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NONEXPANSIVE MAPPINGS AND SEMIGROUPS IN HYPERCONVEX SPACES

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Abstract. We present several new results on the existence of fixed points of nonexpansive mappings and semigroups in hyperconvex spaces.

A metric space (X, d) is said to have the binary intersection property if the intersection of every collection of mutually intersecting closed balls in X is nonempty. It is said to be metrically convex if for each x and y in X and $0 < t < 1$, there is a point z in X such that $d(x, z) = td(x, y)$ and $d(y, z) = (1-t)\, d(x, y)$. A metrically convex metric space which has the binary intersection property is said to be hyperconvex [1].

A nonexpansive self-mapping of (X, d) is a mapping $T\colon X \to X$ for which $d(Tx, Ty) \le d(x, y)$ for all x and y in X. A (continuous) nonlinear semigroup (of nonexpansive mappings) on (X, d) is a family of mappings $S(t)\colon X \to X, 0 \le t < \infty$, such that $S(t_1+t_2) = S(t_1)S(t_2), S(0)$ is the identity, $d(S(t)x, S(t)y) \le d(x, y)$ for $t \ge 0$ and $x, y \in X$, and $S(t)x$ is continuous in $t \ge 0$ for each x in X.

We say that a nonexpansive mapping (semigroup) has bounded orbits (or is bounded) if for each point x in X there is a constant $M(x)$ such that $d(x, T^n x) \le M(x)(d(x, S(t)x) \le M(x))$ for all $n \ge 1 (t \ge 0)$. A sequence $\{Y_n\}$ of subsets of a metric space X is said to have a bounded selection if there is a bounded sequence $\{y_n\}$ such that $y_n \in Y_n$ for all n.

Our main purpose in this note is to present several new results on the existence of fixed points of nonexpansive mappings and semigroups in hyperconvex spaces. These results are of interest in view of analogous results in Banach spaces (see, for example, [10]) and in view of recent activity in the fixed point theory of hyperconvex spaces and its applications (see, for example, [2], [4], [5], [8], [12] and [13, 14]). We also include a new proof of a related result in Banach spaces.

We begin with a proposition which will be needed later.

Proposition 1. *If a nonlinear semigroup on a hyperconvex space has a periodic point, then it also has a fixed point.*

Proof. Let S be a nonlinear semigroup on a hyperconvex space H and assume that $S(T)x = x$ for some positive T and $x \in H$. Consider the orbit $A = \{S(t)x : 0 \le t < \infty\}$.

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We observe that $A = \{S(t)x : 0 \leq t \leq T\}$ and that $S(r)(A) = A$ for all $r \geq 0$. Denote the ball $\{y \in H:\ d(a,\ y) \leq R\}$ by $B(a,\ R)$ and let δ be the diameter of A. The intersection

$$C = \cap\{B(a,\ \delta) : a \in A\}$$

is nonempty and hyperconvex. Moreover, if $y \in C$ and $r \geq 0$, then $A \subset B(y,\ \delta)$ and therefore $A = S(r)(A) \subset B(S(r)y, \delta)$. Hence $S(r)y$ also belongs to C and C is invariant under each $S(r)y$, $r \geq 0$. By [2, Proposition 9] C contains a common fixed point of the mappings $\{S(t) : t \geq 0\}$. This completes the proof.

We continue with two lemmata.

Lemma 2. *A metric space M is hyperconvex if and only if it is isometric to a nonexpansive retract of $\ell_\infty(I)$ for some index set I.*

Proof. Since $\ell_\infty(I)$ is hyperconvex, any nonexpansive retract of $\ell_\infty(I)$ is also hyperconvex. Conversely, fix a point x_0 in a metric space M, let $I = M$ and define $Q : M \to \ell_\infty(I)$ by

$$Q_x = \{d(x,\ y) - d(x_0,\ y) : y \in M\}, x \in M.$$

It is not difficult to see that Q is an isometry. If M is hyperconvex, so is $Q(M)$. Therefore there is a nonexpansive retraction of $\ell_\infty(I)$ onto $Q(M)$.

Lemma 3. *Let C be a closed convex subset of a Banach space E and let $T : C \to C$ be nonexpansive. Then for each x in C the initial value problem*

$$\begin{cases} u'(t) + u(t) - Tu(t) = 0, & 0 \leq t < \infty \\ u(0) = x \end{cases}$$

has a unique solution $u \in C^1([0,\ \infty); E)$ with $u(t) \in C$ for all $t \geq 0$.

This lemma is known [9].

Theorem 4. *The following are equivalent.*

(A) *Any nonexpansive mapping with bounded orbits on a hyperconvex space has a fixed point;*

(B) *Any nonexpansive nonlinear semigroup with bounded orbits on a hyperconvex space has a fixed point;*

(C) *Any decreasing sequence of hyperconvex subspaces of a metric space with a bounded selection has a nonempty intersection.*

Proof. We first show that (A) is equivalent to (B).

Assume that (A) holds and let S be a bounded nonlinear semigroup on a hyperconvex space H. Then S has a periodic point and (B) follows by Proposition 1. Conversely, assume that (B) holds and let T be a nonexpansive mapping with bounded orbits on H. Using the notation of Lemma 2, we define a nonexpansive self-mapping V of $\ell_\infty(H)$ by $V = QTQ^{-1}R$. Since $V^n = QT^nQ^{-1}R$, it is clear that V also has bounded orbits. Moreover, T has a fixed point if and only if V has one. Therefore we may assume without any loss of generality that $H = \ell_\infty(I)$ for some index set I. Fix a point x_0 in $\ell_\infty(I)$ and let $|x_0 - T^n x_0| \le M(x_0)$ for all $n \ge 1$. It is not difficult to see that

$$C = \{y \in \ell_\infty(I) : \limsup_{n\to\infty} |T^n x_0 - y| \le M(x_0)\}$$

is a nonempty bounded closed convex subset of $\ell_\infty(I)$ which is invariant under T. By Lemma 3, it is also invariant under the nonlinear semigroup S generated by the accretive operator $I - T$ on $\ell_\infty(I)$. Therefore S is bounded. Its fixed points are also fixed points of T

Now we wish to show that (A) and (C) are equivalent. to this end, assume (A), and let $\{H_n\}$ be a decreasing sequence of hyperconvex subspaces of a metric space with a bounded selection $\{y_n\}$. Consider the sequence space

$$H = \{(x_1, x_2, \cdots) \in \prod_{n=1}^{\infty} H_n : \sup\{d(x_n, y_n) : n = 1, 2, \cdots\} < \infty\}$$

equipped with the metric $d(x,\ z) = \sup\{d(x_n, z_n) : n = 1, 2, \cdots\}$. Define a nonexpansive self-mapping T of H by $T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. It is clear that the orbit of $y = (y_n)$ is bounded. Since H is hyperconvex, T has a fixed point $(y, y, \cdots)$ with $y \in H_n$ for all $n = 1, 2, \cdots$ Conversely, assume that (C) holds and let T be a nonexpansive mapping with bounded orbits on a hyperconvex space H. Again, without any loss of generality we may assume that $H = \ell_\infty(I)$ for some index set I. There is a nonempty closed convex bounded subset C of $\ell_\infty(I)$ which is invariant under T. Therefore the intersections of the approximate fixed point set $H_n = \{x \in \ell_\infty(I) : |x - Tx| \le 1/n\}$ with C are nonempty for each n. Since these approximate fixed point sets are all hyperconvex [13], the intersection of the decreasing sequence $\{H_n\}$ is nonempty and H has a fixed point.

Remark. The problem raised on p. 520 of [12] remains open.

Since it is known [7] that there are weak* compact convex subsets of ℓ_∞ which lack the fixed point property for nonexpansive mappings, the following consequence of Proposition 1 may also be of interest.

Theorem 5. *Let T be a nonexpansive self-mapping of $\ell_\infty(I)$ and let S be a nonlinear semigroup on $\ell_\infty(I)$. If there exists a weak* compact convex subset C of $\ell_\infty(I)$ which is invariant under T and S, then both T and S have fixed points.*

Proof. By Proposition 1 we may restrict our attention to T. Let K be a minimal weak* closed convex subset of C which is invariant under T, and let δ be its diameter. Consider the intersection $A = \cap\{B(x, \delta) : x \in K\}$. It is nonempty (because $K \subset B(x, \delta)$ for all x in K), bounded and hyperconvex. Moreover, it is invariant under T (because the weak* closed convex hull of $T(K)$ coincides with K and, therefore, $K \subset B(T(a), \delta)$ for all a in A). The result follows by [6, Theorem 2].

We conclude this note with a new proof of a related result which is true in any Banach space. This result (a special case of [3, Theorem 1]) is of interest, because it is still an open question whether the fixed point property for nonexpansive mappings implies the common fixed point property for such mappings [11, p. 180].

Proposition 6. *Let C be a weakly compact convex subset of a Banach space E. If each closed convex subset of C has the fixed point property for nonexpansive mappings then C has the fixed point property for nonlinear nonexpansive semigroups.*

Proof. Let S be a nonlinear semigroup on C. By Zorn's Lemma there is a minimal nonempty closed convex subset K of C which is invariant under $S(t)$ for all $0 \leq t \leq 1$. If K is not reduced to a single point, let x_0 be a fixed point of $S(1)$ in K. Consider the orbit $A = \{S(t)x_0 : 0 \leq t \leq 1\}$. Since S is continuous, A is strongly compact. Therefore the closed convex hull of A is also compact and contains a nondiametral point b (that is, a point b such that $r = \sup\{|b - x| : x \in clco(A)\} < diam(A)$). It is clear that b belongs to $K_0 = (\cap\{B(S(t)x_0, r) : 0 \leq t \leq 1\}) \cap K$. Since $S(t)(A) = A$ and K is invariant under $S(t), 0 \leq t \leq 1$, we see that K_0 is also invariant under $S(t), 0 \leq t \leq 1$. The minimality of K now implies that $K = K_0$. Therefore $|y - S(t)x_0| \leq r$ for all $0 \leq t \leq 1$ and $y \in K$. Hence $diam(A) \leq r$. The contradiction we have reached shows that K is a singleton and the result follows.

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Note added in Proof. B. Prus has exhibited a nonexpansive mapping with bounded orbits on ℓ_∞ with no fixed points. In view of Theorem 4, the answer to the problem raised on p.520 of [12] is "No". More related results can be found in the paper entitled "On the fixed points of commuting nonexpansive maps in hyperconvex spaces" by M. A. Khamsi, M. Lin and R. Sine.

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