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KKM and KY fan theorems in modular function spaces

Mohamed Amine Khamsi[1*], Abdul Latif[2] and Hamid Al-Sulami[2]

* Correspondence: mohamed@utep.edu
[1]Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA
Full list of author information is available at the end of the article

Abstract

In modular function spaces, we introduce Knaster-Kuratowski-Mazurkiewicz mappings (in short KKM-mappings) and prove an analogue to Ky Fan s fixed point theorem.
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1. Introduction

The purpose of this paper is to give outlines of the Knaster-Kuratowski-Mazurkiewicz theory for mappings defined on some subsets of modular function spaces which are natural generalization of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. This paper operates within the framework of convex function modulars.

The importance of applications of nonexpansive mappings in modular function spaces lies in the richness of structure of modular function spaces, that is, besides being Banach spaces (or F-spaces in a more general setting)–are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the tools of modular function spaces. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

The theory of contractions and nonexpansive mappings defined on convex subsets of Banach spaces is very well developed (see e.g. [1-5]) and generalized to other metric spaces (see e.g. [6-8]) and modular function spaces (see e.g. [9-11]). The corresponding fixed point results were then extended to larger classes of mappings like asymptotic mappings [12,13], pointwise contractions [14] and asymptotic pointwise contractions and nonexpansive mappings [15-18].

As noted in [18], questions are sometimes asked whether the theory of modular function spaces provides general methods for the consideration of fixed point properties; the situation here is the same as it is in the Banach setting.

In this paper, we introduce the concept of Knaster-Kuratowski-Mazurkiewicz mappings (in short KKM-mappings) in modular function spaces. Then, we prove an analogue to Ky Fans fixed point theorem which can be seen as an extension to Brouwer's and Schauders fixed point theorems. Most of the results proved here are similar to the extension obtained in hyperconvex metric spaces [19]. Reader may also consult [20,21].

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. By $\mathcal{E}$, we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$, we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A, we denote the characteristic function of the set A.

Definition 2.1. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$be a notrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:*

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e. $|f(\omega)| \le |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \le \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e. $\rho(f1_{A\cup B}) \le \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;
(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

As in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. A property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind, we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \ \rho - a.e\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. When no confusion arises, we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2. *Let ρ be a regular function pseudomodular.*

(1) We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ - a.e.;

(2) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ - a.e.;

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Def. 2.1.1 in [22] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [22-24]; see also Musielak [25] for the basics of the general modular theory.

Remark 2.1. *We limit ourselves to convex function modulars in this paper. However, omitting convexity in Definition* 2.1 *or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in* [22].

Definition 2.3. [22-24]*Let ρ be a convex function modular.*

(a) A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$||f||_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \le 1\}.$$

In the following theorem, we recall some of the properties of modular spaces that will be used later on in this paper.

Theorem 2.1. [23,24,22]*Let $\rho \in \Re$.*

(1) $(L_\rho, ||f||_\rho)$ is complete and the norm $||\cdot||_\rho$ is monotone w.r.t. the natural order in $\mathcal{M}$.

(2) $||f_n||_\rho \to 0$ if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.

(3) If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ ρ - a.e.

(4) If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$, then $\rho(\alpha(f_n - f), E) \to 0$ for every $\alpha > 0$.

(5) Let $f_n \to f$ ρ - a.e. There exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every H_k (Egoroff Theorem).

(6) $\rho(f) \le \liminf \rho(f_n)$ whenever $f_n \to f$ ρ - a.e. (Note: this property is equivalent to the Fatou Property).

(7) Defining $L_\rho^0 = \{f \in L_\rho;\ \rho(f, \cdot) \text{ is order continuous}\}$ and $E_\rho = \{f \in L_\rho;\ \lambda f \in L_\rho^0 \text{ for every } \lambda > 0\}$, we have:

(a) $L_\rho \supset L_\rho^0 \supset E_\rho$,

(b) E_ρ has the Lebesgue property, i.e. $\rho(\alpha f, D_k) \to 0$ for $\alpha > 0$, $f \in E_\rho$ and $D_k \downarrow \emptyset$.

(c) E_ρ is the closure of $\mathcal{E}$(in the sense of $||\cdot||_\rho$).

The following definition plays an important role in the theory of modular function spaces.

Definition 2.4. *Let $\rho \in \Re$. We say that ρ has the Δ_2-property if $\sup_n \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{f_n\} \subset \mathcal{M}$ and $\{D_k\} \subset \Sigma$ which decreases to ∅ and $\sup_n \rho(f_n, D_k) \to 0$ as $k \to \infty$.*

Theorem 2.2. *Let $\rho \in \Re$. The following conditions are equivalent:*

(a) ρ has Δ_2-property,

(b) L_ρ^0 is a linear subspace of L_ρ,

(c) $L_\rho = L_\rho^0 = E_\rho$,

(d) if $\rho(f_n) \to 0$, then $\rho(2f_n) \to 0$,

(e) if $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then $||f_n||\rho \to 0$, i.e. the modular convergence is equivalent to the norm convergence.

The following definition is crucial throughout this paper.

Definition 2.5. *Let $\rho \in \Re$.*

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f\ (\rho)$ if and only if $\rho(f_n - f) \to 0$.

(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f\ (\rho)$ implies that f belongs to B.

(d) A set $B \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f \in B, g \in B\} < \infty$.

(e) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = \inf\{\rho(f - g); g \in C\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f$, $\lambda > 1$. Using Theorem 2.1, it is not difficult to prove the following

Proposition 2.1. *Let $\rho \in \Re$.*

(i) L_ρ is ρ-complete,

(ii) ρ-balls $B_\rho(f, r) = \{g \in L_\rho; \rho(f - g) \le r\}$ are ρ-closed.

In this work, we will need the following definition.

Definition 2.6. *A subset $A \subset L_\rho$ is called finitely ρ-closed if for every $f_1, f_2, ..., f_n \in L_\rho$, the set $\overline{conv}_\rho(\{f_1, \ldots, f_n\}) \cap A$ is ρ-closed.*

Note that if A is ρ-closed, then obviously it is also finitely closed.

The following property plays in the theory of modular function spaces a role similar to the reflexivity in Banach spaces (see e.g. [10]).

Definition 2.7. *We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.*

A more general definition of ρ-compactness is given in the following definition.

Definition 2.8. *A nonempty subset K of L_ρ is said to be ρ-compact if for any family $\{A_\alpha; A_\alpha \in 2^{L_\rho}, \alpha \in \Gamma\}$ of ρ-closed subsets with $K \cap A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$, for any $\alpha_1, ..., \alpha_n \in \Gamma$, we have*

$$K \cap \left(\bigcap_{\alpha \in \Gamma} A_\alpha \right) \neq \emptyset.$$

Let us finish this section with the modular definition of nonexpansive mappings. The definition are straightforward generalizations of their norm and metric equivalents, [12,15-17].

Definition 2.9. *Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty. A mapping $T : C \to C$ is called a nonexpansive mapping if*

$$\rho(T(f) - T(g)) \le \rho(f - g) \text{ for any } f, g \in C.$$

The fixed point set of T is defined by

$$Fix(T) = \{f \in C; T(f) = f\}.$$

3. KKM-maps and Ky Fan theorem

Among the results equivalent to the Brouwer's fixed point theorem, the theorem of Knaster-Kuratowski-Mazurkiewicz [26] occupies a special place. Let $\rho \in \Re$ and let $C \subset$

L_ρ be nonempty. The set of all subsets of C is denoted 2^C. The notation $conv(A)$ describes the convex hull of A, while $\overline{conv}_\rho(A)$ describes the smallest ρ-closed convex subset of L_ρ which contains A. Recall that a family $\{A_\alpha; A_\alpha \in 2^{L_\rho},\ \alpha \in \Gamma\}$ is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

Definition 3.1. *Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty. A multivalued mapping $G : C \to 2^{L_\rho}$ is called a Knaster-Kuratowski-Mazurkiewicz mapping (in short KKM-mapping) if*

$$conv(\{f_1, \ldots, f_n\}) \subset \bigcup_{1 \le i \le n} G(f_i)$$

for any $f_1, \ldots, f_n \in C$.

Now we are ready to prove the following result:

Theorem 3.1. *Let $\rho \in \Re$. Let $C \subset L_\rho$ be nonempty and $G : C \to 2^{L_\rho}$be a KKM-mapping such that for any $f \in C$, $G(f)$ is nonempty and finitely ρ-closed. Then, the family $\{G(f); f \in C\}$ has the finite intersection property.*

Proof. Assume not, i.e. there exist $f_1, \ldots, f_n \in C$ such that $\bigcap_{1 \le i \le n} G(f_i) = \emptyset$. Set $L = \overline{conv}_\rho(\{f_i\})$ in L_ρ. Our assumptions imply that $L \cap G(f_i)$ is ρ-closed for every $i = 1, 2, \ldots, n$. Using Theorem 2.1 (2) with $\alpha = 1$, $L \cap G(f_i)$ is closed for the Luxemburg norm $\|\cdot\|_\rho$ for any $i \in \{1, \ldots, n\}$. Thus for every $f \in L$, there exists i_0 such that f does not belong to $L \cap G(f_{i_0})$ since $L \bigcap \left(\bigcap_{1 \le i \le n} G(f_i)\right) = \emptyset$.

Hence

$$d\left(f, L \cap G(f_{i_0})\right) = \inf\{\|f - g\|_\rho;\ g \in L \cap G(f_{i_0})\} > 0,$$

because $L \cap G(f_{i_0})$ is closed. We use the function

$$\alpha(f) = \sum_{1 \le i \le n} d\left(f, L \bigcap G(f_i)\right) > 0$$

where $f \in K = conv\{f_1, \ldots, f_n\}$ to define the map $T : K \to K$ by

$$T(f) = \frac{1}{\alpha(f)} \sum_{1 \le i \le n} d\left(f, L \bigcap G(f_i)\right) f_i \,.$$

Clearly, T is a continuous map. Since K is a compact convex subset of the Banach space $(L_\rho, \|f\|_\rho)$, Brouwer's theorem implies the existence of a fixed point $f_0 \in K$ of T, i.e. $T(f_0) = f_0$. Set

$$I = \left\{i; d\left(f_0, L \bigcap G(f_i)\right) \neq 0\right\} \,.$$

Clearly,

$$f_0 = \frac{1}{\alpha(f_0)} \sum_{i \in I} d\left(f_0, L \bigcap G(f_i)\right) f_i \,.$$

Hence, $f_0 \notin \bigcup_{i \in I} G(f_i)$ and $f_0 \in conv(\{f_i; i \in I\})$ as this contradicts the assumption

$$conv\left(\{f_i; i \in I\}\right) \subset \bigcup_{i \in I} G(f_i) \,.$$

□

As an immediate consequence, we obtain the following result:

Theorem 3.2. *Let $\rho \in \Re$. Let $C \subset L_\rho$ be nonempty and $G : C \to 2^{L_\rho}$ be a KKM-mapping such that for any $f \in C$, $G(f)$ is nonempty and ρ-closed. Assume there exists $f_0 \in C$ such that $G(f_0)$ is ρ-compact. Then, we have*

$$\bigcap_{f \in C} G(f) \neq \emptyset .$$

Notice that the ρ-compactness of $G(f_0)$ may be weakened, i.e. we can still reach the conclusion if one involves an auxiliary multivalued map and a suitable topology on L_ρ.

Theorem 3.3. *Let $\rho \in \Re$. Let $C \subset L_\rho$ be nonempty and $G : C \to 2^{L_\rho}$ a KKM-mapping such that for any $f \in C$, $G(f)$ is nonempty and finitely ρ-closed. Assume there is a multivalued map $K : C \to 2^{L_\rho}$ such that $G(f) \subset K(f)$ for every $f \in C$ and*

$$\bigcap_{f \in C} K(f) = \bigcap_{f \in C} G(f) .$$

If there is some topology τ on L_ρ such that each $K(f)$ is τ-compact, then

$$\bigcap_{f \in C} G(f) \neq \emptyset .$$

Proof. The proof is obvious. □

Before we state an analogue to Ky Fan fixed point result [26], we need the following definition

Definition 3.2. *Let $\rho \in \Re$. Let $C \subset L_\rho$ be a nonempty ρ-closed subset. Let $T : C \to L_\rho$ be a map. T is called ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f. Also T will be called strongly ρ-continuous if T is ρ-continuous and*

$$\liminf_{n \to \infty} \rho(g - T(f_n)) = \rho(g - T(f)),$$

for any sequence $\{f_n\} \subset C$ which ρ-converges to f and for any $g \in C$.

It is not clear for what type of modular ρ, ρ-continuity implies strong ρ-continuity. The Δ_2-property is enough to provide this implication. The following technical lemma is needed to prove the analogue of Ky Fan fixed point result.

Lemma 3.1. *Let $\rho \in \Re$. Let $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $T : K \to L_\rho$ be strongly ρ-continuous. Then, there exists $f_0 \in K$ such that*

$$\rho(f_0 - T(f_0)) = \inf_{f \in K} \rho\left(f - T(f_0)\right) .$$

Proof. Consider the map $G : K \to 2^{L_\rho}$ defined by

$$G(g) = \left\{f \in K; \rho(f - T(f)) \leq \rho(g - T(f))\right\} .$$

Since T is strongly ρ-continuous, for any sequence $\{f_n\} \subset G(g)$ which ρ-converges to f, we have

$$\rho(f - T(f)) \leq \liminf_{n \to \infty} \rho(f_n - T(f_n)) \leq \liminf_{n \to \infty} \rho(g - T(f_n)) = \rho(g - T(f)),$$

on the basis of the Fatou property and the continuity of T. Clearly, this implies that $G(g)$ is ρ-closed for any $g \in K$. Next, we show that G is a KKM-mapping. Assume not. Then, there exists $\{g_1, ..., g_n\} \subset K$ and $f \in conv(\{g_i\})$ such that $f \notin \bigcup_{1 \leq i \leq n} G(g_i)$. This clearly implies

$$\rho(g_i - T(f)) < \rho(f - T(f)) , \quad \text{for } i = 1, \ldots, n .$$

Let $\varepsilon > 0$ be such that $\rho(g_i - T(f)) \le \rho(f - T(f)) - \varepsilon$, for $i = 1, 2, ..., n$. Since ρ is convex, for any $g \in conv(\{g_i\})$, we have

$$\rho(g - T(f)) \le \rho(f - T(f)) - \varepsilon.$$

As $f \in conv(\{g_i\})$, so we get $\rho(f - T(f)) \le \rho(f - T(f)) - \varepsilon$. Contradiction. Therefore, G is a KKM-mapping. By the ρ-compactness of K, we deduce that $G(g)$ is compact for any $g \in K$. Theorem 3.2 implies the existence of $f_0 \in \bigcap_{g\in K} G(g)$. Hence, $\rho(f_0 - T(f_0)) \le \rho(g - T(f_0))$ for any $g \in K$. In particular, we have

$$\rho(f_0 - T(f_0)) = \inf_{g\in K} \rho\left(g - T(f_0)\right).$$

□

We are now ready to state Ky Fan fixed point theorem [26] in modular function spaces.

Theorem 3.4. *Let $\rho \in \Re$. Let $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $T : K \to L_\rho$ be strongly ρ-continuous. Assume that for any $f \in K$, with $f \ne T(f)$, there exists $\alpha \in (0, 1)$ such that*

$$(*) \quad K \cap B_\rho\left(f, \alpha\rho(f - T(f))\right) \cap B_\rho\left(T(f), (1-\alpha)\rho(f - T(f))\right) \ne \emptyset .$$

Then, T has a fixed point, i.e. $T(g) = g$ for some $g \in K$.

Proof. From the previous lemma, there exists $f_0 \in K$ such that

$$\rho(f_0 - T(f_0)) = \inf_{g\in K} \rho\left(g - T(f_0)\right).$$

We claim that f_0 is a fixed point of T. Assume not, i.e. $f_0 \ne T(f_0)$. Then, our assumption on K implies the existence of $\alpha \in (0, 1)$ such that

$$K_0 = K \cap B_\rho\left(f_0, \alpha\rho(f_0 - T(f_0))\right) \cap B_\rho\left(T(f_0), (1-\alpha)\rho(f_0 - T(f_0))\right) \ne \emptyset .$$

Let $g \in K_0$. Then, $\rho(g - T(f_0)) \le (1 - \alpha)\, \rho(f_0 - T(f_0))$. This implies a contradiction to the property satisfied by f_0.

□

Note that the condition (*) is satisfied if $T(K) \subset K$ which implies the following result:

Theorem 3.5. *Let $\rho \in \Re$. Let $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $T : K \to K$ be strongly ρ-continuous. Then, T has a fixed point, i.e. $T(g) = g$ for some $g \in K$.*

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Author details

[1]Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA [2]Department of Mathematics, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Authors' contributions

All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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