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Sadovskii's fixed point theorem without convexity

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Abstract

The abstract formulation of Kirk's fixed point theorem by Penot played a major role in developing fixed point theorems in nonconvex setting. In this work, we similarly give an abstract formulation to Sadovskii's fixed point theorem using convexity structures. As an example, we discuss these new ideas in the hyperconvex metric setting.

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1. Introduction

May be one of the most interesting result in *metric fixed point theory* is Kirk's theorem [9]. The initial attempts to extend it to the nonlinear case were not very successful. Penot's formulation [12] of this theorem may be considered as a gateway to some amazing new results. Specially, the extension of normal structure ideas to discrete sets (see [7] for more details). In this work we consider, as Penot did, the notion of convexity structures and discuss Sadovskii's fixed point theorem [13] in this setting. Then we give the interesting example of hyperconvex metric spaces. It is

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worth to mention that an extensive research on hyperconvex metric spaces is underway (see [5]).

2. Basic definitions

Recall that Sadovskii's fixed point theorem states that if M is a nonempty, bounded, closed and convex subset of a Banach space X, and $T: M \to M$ is a continuous condensing map, then T has a fixed point, i.e. there exists $x \in M$ such that $T(x) = x$. The condensing behavior is related to the notion of measure of noncompactness. In this work, we will not discuss the general theory of measure of noncompactness. The interested reader may consult [2]. For the purpose of illustrating our ideas, we will only consider the measures of noncompactness introduced by Hausdorff and Kuratowski.

Definition 1. Let (M, d) be a metric space and let $\mathscr{B}(M)$ denote the collection of nonempty, bounded subsets of M.

(1) The Kuratowski measure of noncompactness $\alpha: \mathscr{B}(M) \to [0, \infty)$ is defined by

$$\alpha(A) = \inf\left\{\varepsilon > 0; A \subset \bigcup_{i=1}^{i=n} A_i \text{ with } A_i \in \mathscr{B}(M) \text{ and } diam(A_i) \leqslant \varepsilon\right\}.$$

(2) The Hausdorff (or ball) measure of noncompactness $\chi: \mathscr{B}(M) \to [0, \infty)$ is defined by

$$\chi(A) = \inf\left\{r > 0; A \subset \bigcup_{i=1}^{i=N} B(x_i, r) \text{ with } x_i \in M\right\},$$

where $B(x, r)$ denote the closed ball centered at x with radius r.

(3) The map $T: M \to M$ is said to be α-*condensing* (resp. χ-condensing) if and only if

$$\alpha(T(A)) < \alpha(A) \quad (\text{resp. } \chi(T(A)) < \chi(A))$$

for any $A \in \mathscr{B}(M)$.

The following properties hold in the general case:

(1) For any $A \in \mathscr{B}(M)$, we have

$$0 \leqslant \alpha(A) \leqslant \delta(A) = \text{diameter of } A.$$

(2) For any $A \in \mathscr{B}(M)$, we have

$$\alpha(A) = 0 \quad \text{iff } A \text{ is precompact}.$$

(3) For any $A \in \mathscr{B}(M)$ and $B \in \mathscr{B}(M)$, we have

$$\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}.$$

(4) For any $A \in \mathscr{B}(M)$, we have

$$\chi(A) \leqslant \alpha(A) \leqslant 2\chi(A).$$

(5) If (A_i) is a decreasing chain of nonempty closed bounded sets such that $\inf_i\{\alpha(A_i)\} = 0$, then $\cap A_i$ is not empty and is compact (i.e. $\alpha(\cap A_i) = 0$).

For our abstract formulation, we need the following definition:

Definition 2. Let M be a metric space and $\mathscr{F}$ a family of bounded subsets of M. We will say

(1) $\mathscr{F}$ has the *intersection property* (IP) if and only if $A \cap B \in \mathscr{F}$ provided $A \in \mathscr{F}$ and $B \in \mathscr{F}$.
(2) $\mathscr{F}$ has the *chain intersection property* (CIP) if and only if $\cap A_i \in \mathscr{F}$ provided (A_i) is a decreasing chain of elements in $\mathscr{F}$.

In both cases, we may talk about the $\mathscr{F}$-closure of $A \in \mathscr{B}$, which we will denote $co_{\mathscr{F}}(A)$. Indeed, if $\mathscr{F}$ has IP, then we set

$$co_{\mathscr{F}}(A) = \bigcap_{B \in \mathscr{F}(A)} B,$$

where $\mathscr{F}(A) = \{B \in \mathscr{F}; A \subset B\}$. And if $\mathscr{F}$ has CIP, then the subfamily $\mathscr{F}(A)$ satisfies the assumptions of Zorn's lemma. Therefore, $\mathscr{F}(A)$ has minimal elements. We will still use the notation $co_{\mathscr{F}}(A)$ to designate such minimal elements.

Example. Let C be a closed bounded convex of a normed linear space. Consider $\mathscr{F}$ to be the family of all the closed convex subsets of C. Then clearly $\mathscr{F}$ satisfies IP.

In the next section, we will discuss the case of hyperconvex metric spaces in which a natural family of subsets may be found which satisfies CIP.

In the proof of Sadovskii theorem, one crucial step is the invariance of the measure of noncompactness with respect to the convex closure of a set. Indeed, if M is a normed linear space, then we have

$$\alpha(\overline{conv}(A)) = \alpha(A),$$

where $\overline{conv}(A)$ is the convex closure of $A \in \mathscr{B}(M)$. This property suggests the following definition.

Definition 3. Let M be a metric space and $\mathscr{F}$ a family of closed bounded subsets of M. We will say that $\mathscr{F}$ is α-invariant if and only if for any $A \in \mathscr{B}$, we have

$$\alpha(co_{\mathscr{F}}(A)) = \alpha(A).$$

Of course, we do assume that $co_{\mathscr{F}}(A)$ exists for any $A \in \mathscr{B}(M)$.

In the next section, we will discuss an interesting example which will help us shed some light on the difference between IP and CIP.

3. The hyperconvex case

Recall that a metric space M is said to be *hyperconvex* if and only if for any family $(x_i)_{i\in I}$ of points in M and any family $(r_i)_{i\in I}$ of positive numbers such that $d(x_i,x_j)\leqslant r_i+r_j$, for all $i,j\in I$, then we have

$$\bigcap_{i\in I} B(x_i,r_i)\neq\emptyset.$$

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [1] who proved that a hyperconvex space is a nonexpansive absolute retract, i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin (see for instance [11, p. 92]). The nonlinear theory is still developing. The recent interest into these spaces goes back to the results of Sine [14] and Soardi [15] who proved independently that fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many interesting results have been shown to hold in hyperconvex spaces (see [5]).

One of the most elegant results in this direction was discovered by Baillon [3] who showed that if M is a hyperconvex metric space and (H_i) is a decreasing chain of bounded hyperconvex subsets of M, then $\bigcap_i H_i$ is not empty and is hyperconvex. Therefore, the family

$$\mathscr{H}=\{H\subset\mathscr{B}(M);H\neq\emptyset \text{ and is hyperconvex}\}$$

satisfies CIP (but fails to satisfy IP, i.e. the intersection of two hyperconvex is not necessarily hyperconvex). Another natural family considered by many is the family of admissible subsets of M, denoted by $\mathscr{A}(M)$, and defined by

$$A\in\mathscr{A}(M)\quad\text{if and only if } A \text{ is a nonempty intersection of closed balls.}$$

It is easy to see that $\mathscr{A}(M)$ satisfies IP. This family was used extensively in the study of metric fixed point property, the theory which deals with the existence of fixed points of nonexpansive mappings. But when it comes to condensing mappings, Espinola [4] noticed that it is not a natural family to work with. Indeed, one may easily come up with a compact set A in l_∞ such that $co_{\mathscr{A}(l_\infty)}(A)$ is not compact. Therefore, the family $\mathscr{A}(l_\infty)$ is not α-invariant.

First let us note the following property of the two measures of noncompactness α and χ.

Proposition 1. *Let H be a hyperconvex metric space and A a bounded subset of H. Then we have*

$$\alpha(A)=2\chi(A).$$

Proof. It is enough to prove that

$$2\chi(A)\leqslant\alpha(A).$$

Let $\varepsilon > \chi(A)$. Then there exists $A_1, \ldots, A_n$ subsets of A such that

$$A = \bigcup_{1 \leqslant i \leqslant n} A_i$$

with $\delta(A_i) \leqslant \varepsilon$, for $i = 1, \ldots, n$. Since H is hyperconvex, then there exists $h_i \in H$, for $i = 1, \ldots, n$, such that

$$A_i \subset B\left(h_i, \frac{\varepsilon}{2}\right).$$

Hence

$$A \subset \bigcup_{1 \leqslant i \leqslant n} B\left(h_i, \frac{\varepsilon}{2}\right),$$

which gives $\alpha(A) \leqslant \varepsilon/2$. Therefore

$$\alpha(A) \leqslant \frac{\chi(A)}{2},$$

which completes the proof of Proposition 1. □

Note that a similar result exists in any infinite-dimensional Banach spaces for closed balls. Indeed, in these spaces, we have

$$\alpha(B(x,r)) = 2\chi(B(x,r)) = 2r.$$

Before we show that for a hyperconvex metric space, the natural family $\mathscr{H}$ is α-invariant, we need some basic results due to Isbell [6]. Indeed, let H be a bounded hyperconvex metric space and $\mathscr{H}$ the above natural family. Isbell has shown that for any subset A of H, all the minimal elements of the subfamily

$$\mathscr{H}(A) = \{C \in \mathscr{H}; A \subset C\}$$

are isometric to the set $\varepsilon(A)$, called the *injective envelope* of A. The set $\varepsilon(A)$ is the set of all extremal functions defined on A. Recall that the function $f : A \to [0, \infty)$ is extremal if

$$d(x, y) \leqslant f(x) + f(y) \quad \text{for all } x, y \text{ in } A$$

and is pointwise minimal, i.e. if $g : A \to [0, \infty)$ such that

$$d(x, y) \leqslant g(x) + g(y) \quad \text{for all } x, y \text{ in } A$$

and $g(x) \leqslant f(x)$ for all $x \in A$, then we must have $f = g$. Note that if $f \in \varepsilon(A)$, then it satisfies

$$f(x) \leqslant d(x, y) + f(y) \quad \text{for all } x, y \text{ in } A.$$

This inequality implies that $\varepsilon(A) \subset Lip_1(A)$, where $Lip_1(A)$ is the set of all Lipschitzian functions defined on A with Lipschitz constant 1. Before, we show the α-invariance of the family $\mathscr{H}$, we need the following technical lemma (which appeared first in [4] with a different proof). Note that this result may be seen as an adaptation of the classical Arzela–Ascoli Theorem.

Lemma 1. *Let M be a metric space. Consider the space $\lambda_{[a,b]}(M)$ of Lipschitzian real-valued functions defined on M with Lipschitz constant less than λ with values in the interval $[a,b]$. Then we have*

$$\alpha(\lambda_{[a,b]}(M)) \leqslant 2\lambda\chi(M).$$

Proof. Let $\varepsilon_0 > \chi(M)$. Without loss of generality, we may assume that there exists $x_1,\ldots,x_n$ in M such that for any $x \in M$, there exists $i \in [1,n]$ such that $d(x,x_i) \leqslant \varepsilon_0$. Since $[a,b]$ is compact, for any $\varepsilon > 0$, there exists $c_1,\ldots,c_m$ in $[a,b]$ such that for any $c \in [a,b]$ there exists $i \in [1,m]$ such that

$$|c - c_i| \leqslant \varepsilon.$$

Let ψ: $\{1,\ldots,n\} \to \{1,\ldots,m\}$ be an application. Define

$$\lambda_\psi = \{f \in \lambda_{[a,b]}(M);\ \sup_{1 \leqslant i \leqslant n} |f(x_i) - c_{\psi(i)}| \leqslant \varepsilon\}.$$

This set may be empty. On the other hand, we have

$$\lambda_{[a,b]}(M) = \bigcup_{\psi \in \{1,\ldots,m\}^{\{1,\ldots,n\}}} \lambda_\psi.$$

Let $f,g \in \lambda_\psi$. For any $x \in M$, there exists $i \in [1,n]$ such that $d(x,x_i) \leqslant \varepsilon_0$. Then we have

$$|f(x) - g(x)| \leqslant |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)|,$$

which gives

$$|f(x) - g(x)| \leqslant \lambda\varepsilon_0 + 2\varepsilon + \lambda\varepsilon_0.$$

Hence

$$\sup_{x \in M} |f(x) - g(x)| \leqslant 2\lambda\varepsilon_0 + 2\varepsilon.$$

Since the set $\{1,\ldots,m\}^{\{1,\ldots,n\}}$ is finite, we get

$$\alpha(\lambda_{[a,b]}(M)) \leqslant 2\lambda\varepsilon_0 + 2\varepsilon.$$

Since ε was chosen arbitrarily positive, we get

$$\alpha(\lambda_{[a,b]}(M)) \leqslant 2\lambda\varepsilon_0,$$

which clearly implies

$$\alpha(\lambda_{[a,b]}(M)) \leqslant 2\lambda\chi(M). \qquad \square$$

From this lemma, we deduce the following result.

Corollary. *Let M be any bounded metric space and $\varepsilon(M)$ its injective envelope. Then we have*

$$\chi(\varepsilon(M)) = \chi(M).$$

Proof. Since $\varepsilon(M) \subset \lambda_{[0,\delta]}(M)$, where $\lambda = 1$ and δ is the diameter of M, then we have

$$\alpha(\varepsilon(M)) \leqslant 2\chi(M).$$

Using Proposition 1, we get

$$\chi(\varepsilon(M)) \leqslant \chi(M).$$

Since M can be isometrically embedded into $\varepsilon(M)$, then

$$\chi(M) \leqslant \chi(\varepsilon(M)). \qquad \square$$

In particular, if A is a bounded subset of a hyperconvex metric space H, and $h(A)$ is any minimal element of the family

$$\mathscr{H}(A) = \{C \in \mathscr{H}; A \subset C\},$$

then Isbell [6] proved that $h(A)$ is isometric to $\varepsilon(A)$. Hence we have

$$\chi(h(A)) = \chi(\varepsilon(A)) = \chi(A).$$

Using Proposition 1, we get

$$\alpha(h(A)) = \alpha(A).$$

This clearly implies that the family $\mathscr{H}$, associated to any hyperconvex metric space H, is α-invariant.

4. Abstract formulation of Sadovskii's theorem

Let M be a metric space and $\mathscr{F}$ a family of bounded subsets of M. We will say that $\mathscr{F}$ satisfies the property (S) (for Schauder) if and only if for any $C \in \mathscr{F}$ nonempty compact and any $T : C \to C$ continuous map, there exists $x \in C$ such that $T(x) = x$, i.e. T has a fixed point.

Khamsi [8] has shown that if M is hyperconvex, then the family $\mathscr{H}$ satisfies (S).

Theorem 1. *Let M be a metric space and $\mathscr{F}$ a family of bounded subsets of M. Assume $\mathscr{F}$ satisfy IP (or CIP), and the property (S). We will also assume that $\mathscr{F}$ is α-invariant. Then, for any nonempty $C \in \mathscr{F}$ and any continuous $T : C \to C$, which is condensing, has a fixed point.*

Proof. First let us give the proof of this theorem when $\mathscr{F}$ satisfies IP. Let $m \in C$ and define

$$\mathscr{F}(m, T) = \{D \in \mathscr{F}; m \in D \text{ and } T(D) \subset D\}.$$

Let

$$C(m) = \bigcap_{D \in \mathscr{F}(m,T)} D.$$

Since $C \in \mathscr{F}(m,T)$, then $C(m)$ does exist. It is easy to see that $C(m)$ is T-invariant (i.e. $T(C(m)) \subset C(m)$) and is not empty since $m \in C(m)$. Let us show that $C(m)$ is compact. Indeed, we have $co_{\mathscr{F}}(T(C(m)) \cup \{m\}) \subset C(m)$. Hence we deduce

$$T(co_{\mathscr{F}}(T(C(m))) \cup \{m\}) \subset T(C(m)) \subset co_{\mathscr{F}}(T(C(m)) \cup \{m\}).$$

By minimality of $C(m)$, we deduce that

$$C(m) = co_{\mathscr{F}}(T(C(m)) \cup \{m\}).$$

Since $\mathscr{F}$ is α-invariant, we get

$$\alpha(co_{\mathscr{F}}(T(C(m)) \cup \{m\})) = \alpha(T(C(m)) \cup \{m\}) = \alpha(T(C(m))).$$

Therefore, we have

$$\alpha(C(m)) = \alpha(T(C(m))).$$

Since T is condensing, we deduce that $C(m)$ is compact. Using the property (S), we conclude that T has a fixed point (in $C(m)$). When $\mathscr{F}$ satisfies the property CIP, the proof is identical. We just need to be a little more careful about $co_{\mathscr{F}}$. Indeed, once $C(m)$ is defined, we consider the family

$$\mathscr{F}(C(m),T) = \{D \in \mathscr{F}; T(C(m)) \cup \{m\} \subset D, \text{ and } T(D) \subset D\}.$$

This family is not empty since $C(m) \in \mathscr{F}(C(m),T)$. Let $C^*(m)$ be a minimal element of $\mathscr{F}(C(m),T)$. This element exists since $\mathscr{F}$ satisfies CIP. By minimality of $C(m)$, we conclude that $C(m) = C^*(m)$. In other words,

$$co_{\mathscr{F}}(T(C(m)) \cup \{m\}) = C(m).$$

The end of the proof is similar to the case described above. □

As a corollary, we get the following result.

Corollary. *Let H be a bounded hyperconvex metric space and $T : H \to H$ a continuous condensing map. Then T has a fixed point.*

Note that this result was first obtained by Kirk and Shin [10].

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