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COMPACTNESS OF CONVEXITY STRUCTURES IN METRIC SPACES

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ABSTRACT. In this work, we discuss the compactness of convexity structures in metric spaces. We also discuss a problem posed by Kirk on an extension of Caristi's classical theorem.

1. Introduction. The classical fixed point theorems [1, 6] involve an interplay between the geometric sturcture of the underlying metric spaces and a compactness assumption on the domain of the mapping. The compactness assumption is used to insure the existence of minimal elements. The existence of such elements is crucial to many fixed point theorems. That is why recently Buber and Kirk [3] investigated the existence of minimal element which requires a weakened form of compactness. In this work, we will prove that under the assumptions of [3], we have in fact compactness and therefore Zorn's Lemma applies. We will also discuss a problem posed by Kirk [5] on a generalization of Caristi's fixed classical point theorem.

2. Basic Definitions and Results. We begin by describing Penot's framework [10]. Let (M,d) be a metric space. We shall use $B(a,r)$ to denote the closed ball centered at $a \in M$ with radius $r \leq 0$.

Definition 1. Let $\mathcal{F}$ be a nonempty family of subsets of M. $\mathcal{F}$ is said to be a convexity structure on M if $\mathcal{F}$ is stable by intersection and contains the closed balls.

Examples:

(1) Let X be a Banach space and C a closed bounded convex subset of X. Let $\mathcal{F}$ be the set of closed convex subsets of C. Then $\mathcal{F}$ defines a convexity structure on C.
(2) Let M be the unit ball of ℓ_∞ and

$$\mathcal{F} = \left\{ \bigcap_{i\in I} B(a_i, r_i); a_i \in M \ \text{ and } \ r_i \geq 0 \right\}.$$

Then $\mathcal{F}$ defines a convexity structure on M.

The smallest convexity structure $\mathcal{A}(M)$ on M that contains the closed balls is the family of admissible subsets of M. Recall that $A \subset M$ is an admissible set if $A = \bigcap_{i\in I} B(a_i, r_i)$.

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Let B be a bounded subset of M. Set

$$r(x,B) = \sup\{d(x,y);\ \ y \in B\} \quad \text{for } x \in M$$
$$diam(B) = \sup\{r(x,B);\ \ x \in B\}$$
$$R(B) = \inf\{r(x,B);\ \ x \in B\}$$

Definition 2. Let $\mathcal{F}$ be a convexity structure on M.

(i) We will say that $\mathcal{F}$ is normal if for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) < diam(A)$.
(ii) We will say that $\mathcal{F}$ is uniformly normal if there exists $c \in (0,1)$ such that for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) \leq c\ diam(A)$.

A key assumption in the proof of classical fixed theorems in metric spaces, is a compactness argument. The following definition originated in [10].

Definition 3.

(i) $\mathcal{F}$ is said to be χ-compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$, with $card(\Gamma) \leq \chi$, has a nonempty intersection provided $\bigcap\limits_{\alpha\in F} C_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.
(ii) $\mathcal{F}$ is said to be countably compact (resp. compact) if $\mathcal{F}$ is χ_0-compact (rest. χ-compact for any cardinal χ).

Note that since $card\{F \subset \Gamma; F \text{ is finite }\} = card(\Gamma)$, one can assume also that the family $(C_\alpha)_{\alpha\in\Gamma}$ is decreasing and Γ is downward directed.

Recall that a mapping $T : M \to M$ is said to be nonexpansive if $d(T(x),T(y)) \leq d(x,y)$ for all x, y in M. We will say that M has the fixed point property if any nonexpansive self-map T defined on M, has a fixed point (i.e. there exists $x \in M$ such that $T(x) = x$).

3. Main Results.

Main theorem. *Let M be a bounded metric space and D be a dense subset in M. Assume that $card(D) \leq \chi$ and $\mathcal{A}(M)$ is χ-compact. Then $\mathcal{A}(M)$ is in fact compact.*

Let $D = \{x_\alpha\}_{\alpha\in\Gamma}$ be such that $card(\Gamma) \leq \chi$, and let $(A_i)_{i\in I}$ be a decreasing family of nonempty elements of $\mathcal{A}(M)$, where I is downward directed. Let us prove that $\bigcap\limits_{i\in I} A_i \neq \emptyset$.

Lemma. *For every $\epsilon > 0$, there exists $I_\epsilon \subset I$ such that:*

$$d(x,A_i) \leq d(x,A_\omega) + \epsilon, \quad \textit{for every } i \geq k,$$

for all $k \in I_\epsilon$ and $x \in M$, where $A_\omega = \bigcap\limits_{i\in I_\epsilon} A_i$.

Proof of Lemma. Note $\lambda(x) = \sup\{d(x,A_i);\ i \in I\}$. For any $\alpha \in \Gamma$, there exists $i(\alpha) \in I$ such that:

$$\lambda(x_\alpha) - \epsilon \leq d(x, A_{i(\alpha)}) \leq \lambda(x_\alpha).$$

Since $card\{i(\alpha);\ \alpha \in \Gamma\} \leq \chi$ and $\mathcal{F}$ is χ-compact, then $A_\omega = \bigcap\limits_{\alpha\in\Gamma} A_{i(\alpha)} = \emptyset$.
Let $i \geq k$ for $k \in I_\epsilon = \{i(\alpha);\ \alpha \in \Gamma\}$. Then $A_i \subset A_{i(\alpha)}$ for all $\alpha \in \Gamma$. Hence $A_i \subset A_\omega$. Therefore,

$$d(x_\alpha, A_i) \leq \lambda(x_\alpha) \leq d(x_\alpha, A_{i(\alpha)}) + \epsilon \leq d(x_\alpha, A_\omega) + \epsilon \leq d(x_\alpha, A_i) + \epsilon.$$

Since $\{x_\alpha, \alpha \in \Gamma\}$ is dense in M and the function $x \to d(x, A_i)$ is uniformly continuous, we get

$$d(x, A_i) \leq d(x, A_\omega) + \epsilon \quad \text{for every } x \in M. \quad \spadesuit$$

Proof of Main theorem. Let $\epsilon_1 = 1$ and consider $A_{\omega_1} = \bigcap_{i \in I_1} A_i$ given by the lemma. Let also $\overline{I}_1 = \{i \in I; i \geq k \quad \text{for all } k \in I_1\}$.

Case 1. $\overline{I}_1 = \emptyset$. In this case, we have $\bigcap_{i \in I} A_i = \bigcap_{i \in I_1} A_i = A_{\omega_1}$.

Case 2. $\overline{I}_1 \neq \emptyset$.

Subcase 2.1: If $card(\overline{I}_1) \leq \chi$, then $\bigcap_{i \in \overline{I}_1} A_i \neq \emptyset$. Therefore, we have

$$\bigcap_{i \in I} A_i = \bigcap_{i \in \overline{I}_1} A_i \neq \emptyset.$$

Subcase 2.2: If $card(\overline{I}_1) > \chi$. Let $\epsilon_2 = \frac{1}{2}$, by the lemma, there exists $I_2 \subset \overline{I}_1$ such that:

$$d(x, A_i) \leq d(x, A_{\omega_2}) + \frac{1}{2}, \quad \text{for every } i \geq k,$$

for all $k \in I_2$ and $x \in M$, where $A_{\omega_2} = \bigcap_{i \in I_2} A_i \neq \emptyset$.

As above note $\overline{I}_2 = \{i \in \overline{I}_1; i \geq k \text{ for all } k \in I_2\}$.

Case 1. $\overline{I}_2 = \emptyset$. Then $\bigcap_{i \in \overline{I}_1} A_i = \bigcap_{i \in I_2} A_i \neq \emptyset$.

Case 2. $\overline{I}_2 \neq \emptyset$. Then:

Subcase 2.1 : $card(\overline{I}_2) \leq \chi$, in this case, we have

$$\bigcap_{i \in I} A_i = \bigcap_{i \in \overline{I}_2} A_i \neq \emptyset,$$

Subcase 2.2 : otherwise if $card(\overline{I}_2) > \chi$, let $\epsilon_3 = \frac{1}{3}$ and repeat the process.

Assume that $I_1, \overline{I}_1, I_2, \overline{I}_2, \ldots, I_n, \overline{I}_n$ are constructed. Then:

Case 1. $card(\overline{I}_n) \leq \chi$, in this case $\bigcap_{i \in I} A_i = \bigcap_{i \in \overline{I}_n} A_i \neq \emptyset$.

Case 2. $card(\overline{I}_n) > \chi$. Let $\epsilon_{n+1} = \frac{1}{n+1}$ and consider $A_{\omega_{n+1}} = \bigcap_{i \in I_{n+1}} A_i$ given by the lemma with $I_{n+1} \subset \overline{I}_n$. Consider $\overline{I}_{n+1} = \{i \in \overline{I}_n;\ i \geq k \text{ for all } k \in I_{n+1}\}$. If $\overline{I}_{n+1} = \emptyset$ then $\bigcap_{i \in I} A_i = \bigcap_{i \in I_{n+1}} A_i$, otherwise $\overline{I}_{n+1} \neq \emptyset$ and start over.

If the process stops then $\bigcap_{i \in I} A_i \neq \emptyset$. If the process does not stop, then we construct a sequence $I_1, \overline{I}_1, I_2, \overline{I}_2, \ldots, I_n, \overline{I}_n, \ldots$ such that

$$(*) \qquad dist(x, A_i) \leq dist(x, A_{\omega_n}) + \frac{1}{n},$$

for any $k \in I_n$, $i \geq k$ and $n \geq 1$. Note that $A_{\omega_{n+1}} \subset A_{\omega_n}$ since for any $i \in I_{n+1}$, we have $A_i \subset \bigcap_{k \in I_n} A_k = A_{\omega_n}$. Let $\bigcap_n A_{\omega_n} = A_\omega$. Since $\mathcal{F}$ is χ-compact and $\chi \geq \chi_0$, then A_ω is not empty. We claim that

$$A_\omega \subset \bigcap_{i \in I} A_i.$$

Indeed, let $i \in I$. If there exists $n \geq 1$ and $k \in I_n$ such that $i \leq k$, then $A_\omega \subset A_{\omega_n} \subset A_k \subset A_i$. Otherwise, assume that for any $n \geq 1$ and any $k \in I_n$, we have $k \leq i$. Using (*), we get

$$dist(x, A_i) \leq dist\,(x, A_{\omega_n}) + \frac{1}{n},$$

for any $x \in M$ and $n \geq 1$. In particular, we have

$$dist(x, A_i) \leq dist\,(x, A_\omega) + \frac{1}{n},$$

which implies $dist(x, A_i) \leq dist\,(x, A_\omega)$ for every $x \in M$. Therefore, if $x \in A_\omega$, then $dist(x, A_i) \leq 0$, which implies that $x \in A_i$. Hence, we have

$$A_\omega \subset A_i.$$

Our claim is therefore proved which completes the proof. ♠

As a corollary, we get Buber and Kirk's result [3].

Corollary 4. *Let M be a separable metric space for which $\mathcal{A}(M)$ is countably compact. Then $\mathcal{A}(M)$ has minimal elements.*

Recall the classical Kirk's fixed point theorem [8].

Theorem [8]. *Let M be a bounded metric space. Assume $\mathcal{A}(M)$ is compact and normal. Then M has the fixed point property.*

The original proof is based on the existence of minimal elements in $\mathcal{A}(M)$. Kirk [9] extended this result to metric spaces for which $\mathcal{A}(M)$ is countably compact and normal. The proof is constructive and does not use Zorn's lemma. Recall that in [7], it is proved that if $\mathcal{A}(M)$ is uniformly normal, then it is countably compact. It is still unknown whether these assumptions imply compactness.

The next result deals with a question that was asked by Kirk [5] on Caristi's classical fixed point theorem [4]. Recall that this theorem states that any map $T : M \to M$ defined on a complete metric space has a fixed point provided that there exists a lower semi-continuous map ϕ mapping M into the nonnegative real numbers such that

$$d(x, T(x)) \leq \phi(x) - \phi(T(x)), \quad \text{for every } x \in M.$$

Before we discuss Kirk's problem, let us start by giving some basic definitions regarding this problem. Let $\phi : M \to [0, \infty)$ be a map. Define the order $\prec_\phi$ [2] on M by

$$x \prec_\phi y \quad \text{iff} \quad d(x, y) \leq \phi(y) - \phi(x),$$

for any x, y in M. It is straightforward that $(M, \prec_\phi)$ is indeed an ordered set. However, it is not clear what are the minimal assumptions on M and ϕ which oblige $(M, \prec_\phi)$ to have minimal elements. As a matter of fact, if a is a minimal element in $(M, \prec_\phi)$ and $T : M \to M$ is any map which satisfies

$$(C) \qquad d(x, T(x)) \leq \phi(x) - \phi(T(x)), \quad \text{for all } x \in M,$$

then $T(a) = a$, i.e. a is a fixed point of T. Caristi [4] noticed that if M is complete and ϕ is lower semi-continuous, then $(M, \prec_\phi)$ satisfies the assumptions of Zorn's Lemma and therefore has minimal elements. In attempting to improve Caristi's result, Kirk has raised the question of whether a map $T: M \to M$ which satisfies

$$(K) \qquad d(x,Tx)^p \leq \phi(x) - \phi(Tx), \quad \text{for all } x \in M,$$

for some $p > 1$, has a fixed point. We answer this question by the negative. Indeed, let $M = \{x_n; n \geq 1\} \subset [0,\infty)$ be defined by

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

for all $n \geq 1$. Then M is a closed subset of $[0,\infty)$ and therefore is complete. Define $T: M \to M$ by $Tx_n = x_{n+1}$ for all $n \geq 1$. Then,

$$d(x,Tx)^p = \frac{1}{(n+1)^p} = \phi(x) - \phi(Tx),$$

where $\phi(x_n) = \sum_{n+1 \leq i} \frac{1}{i^p}$. for all $n \geq 1$. It is easy to see that ϕ is lower semicontinuous. Furthermore one can also show that T is nonexpansive and fails to have a fixed point. It is commonly known that any contraction map $T: M \to M$ (i.e. $d(Tx,Ty) \leq kd(x,y)$ with $k \in (0,1)$, and more generally any map T which satisfies for any $x \in M$

$$d(T^2(x),T(x)) \leq kd(x,Tx) \text{ with } k \in (0,1)\ )$$

satisfine (C) with $\phi(x) = \frac{1}{1-k}d(x,Tx)$. Clearly one can deduce that, in this case, $\phi(Tx) \leq k\phi(x)$ for all $x \in M$. More generally, let T be a selfmap defined on a complete metric space M which satisfies (K) and such that $\phi(Tx) \leq k\phi(x)$ holds with $k \in (0,1)$. Then the Picard iterates $(T^n(x))$ (for any $x \in M$) converges to a fixed point. Indeed, one can easily show that T satisfies (C) where the new function ϕ' is defined by $\phi'(x) = K\phi^{\frac{1}{p}}(x)$ for every $x \in M$, where K is a constant. So for any $x \in M$ $(T^n(x))$ is a Cauchy sequence, which converges to $a \in M$. Clearly $\phi(a) \leq \liminf \phi(T^n(x))$. And since $\phi(T^n(x)) \leq k^n\phi(x)$, we obtain $\phi(a) = 0$. This obviously implies that $d(a,Ta)^p = 0$, i.e. $Ta = a$.

In all the previous results, the main question was about the existence of minimal elements in ordered sets. Although in many instances this can be insured by applying Zorn's Lemma, It is well known that there exists ordered sets with minimal elements but fails to satisfy the assumptions of Zorn's Lemma. Therefore a natural question is to characterize ordered set in which the existence of a minimal element is insured. The following theorem answers this question.

Theorem 5. *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.*

(1) *A contains a minimal element,*
(2) *Any set valued map T defined on A which satisfies*

$$x \in A \text{ and } y \in T(x) \Longrightarrow y \prec x,$$

has a fixed point, i.e. there exists a in A such that $a \in Ta$.

Proof. (1) $\Rightarrow$ (2) Obviously any minimal element is fixed by T. We complete the proof of Theorem 1 by showing that (2) $\Rightarrow$ (1). Assume that A fails to have a minimal element. Define the set valued map T on A by

$$T(x) = \{y \in A;\ y \prec x \text{ with } y \neq x\},$$

for any $x \in A$. Clearly our assumption on A implies that $T(x)$ is not empty for any $x \in A$. (2) will imply that T has a fixed point $a \in A$. Contradiction with the definition of T. So, the proof of Theorem 5 is complete. ♠

Remark. Recall that in [11], it is proved that Zorn's Lemma is equivalent to:

> *Let $\mathcal{F}$ be a family of selfmappings defined on a partially ordered set such that $x \leq f(x)$ (resp. $f(x) \leq x$), for all $x \in A$ and all $f \in \mathcal{F}$. If each chain in A has an upper bound (resp. lower bound), then the family $\mathcal{F}$ has a common fixed point.*
>
> As it was noted to us by the referee, Taskovic's proof is not correct since it uses the axiom of choice (see the review of paper [11] in Mathematical Reviews) but the conclusion is still true.

The authors would like to thank the referee for pointing out that the proof of the main theorem can be easily derived using a Lindelof argument. The technical proof given in this work can be of some interest to people working in modular spaces (see [12, 13, 14, 15, 16]). Since this proof can be adapted in this setting but not the Lindelof argument.

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