n\geq 1} L^n_\beta \neq \emptyset \text{ for all } \beta \geq \beta'.$$

Clearly $\cap L^n_\beta$ is reduced to one point, say h_β for all $\beta \geq \beta'$. Since $L^n_\beta \subset L^n_\gamma$ for any $\beta \geq \gamma \geq \beta'$ and any $n \geq 1$, we obtain $h_\beta = h_\gamma$ for every $\beta \geq \gamma \geq \beta'$. Therefore, $h_{\beta'} \in L^n_\beta \subset K_\beta$ for all $n \geq 1$ and all $\beta \geq \beta'$, which clearly implies

$$\emptyset \neq \bigcap_{\beta \geq \beta'} K_\beta \subset \bigcap_{\beta \in \Gamma} A_\beta.$$

This completes the proof of our claim.

The next result discuss the ρ-normal structure in uniformly ρ-noncompact convex modular spaces. Let us first recall some basic definitions.

Definition 8. We will say that X_ρ has the ρ-normal structure if and only if for any nonempty C ρ-closed ρ-bounded convex subset of X_ρ, not reduced to one point, there exists $f \in C$ such that

$$\sup\{\rho(f-g); g \in C\} < diam_\rho(C).$$

Theorem 3. *Let X_ρ be a ρ-complete modular space. Assume that ρ is convex, satifies the Fatou property and $\epsilon_\alpha(X_\rho) < 1$.*
Then X_ρ has the ρ-normal structure proveded that X_ρ has the property (R').

Proof. Assume to the contrary that X_ρ fails to satisfy the ρ-normal structure. Then there exists a nonempty K ρ-closed, ρ-bounded convex subset, not reduced to one point, such that

$$\sup\{\rho(f-g); g \in K\} diam_\rho(K) = d,$$

for all $f \in K$. Fix $f_1 \in K$ and let $f_2 \in K$ such that

$$\rho(f_2 - f_1) \geq d(1 - \frac{1}{2^3}).$$

Assume that $(f_1, .., f_n)$ has been constructed, then let $f_{n+1} \in K$ such that

$$\rho(f_{n+1} - \frac{1}{n}\sum_{i=1}^{n} f_i) \geq d(1 - \frac{1}{(n+1)^3}).$$

So there exists a sequence $(f_n) \subset K$ such that

$$d(1 - \frac{1}{(n+1)^3}) \leq \rho(f_{n+1} - \frac{1}{n}\sum_{i=1}^{n} fi) \leq d,$$

for every $n \geq 1$. Using the convexity of ρ, we can get

$$(***) \quad d(1 - \frac{1}{(n+1)^2}) \leq \rho(f - f_{n+1}) \leq d,$$

for every $f = \sum\limits_{1\leq i\leq n} \alpha_i f_i$ with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Indeed put $\alpha = \max(\alpha_i)$, then we have

$$\frac{1}{n}\sum_{1\leq i\leq n} f_i = \frac{1}{n\alpha} f + \sum_{1\leq i\leq n} (\frac{1}{n} - \frac{\alpha_i}{n\alpha}) f_i.$$

Since $\frac{1}{n\alpha} + \sum(\frac{1}{n} - \frac{\alpha_i}{n\alpha}) = 1$, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} f_i - f_{n+1} = \frac{1}{n\alpha}(f - f_{n+1}) + \sum_{1\leq i\leq n} (\frac{1}{n} - \frac{\alpha_i}{n\alpha})(f_i - f_{n+1}).$$

Therefore,

$$d(1 - \frac{1}{(n+1)^3}) \leq \frac{1}{n\alpha}\rho(f - f_{n+1}) + \sum_{1\leq i\leq n} (\frac{1}{n} - \frac{\alpha_i}{n\alpha})\rho(f_i - f_{n+1}),$$

and then

$$d(1 - \frac{1}{(n+1)^3}) \leq \frac{1}{n\alpha}\rho(f - f_{n+1}) + (1 - \frac{1}{n\alpha})d,$$

which implies

$$d(1 - \frac{n\alpha}{(n+1)^3}) \leq \rho(f - f_{n+1}).$$

Since $n\alpha \leq n + 1$, we obtain the inequality $(***)$.
In partisular, we have

$$(\zeta) \qquad \rho(f_n - f_m) \geq d(1 - \frac{1}{m^2}),$$

for every $m > n \geq 1$. Since X_ρ satisfies the property (R'), there exists a subsequence $(f_{n'})$ of (f_n) such that

$$\bigcap_{n\geq 1} cl(conv\{f_{i'}; i \geq n\}) = \{h\}.$$

Denote $C_n = cl(conv\{f_{i'}; i \geq n\})$. Using (ζ), it is not hard to show that $\alpha(C_n) \geq d$ for every $n \geq 1$. Let $f \in K$, then $C_n \subset B_\rho(f, d)$ since $diam_\rho(K) = d$. Because $\epsilon_\alpha(X_\rho) < 1$, one can find $\Delta > 0$ such that

$$dist_\rho(f, C_n) \leq (1 - \Delta)d.$$

Theorem 1 implies that

$$C_n \cap B_\rho(f, (1-\Delta)d) = K_n,$$

is nonempty for all $n \geq 1$. Since (K_n) is a decreasing sequence of ρ-closed ρ-bounded nonempty convex subsets we obtain, using Theorem 2,

$$\bigcap_n K_n = \bigcap_n C_n \cap B_\rho(f, (1-\Delta)d)$$

is nonempty. This clearly implies that

$$\rho(f-h) \leq (1-\Delta)d.$$

Since this is true for any $f \in K$, we get

$$\sup\{\rho(f-h); f \in K\} \leq (1-\Delta)d.$$

Contradicting our assumption on K, which completes the proof of Theorem 3.

The last result of this work gives an analoguous of Kirk's fixed point theorem [7].

Theorem 4. *Let X_ρ be a ρ-complete modular space. Assume that ρ is convex and satisfies the Fatou property. Moreover, we will assume that X_ρ has the ρ-normal structure and has the property (R). Let C be any ρ-closed ρ-bounded convex nonempty subset of X_ρ.* Then any $T: C \to C$ ρ-nonexpansive (i.e. $\rho(Tf - Tg) \leq \rho(f-g)$ for all $f, g \in C$) has a fixed point, i.e. there exists $f \in C$ such that $Tf = f$.

Proof. Let $T: C \to C$ be a ρ-nonexpansive map. For any ρ-closed convex nonempty D subset of C, define

$$R_\rho(D) = \inf\{r_\rho(f, D); f \in D\},$$

where $r_\rho(f, D) = \sup\{\rho(f-g); g \in D\}$.

The following lemma will be very helpful for the proof of Theorem 4. Its proof can be found in [6,8].

Lemma. Under the above assumptions, for any D ρ-closed convex nonempty subset of C, which is T-invariant (i.e. $TD \subset D$), there exists D^* ρ-closed nonempty convex subset of D which is T-invariant and

$$diam_\rho(D^*) \leq \frac{1}{2}(diam_\rho(D) + R_\rho(D)).$$

In order to compete the proof of Theorem 4, let

$$\mathcal{F} = \{D \subset C; D \text{ is } \rho\text{-closed convex nonempty with } TD \subset D\}.$$

It is clear that $C \in \mathcal{F}$. Define $\delta_0 : \mathcal{F} \to [0, \infty)$ by

$$\delta_0(D) = \inf\{diam_\rho(K); K \in \mathcal{F} \text{ and } K \subset D\}.$$

Put $D_1 = C$ and let $D_2 \in \mathcal{F}$ with $D_2 \subset D_1$ such that

$$diam_\rho(D_2) \leq \delta_0(D_1) + \frac{1}{2}.$$

Suppose that $D_1, .., D_n$ have been constructed. Let $D_{n+1} \in \mathcal{F}$ with $D_{n+1} \subset D_n$ such that

$$(\beta) \qquad diam_\rho(D_{n+1}) \leq \delta_0(D_n) + \frac{1}{n+1}.$$

Therefore, one can construct a sequence $(D_n) \subset \mathcal{F}$ such that (β) is satisfied for every $n \geq 1$. Since X_ρ satisfies the property (R), we deduce that

$$D_\infty = \bigcap_{n \geq 1} D_n$$

is nonempty and belongs to $\mathcal{F}$. Let D^*_∞ be the subset of D_∞ given by the Lemma. Then we have

$$diam_\rho(D^*_\infty) \leq \frac{1}{2}(diam_\rho(D_\infty) + R_\rho(D_\infty)).$$

But

$$diam_\rho(D^*_\infty) \leq diam_\rho(D_\infty) \leq diam_\rho(D_{n+1}) \leq \delta_0(D_n) + \frac{1}{n+1}.$$

Therefore using the definition of δ_0 we get

$$diam_\rho(D^*_\infty) \leq diam_\rho(D_\infty) \leq diam_\rho(D^*_\infty) + \frac{1}{n+1},$$

for every $n \geq 1$. So $diam_\rho(D_\infty) = diam_\rho(D^*_\infty)$ holds. This clearly implies that

$$R_\rho(D_\infty) = diam_\rho(D_\infty).$$

Using the ρ-normal structure assumption, we deduce that D_∞ is reduced to one point which is a fixed point for T. The proof of Theorem 4 is therefore complete.

Remark. If we assume that X_ρ is α-uniformly ρ-noncompact convex in Theorem 4, the proof will be easier. Indeed, let $T : C \to C$ be a ρ-nonexpansive map. Define

$$\mathcal{F} = \{D \subset C; D \text{ is } \rho\text{-closed convex nonempty with } TD \subset D\}.$$

Using the remark following Theorem 2, we deduce that $\mathcal{F}$ satifies the Zorn's lemma assumptions. Therefore $\mathcal{F}$ has minimal elements. Let D be a minimal element and let us show that D is reduced to one point which is therefore a fixed point for T. Assume to the contrary that $diam_\rho(D) > 0$. Set

$$D_0 = cl(conv(TD)).$$

Then D_0 is a nonempty ρ-closed convex subset of D which is T-invariant since $TD_0 \subset TD \subset D_0$. The minimality of D implies $D_0 = D$. On the other hand, let $f \in D$ and put

$$r(f) = \sup\{\rho(f - g); g \in D\}.$$

Then $D \subset B_\rho(f, r)$ holds. Since T is ρ-nonexpansive, we deduce that $TD \subset B_\rho(Tf, r)$. Therefore

$$D = cl(conv(TD)) \subset B_\rho(Tf, r)$$

holds, which means that $r(Tf) \leq (f)$ for every $f \in D$. Put

$$D^* = \{f \in D; r(f) \leq r(f_0)\}$$

for a fixed $f_0 \in D$. It is not hard to show that D^* is a nonempty ρ-closed convex subset of D which is T-invariant. The minimality of D will imply that $D = D^*$. This clearly implies that the function r is constant on D. Therefore, we obtain

$$R_\rho(D) = r(f) = diam_\rho(D)$$

for every $f \in D$. This obviously contradicts the ρ-normal structure satisfied by X_ρ, which completes the proof of our claim.

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