A convexity property in Modular function spaces

Mohamed A. Khamsi,
The University of Texas at El Paso,
Department of Mathematical Sciences,
El Paso, Texas 79968-0514.

Abstract

In this paper we prove that a modular function space satisfies a property that implies uniform-Opial and uniform-Kadec-Klee properties with respect to convergence almost everywhere. A fixed point result is also established generalizing recent works in L^1.

1980 Mathematics Subject classification: Primary 46E30, 47E10.
Key words and phrases: Opial property, Kadec-Klee property, fixed point property, nonexpansive mappings, modular function spaces.

1 Introduction and Preliminaries.

A problem that mathematician dealt with for almost fifty years is how to generalize the classical function spaces L^p. A first attempt was made by Orlicz and Birnbaum [3]. Their approach consisted in considering spaces of functions with some growth properties different from the power type growth control provided by the L^p-norm. This generalization found many applications in differential and integral equations with kernels of nonpower types. Another generalization was given by Luxemburg [20] (see also [21]). The main idea is to consider, in a measure space, a functional that has the properties of a norm plus a monotony condition. The more abstract generalization was given by Nakano [24] based on replacing the particular integral form of the functional by an abstractly one that satisfy some good properties. This functional was called a modular. Let us add that this notion was redefined and generalized by Orlicz and Musielak [23]. In this work we consider the formulation given by Kozlowski [16].

We start with a brief recollection of basic concepts and facts of the theory of modular spaces.

Definition 1.1. *Let X be an arbitrary vector space.*

(a) A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary x,y in X,

- *(i) $\rho(x) = 0$ iff $x = 0$,*
- *(ii) $\rho(\alpha\, x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,*
- *(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \ge 0, \beta \ge 0$.*

(b) If (iii) is replaced by

(iii)' $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \ge 0, \beta \ge 0$,

we say that ρ is a convex modular.

(c) A modular ρ defines a corresponding modular space, i.e the vector space X_ρ given by

$$X_\rho = \{x \in X; \rho(\lambda x) \to 0 \ \text{ as } \ \lambda \to 0\}.$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an F-norm. Recall that a functional $||.|| : X \to [0, \infty]$ defines an F-norm if and only if

(1) $||x|| = 0$ if and only if $x = 0$,

(2) $||\alpha x|| = ||x||$ whenever $|\alpha| = 1$,

(3) $||x + y|| \leq ||x|| + ||y||$,

(4) $||\alpha_n x_n - \alpha x|| \to 0$ if $\alpha_n \to \alpha$ and $||x_n - x|| \to 0$.

An F-norm defines a distance on X by

$$d(x, y) = ||x - y||.$$

The linear metric space (X, d) is called an F-space if d is complete.

__**Definition 1.2.**__ *The modular space X_ρ can be equipped with an F-norm defined by*

$$||x||_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}.$$

When ρ is convex the formula

$$||x||_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}$$

defines a norm which is frequently called the Luxemburg norm.

It is clear that $||x_n||_\rho \to 0$ if and only if $\rho(\beta x_n) \to 0$ for every $\beta > 0$. One can easily observe that $\alpha \to \rho(\alpha x)$ is increasing for every $x \in X$.
As a classical example we may give the Orlicz' modular defined for every measurable real function f by the formula

$$\rho(f) = \int_{\Re} \varphi(|f(t)|) dm(t),$$

where m denotes the Lebesgue measure in $\Re$ and $\varphi : \Re \to [0, \infty)$ is continuous, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.
The modular space induced by the Orlicz' modular ρ_φ is called the Orlicz space L^φ.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. In an other word, the family $\mathcal{P}$ plays the role of the δ-ring of subsets of finite measure. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e. all functions $f : \Omega \to \Re$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Let us add that a set function $\mu : \Sigma \to [0, \infty]$ is called a σ-subadditive measure if

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \leq \mu(B)$ for any $A \subset B$,

(iii) $\mu(\bigcup A_n) \leq \sum \mu(A_n)$ for any sequence of sets $A_n \in \Sigma$.

<u>**Definition 1.3**</u> *A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if*

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega) \leq |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$,

(P_3) $\rho(f, .) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,

(P_6) for any $\alpha > 0$ $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E); g \in \mathcal{E},\ |g(\omega)| \leq |f(\omega)|\ \ \omega \in \Omega\}.$$

This will enable us to define $\rho(\alpha, E)$ for sets E not in $\mathcal{P}$; for the sake of simplicity, we write $\rho(f)$ instead of $\rho(f, \Omega)$.

Definition 1.4 *A set E is said to be ρ-null if and only if $\rho(\alpha, E) = 0$ for $\alpha > 0$. A property $p(\omega)$ is said to hold almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega;\ p(\omega)$ does not hold $\}$ is ρ-null. For example we will say frequently $f_n \to f$ ρ-a.e.*

Note that a countable union of ρ-null sets is still ρ-null [16, pages 15-16]. In the sequel we will identify sets A and B whose symmetric difference $A\Delta B$ is ρ-null; similarly we will identify measurable functions which differ only on a ρ-null set.

It is easy to see that the functional $\rho : \mathcal{M} \to [0, \infty]$ is a modular in the sense of Definition 1.1. The modular space determined by ρ will be called a modular function space and will be denoted by L_ρ. Recall that

$$L_\rho = \{f \in \mathcal{M};\ \lim_{\alpha \to 0} \rho(\alpha\ f) = 0\}.$$

Let us recall some basic facts about modular function spaces. For proofs and details the reader is referred to [16,22].

Theorem 1.1

(1) $(L_\rho, ||.||_\rho)$ is a complete space and the F-norm $||.||_\rho$ is monotone with respect to the natural order in $\mathcal{M}$.

(2) If there is a number $\alpha > 0$ such that $\rho(\alpha(f_n - f)) \to 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to f$ ρ-a.e.

(3) (Egoroff's Theorem) If $f_n \to f$ ρ-a.e. then there exists an increasing sequence of sets $H_k \in \mathcal{P}$ such that $\Omega = \bigcup H_k$ and $\{f_n\}$ converges uniformly to f on every H_k.

(4) Define

$$L_\rho^0 = \{f \in \mathcal{M}; \rho(f, .) \text{ is order continuous}\}$$

and

$$E_\rho = \{f \in \mathcal{M}; \alpha f \in L_\rho^0 \text{ for every } \alpha > 0\}.$$

Then

(4.1) $E_\rho \subset L^0_\rho \subset L_\rho$,

(4.2) E_ρ *has the Lebesgue property, i.e.* $||f\ 1_{D_n}||_\rho \to 0$ *if* $f \in E_\rho$ *and* D_n *decreases to* $\emptyset$,

(4.3) E_ρ *is the closure of* $\mathcal{E}$ *(in the sense of* $||.||_\rho$*).*

(5) (Vitali's Theorem) If $f_n \in E_\rho$ *and* $f_n \to f \in L_\rho$ ρ*-a.e., then the following conditions are equivalent*

(i) $f \in E_\rho$ *and* $||f_n - f||_\rho \to 0$,

(ii) for every $\alpha > 0$ *the subadditive measures* $\rho(\alpha f_n, .)$ *are equicontinuous, i.e.*

$$\lim_{k\to\infty} \sup_n \rho(\alpha f_n, D_k) = 0,$$

for every sequence $\{D_k\} \in \Sigma$ *that decreases to* $\emptyset$.

(6) (Lebesgue's Theorem) If $f_n, f \in \mathcal{M}$, $f_n \to f$ ρ*-a.e. and there exists a function* $g \in E_\rho$ *such that* $|f_n| \le |g|$ ρ*-a.e. for all* n*, then* $||f_n - f||_\rho \to 0$.

(7) For $f_n, f \in \mathcal{M}$*, the following conditions are equivalent*

(i) ρ *has the Fatou property, i.e.*

$$\rho(f_n) \uparrow \rho(f) \quad \textit{whenever} \ \ |f_n| \uparrow |f| \ \ \rho - a.e.$$

(ii) ρ *is a left continuous modular, i.e.*

$$\rho(\alpha_n\ f) \uparrow \rho(f) \quad \textit{whenever} \ \ \alpha_n \uparrow 1.$$

(iii) $\rho(f) \le \liminf \rho(f_n)$ *whenever* $f_n \to f$ ρ*-a.e.*

A function modular is said to satisfy the Δ_2-condition if $\sup \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever D_k decreases to $\emptyset$ and $\sup \rho(f_n, D_k) \to 0$. It was proved in [16] that Δ_2 is equivalent to the equality $E_\rho = L_\rho$. The other characterization is as follows: ρ satisfies the Δ_2 condition if and only if F-norm convergence and modular convergence are equivalent.

<u>Definition 1.5.</u>

(a) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty,$$

(b) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$,

(c) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞,

(d) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(e) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(f) A subset D of L_ρ is said to be ρ-complete if every ρ-Cauchy sequence is ρ-convergent in D.

The above terminology is used because of its formal similarity to the metric case. Since ρ is far from behaving as a distance, one should be very careful when dealing with these notions.

Before we give few examples of modular function spaces we will need the following definition.

Definition 1.6 *Let (Ω, Σ, μ) be a measure space. A real function φ defined on $\Omega \times \Re_+$ will be said to belong to the class Φ if the following conditions are satisfied*

(i) $\varphi(\omega, u)$ is a nondecreasing continuous function of u such that $\varphi(\omega, 0) = 0, \varphi(\omega, u) > 0$ for $u > 0$ and $\varphi(\omega, u) \to \infty$ as $u \to \infty$,

(ii) $\varphi(\omega, u)$ is a Σ-measurable function of ω for all $u \geq 0$,

(iii) $\varphi(\omega, u)$ is a convex function of u, for all $\omega \in \Omega$.

For the sake of generality some authors will not assume that $\varphi(\omega, u)$ is a convex function of u. Although the results in this work can be easily generalized into their setting, it is not the feeling of the author that this will change anything to the general idea.

Examples.

(1) It is easy to check that Orlicz spaces are modular function spaces. Similarly Musielak-Orlicz spaces, i.e. spaces determined by a modular of the form
$$\rho(f, E) = \int_E \varphi(t, |f(t)|) d\mu(t),$$
are modular function spaces, provided φ belongs to the class Φ. For the precise definitions and properties of Musielak-Orlicz spaces see the book by Musielak [22], where they are called generalized Orlicz spaces. The particular case when
$$\varphi(t, s) = s^p, \quad \text{for } 1 \leq p < \infty,$$
gives the classical L^p spaces. The Luxemburg's norm is the classical L^p-norm. Moreover we have
$$\rho(f) = ||f||^p_{L^p}.$$
Let us add that Musielak-Orlicz modular spaces are complete for the modular [14].

(2) Suppose $\mathcal{M}$ is a family of σ-additive measures on (Ω, Σ), and $\varphi \in \Phi$. One can prove that
$$\rho(f, E) = \sup_{\tau \in \mathcal{M}} \int_E \varphi(t, |f(t)|) d\mu_\tau(t),$$
is a function modular. As an example of function spaces determined by a function modular of this type we can mention the Lorentz type L^p-spaces, where
$$\rho(f, E) = \sup_{\tau \in \mathcal{T}} \int_E |f(t)|^p d\mu_\tau(t).$$
Here μ is a fixed σ-finite measure on Ω, $\mathcal{T}$ is any set of measurable, invertible transformations $\tau : \Omega \to \Omega$ and $\mu_\tau(E) = \mu(\tau^{-1}(E))$.

2 Main result

This work was motivated by recent results [2,19] on uniform Kadec-Klee property and uniform Opial condition. Looking at the proof it was clear that the results are of measure theoretical nature. Finally after looking into the problem more closely one would say that the original result can be found in [4] as the authors in [18] did notice. We will consider the original approach in the convex case. Let us add that one can easily adapt the original ideas into a more general case. As we mentioned before it is our feeling that this will not add anything substantial to the main result.

Throughout this work L_ρ will be a modular function space where ρ is assumed to be convex. Before we give the main result of this work we need the following technical lemma; see also [4].

Lemma 2.1 *Let $\epsilon > 0$ and $k > 1$ be such that $k\,\epsilon < 1$. Then for every $f, g \in L_\rho$ such that $\rho(kf) < \infty$ and $\rho(1/\epsilon\ (k-1)g) < \infty$, we have*

$$|\rho(f+g) - \rho(f)| \leq \epsilon\ |\rho(kf) - k\rho(f)| + 2\ \rho(C_\epsilon\ g),$$

where $C_\epsilon = 1/\epsilon\ (k-1)$.
Proof. Put

$$\alpha = 1 - k\epsilon, \quad \beta = \epsilon, \quad \gamma = (k-1)\epsilon.$$

Then we have $\alpha + \beta + \gamma = 1$ and

$$f + g = \alpha\ f + \beta k f + \gamma C_\epsilon g.$$

Since ρ is convex we get

$$\rho(f+g) \leq \alpha\ \rho(f) + \beta\rho(kf) + \gamma\rho(C_\epsilon g).$$

Hence

$$\rho(f+g) - \rho(f) \leq \epsilon\ \left(\rho(kf) - k\rho(f)\right) + (k-1)\epsilon\rho(C_\epsilon g)$$

which implies

$$\rho(f+g) - \rho(f) \leq \epsilon\ \left(\rho(kf) - k\rho(f)\right) + \rho(C_\epsilon g).$$

On the other hand if we put

$$\alpha = \frac{1}{1+k\epsilon}, \quad \beta = \frac{\epsilon}{1+k\epsilon}, \quad \gamma = \frac{\epsilon(k-1)}{1+k\epsilon},$$

then $f = \alpha(f+g) + \beta kf + \gamma(-C_\epsilon g)$. Hence

$$\rho(f) - \rho(f+g) \leq \epsilon \ [\rho(kf) - k\rho(f)] + \epsilon(k-1)\rho(-C_\epsilon g).$$

This will imply

$$\rho(f) - \rho(f+g) \leq \epsilon \ [\rho(kf) - k\rho(f)] + \rho(C_\epsilon g).$$

The proof is therefore complete.

From now on we will assume that ρ is additive, i.e.

$$\rho(f, A \cup B) = \rho(f, A) + \rho(f, B),$$

whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$. Clearly this implies $\rho(f, A) = \rho(f \ 1_A)$. This may seem strong, but many interesting examples lead to additive modulars. For example, any modular generated by a functionnal measure.

The next result states the main result of this work.

Theorem 2.1. *Let $\{f_n\} \subset L_\rho$ be ρ-a.e. convergent to 0. Assume there exists $k > 1$ such that*

$$\sup_n \rho(kf_n) = M < \infty.$$

Let $g \in E_\rho$, then we have

$$\lim_{n\to\infty} \Big(\rho(f_n + g) - \rho(f_n)\Big) = \rho(g).$$

Proof. Since $\{f_n\}$ converges ρ-a.e. to 0, then by Egoroff's Theorem, there exists an increasing sequence of sets $H_k \in \mathcal{P}$ such that $\Omega = \bigcup H_k$ and $\{f_n\}$ converges uniformly to f on every H_k. On the other hand we have

$$|\rho(f_n + g) - \rho(f_n) - \rho(g)| \leq |\rho(f_n + g, H_m) - \rho(f_n, H_m) - \rho(g, H_m)|$$

$$+|\rho(f_n+g,H_m^c)-\rho(f_n,H_m^c)-\rho(g,H_m^c)|\,,$$

where A^c denotes the complement of the subset A. Using Lemma 2.1 we get

$$|\rho(f_n+g,H_m)-\rho(g,H_m)|\le\epsilon\,(\rho(kg,H_m)-k\rho(g,H_m))+2\rho(C_\epsilon f_n,H_m),$$

for every $\epsilon>0$ such that $\epsilon\ k<1$. Since $\{f_n\}$ converges uniformly to 0 on every H_m we have

$$\limsup_{n\to\infty}|\rho(f_n+g,H_m)-\rho(f_n,H_m)-\rho(g,H_m)|\le\epsilon\rho(kg).$$

Using the same ideas we get

$$\limsup_{n\to\infty}|\rho(f_n+g,H_m^c)-\rho(f_n,H_m^c)-\rho(g,H_m^c)|\le\epsilon\ \limsup_{n\to\infty}\rho(kf_n)+2\rho(C_\epsilon g,H_m^c)+\rho(g,H_m^c).$$

Hence

$$\limsup_{n\to\infty}|\rho(f_n+g,H_m^c)-\rho(f_n,H_m^c)-\rho(g,H_m^c)|\le\epsilon\ \sup_n\rho(kf_n)+2\rho(C_\epsilon g,H_m^c)+\rho(g,H_m^c).$$

Therefore

$$\limsup_{n\to\infty}|\rho(f_n+g)-\rho(f_n)-\rho(g)|\le\epsilon\ \rho(kg)+\epsilon\ \sup_n\rho(kf_n)+2\rho(C_\epsilon g,H_m^c)+\rho(g,H_m^c).$$

Let m goes to ∞ and using the fact that $g\in E_\rho$, we get

$$\limsup_{n\to\infty}|\rho(f_n+g)-\rho(f_n)-\rho(g)|\le\epsilon\ \rho(kg)+\epsilon\ \sup_n\rho(kf_n).$$

Finally we let ϵ goes to 0 to get

$$\limsup_{n\to\infty}|\rho(f_n+g)-\rho(f_n)-\rho(g)|\le 0.$$

The proof is therefore complete.

As a corollary to this theorem one can get the following.

Corollary 2.1 *Let $p\ge 1$ and $\{f_n\}$ be a sequence of L^p-uniformly bounded functions on a measure space. Assume that $\{f_n\}$ converges almost everywhere to $f\in L^p$. Then*

$$\liminf_{n\to\infty}||f_n||^p=\liminf_{n\to\infty}||f_n-f||^p+||f||^p.$$

When $p = 1$, the conclusion of Corollary 2.1 gives the main result of [2,19].

Let us add that when $p < 1$ the conclusion of Corollary 2.1 is still true. This will not be a simple deduction from Theorem 2.1 since the function $\varphi(t) = t^p$ is not convex. A technical assumption [4] can be added to get a more general result (see also [18]).

Theorem 2.2 *Let $\epsilon > 0$ and $\{f_n\} \subset L_\rho$ be ρ-a.e. convergent to 0. Assume there exists $k > 1$ such that*

$$\sup_n \rho(kf_n) = M < \infty.$$

Let $f \in E_\rho$ such that $\rho(f) \geq \epsilon$, then we have

$$\liminf_{n\to\infty} \rho(f_n) + \epsilon \leq \liminf_{n\to\infty} \rho(f_n + f).$$

The proof is obvious using the conclusion of Theorem 2.1. This is a kind of Opial property. First let us give the following definition.

Definition 2.1 *We will say that L_ρ satisfies the ρ-a.e.-Opial property if for every $\{f_n\} \subset L_\rho$ ρ-a.e. convergent to 0 such that there exists $k > 1$ for which*

$$\sup_n \rho(kf_n) = M < \infty$$

then for every $f \in E_\rho$ not equal to 0 we have

$$\liminf_{n\to\infty} \rho(f_n) < \liminf_{n\to\infty} \rho(f_n + f).$$

We will say that L_ρ satisfies the ρ-a.e.-uniform Opial property if for every $\epsilon > 0$ there exists $\eta > 0$ such that for every $\{f_n\} \subset L_\rho$ ρ-a.e. convergent to 0 such that there exists $k > 1$ for which

$$\sup_n \rho(kf_n) = M < \infty$$

then for every $f \in E_\rho$ such that $\rho(f) \geq \epsilon$ we have

$$\liminf_{n\to\infty} \rho(f_n) + \eta \leq \liminf_{n\to\infty} \rho(f_n + f).$$

Opial's property plays an important role in the study of convergence of iterates of nonexpansive mappings and of the asymptotic behavior of nonlinear semigroups [10,13,17,25,26,27]. Clearly the ρ-a.e.-uniform Opial property implies ρ-a.e.-Opial property.
Therefore the conclusion of Theorem 2.2 means that L_ρ satisfies the ρ-a.e.-uniform Opial property.

The next result deals with a property similar to Kadec-Klee property [6,7,11,12].

Definition 2.2 *We will say that L_ρ satisfies ρ-a.e.-Kadec-Klee property if for every $\epsilon > 0$ and every $r > 0$ there exists $\eta > 0$ such that for every $\{f_n\} \subset E_\rho$ ρ-a.e. convergent to $f \in E_\rho$ such that there exists $k > 1$ for which*

$$\sup_n \rho(k\,[f_n - f]) = M < \infty$$

and $\rho(f_n) \leq r$ for every $n \geq 1$ we have

$$\rho(f) \leq r(1-\eta)$$

provided that

$$sep\left\{\frac{1}{2}f_n\right\} = \inf\left\{\rho\left(\frac{(f_n - f_m)}{2}\right); n \neq m\right\} > r\,\epsilon.$$

We will say that L_ρ satisfies ρ-a.e.-uniform Kadec-Klee property if the above still holds for every ϵ.

Theorem 2.3 *Under the assumptions of Theorem 2.1, the modular function space L_ρ satisfies ρ-a.e.-uniform Kadec-Klee property.*
Proof. Let $\epsilon > 0$, $r > 0$ and $\{f_n\}$ be as in Definition 2.2 ρ-a.e. convergent to f. Theorem 2.1 implies that

$$\liminf_{n\to\infty} \rho(f_n - f) + \rho(f) = \liminf_{n\to\infty} \rho(f_n).$$

Our assumption on $\{f_n\}$ implies that

$$\liminf_{n\to\infty} \rho(f - f_n) \geq r\,\frac{\epsilon}{2}.$$

Therefore

$$\rho(f) \le r\,\left(1 - \frac{\epsilon}{2}\right).$$

The proof is complete.

If the modular ρ is subadditive then one does not need to take $\rho((f_n - f_m)/2)$ in Definition 2.2 we could take $\rho(f_n - f_m)$. This is the case when $L_\rho = L^1$ (see [2,4,19]).

3 Application to fixed point property

Fixed point theory for nonexpansive mappings has its origins in the 1965 existence theorems [5,9,15]. Although such mappings are natural extensions of the contraction mappings, it was clear from the outset that the study of nonexpansive mappings required techniques which go far beyond the purely metric approach. For more on the fixed point property see [1,8,14].

Definition 3.1. *Let C be a subset of a modular space L_ρ and let $T : C \to C$ be an arbitrary mapping.*

(1) T is called a strict ρ-contraction if there exists $\lambda < 1$ such that

$$\rho(T(f) - T(g)) \le \lambda\, \rho(f - g)$$

for all $f, g \in C$.

(2) T is said to be ρ-nonexpansive if

$$\rho(T(f) - T(g)) \le \rho(f - g)$$

for all $f, g \in C$.

(3) $f \in C$ is said to be a fixed point of T if $T(f) = f$. The fixed point set of T will be denoted $Fix(T)$.

C will be said to have the fixed point property if every ρ-nonexpansive selfmap defined on C has a fixed point.

In many cases it is more convenient to verify assumptions related to the modular than the associated norm. Since the latest is given indirectly. Before we give a fixed point result for ρ-nonexpansive mappings, let us show how a ρ-strict contraction has a fixed point.

<u>**Theorem 2.4**</u> *Let C be ρ-complete ρ-bounded subset of L_ρ and $T : C \to C$ be a ρ-strict contraction. Then T has a unique fixed point $z \in C$. Moreover z is the ρ-limit of the iterate of any point in C under the action of T.*
<u>**Proof.**</u> Let $k \in (0, 1)$ be such that

$$\rho(T(x) - T(y)) \le k\ \rho(x - y)$$

for every $x, y \in C$. Let $x \in C$ be fixed. Then we have

$$\rho(T^{(n+h)}(x) - T^n(x)) \le k^n\ \rho(T^h(x) - x) \le k^n \delta_\rho(C)$$

for every $n, h \in N$. Therefore $\{T^n(x)\}$ is ρ-Cauchy. Since C is ρ-complete we deduce that $\{T^n(x)\}$ ρ-converges to $z \in C$. Let us show that $z \in Fix(T)$. Then from $\rho(z - T^n(x)) \to 0$, we get

$$\rho(T(z) - T^n(x)) \le k\rho(z - T^{(n-1)}(x)) \to 0.$$

This clearly implies that $\{T^n(x)\}$ ρ-converges to $T(z)$. But

$$\rho\left(\frac{(z - T(z))}{2}\right) \le \rho(z - T^n(x)) + \rho(T^n(x) - T(z)),$$

for every $n \ge 1$. This implies that $\rho\left(\frac{(z-T(z))}{2}\right) = 0$ which implies that z is a fixed point. It is obvious that z is the only fixed point.

Let C be a ρ-complete, ρ-bounded subset of L_ρ. Assume that C is star-shaped, i.e. there exists $x_0 \in C$ such that $\alpha x + (1-\alpha)x_0 \in C$ provided $x \in C$ and $\alpha \in [0, 1]$. Let $T : C \to C$ be ρ-nonexpansive and $\epsilon < 1$ be a positive number. Set

$$T_\epsilon(x) = (1 - \epsilon)T(x) + \epsilon x_0.$$

Then T_ϵ defines a ρ-strict contraction on C. Theorem 2.4 implies that T_ϵ has a unique fixed point x_ϵ. Clearly we have $x_\epsilon = (1 - \epsilon)T(x_\epsilon) + \epsilon x_0$ which implies

$$\rho(x_\epsilon - T(x_\epsilon)) \le \rho(\epsilon(T(x_\epsilon) - x_0)) \le \epsilon\delta_\rho(C)$$

where we used the fact that ρ is convex. Put $x_n = x_{1/n}$ for $n \geq 1$. Then we have

$$\lim_{n\to\infty} \rho(x_n - T(x_n)) = 0.$$

Theorem 2.5 *Let C be a starshaped subset of E_ρ. Assume that C is ρ-a.e. compact. Suppose that there exists $k > 1$ such that*

$$\delta_\rho(k\ C) = \sup\{\rho(k(x-y));\ x, y \in C\} < \infty.$$

Then any ρ-nonexpansive map $T; C \to C$ has a fixed point.

Proof. There exists a sequence $\{x_n\} \subset C$ such that $x_n = (1-\epsilon_n)T(x_n)+\epsilon_n x_0$ where x_0 is the point from which C is starshaped and $\epsilon_n \to 0$ as $n \to \infty$. Since C is ρ-a.e. compact, there exists $\{x_{n'}\}$ a subsequence of $\{x_n\}$ that is ρ-convergent. Call x its limit. Since $k > 1$, we have $\delta_\rho(C) \leq \delta_\rho(k.C) < \infty$. Then

$$\begin{aligned}
\rho\left(\tfrac{x-T(x)}{2}\right) &\leq \rho(x - x_{n'}) + \rho(x_{n'} - T(x)) \\
&= \rho(x - x_{n'}) + \rho\Big((1-\epsilon_{n'})(T(x_{n'}) - T(x)) + \epsilon_{n'}(x_0 - T(x))\Big) \\
&\leq \rho(x - x_{n'}) + (1-\epsilon_{n'})\rho(x_{n'} - x) + \epsilon_{n'}\delta_\rho(C) \\
&\to 0
\end{aligned}$$

which implies that $x = T(x)$. The proof is therefore complete.

In [6,7] the uniform-Kadec-Klee property is related to the fixed point property through the normal structure property via Kirk's theorem [15]. One can indeed prove an analogous to this in modular function spaces. The result will be weaker than the conclusion of Theorem 2.5.

The author would like to thank the referee for the suggested remarks.

References

[1] A.G. Aksoy, M. A. Khamsi,"Nonstandard Methods in fixed point theory", Springer-Verlag, New York and Berlin, 1990.

[2] M. Besbes,"Points fixes des contractions definies sur un convexe L^0-ferme de L^1", C.R.A. Sc. de Paris, Tome 311, Serie I (1990), 243-246.

[3] Z. Birnbaum, W. Orlicz,"Uber die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen", Studia Math. 3(1931), 1-67.

[4] H. Brezis, E. Lieb,"A relation between pointwise convergence of functions and convergence of functionals", Proc. A.M.S. Vol. 88-3(1983), 486-490.

[5] F.E. Browder,"Nonexpansive nonlinear operators in a Banach space", Proc. Nat. Acad. Sci. USA. 54(1965), 1041-1044.

[6] D. van Dulst, B. Sims,"Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type (KK)", Banach space theory and its applications, Proc. Bucharest 1981, Lecture Notes in Math. 991, Springer, 1983, 35-43.

[7] D. van Dulst, V. de Valk,"(KK)-properties, normal structure and fixed points of nonexpansive mappings in Orlicz spaces", Canad. J. Math. 38 (1986), 728-750.

[8] K. Goebel, W.A. Kirk,"Topics in Metric fixed point theory", Cambridge University Press, Cambridge, 1990.

[9] D. Gohde,"Zum Pinzip der kontraktiven abbildung", Math. Nachr. 30(1965), 251-258.

[10] J. Gorniki," Some remarks on almost convergence of the Picard iterates for nonexpansive mappings in Banach spaces which satisfy the Opial-condition", Comment. Math. 29(1988),59-68.

[11] R. Huff,"Banach spaces which are nearly uniformly convex", Rocky Mountain J. Math. 10 (1980), 743-749.

[12] V.I. Istratescu, J.R. Partington,"On nearly uniformly convex and k-uniformly convex spaces", Math. Proc. Cambridge Philo. Soc. 95(1984), 325-327.

[13] L. Karlovitz,"On nonexpansive mappings", Proc. A.M.S. 55(1976), 321-325.

[14] M.A. Khamsi, W.M. Kozlowski, S. Reich,"Fixed point theory in Modular function spaces", Nonlinear Analysis Th. M. Appl., Vol. 14-11(1990), 935-953.

[15] W.A. Kirk,"A fixed point theorem for mappings which do not increase distances", Amer. Math. Monthly 71(1965), 1004-1006.

[16] W.M. Kozlowski, " Modular function spaces", Dekker, New York, Basel 1988.

[17] T. Kuczumow,"Weak convergence theorems for nonexpansive mappings and semi-groups in Banach spaces with Opial' s property", Proc. Amer. Math. Soc. 93 (1985), 430-432.

[18] E. Lami-Dozo, Ph. Turpin, "Nonexpansive maps in generalized Orlicz spaces", Studia Math. 8691987), 155-188.

[19] C. Lennard,"A new convexity property that implies a fixed point property for L_1", Studia Math. 100-2(1991), 95-108.

[20] W.A.J. Luxemburg, "Banach function spaces", Thesis, Delft 1955.

[21] W.A.J. Luxemburg, A.C. Zaanen, "Notes on Banach function spaces I-XII", Proc. Acad. Sci. Amsterdam, (1963) A-66, 135-153, 239-263-, 496-504, 655-681, (1964) A-64, 101-119, (1964) A-67, 360-376, 493-543.

[22] J. Musielak, " Orlicz spaces and Modular spaces", Lecture Notes in Math. 1034, Springer-Verlag, Berlin, Heidelberg, New York 1983.

[23] J. Musielak, W. Orlicz,"On modular spaces", Studia Math. 18(1959), 49-65.

[24] H. Nakano," modulared semi-ordered spaces", Tokyo 1950.

[25] Z. Opial,"Weak convergence of the sequence of successive approximations for nonexpansive mappings", Bull. Amer. Math. Soc. 73(1967), 591-597.

[26] Z. Opial,"Nonexpansive and Monotone mappings in Banach spaces", Lecture Notes 67-1, Center for Dynamical Systems, Brown University, Providence, R.I., 1967.

[27] S. Prus,“Banach spaces with the uniform Opial property”, Nonlinear Analysis, To appear.