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On asymptotic pointwise nonexpansive mappings in modular function spaces

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ABSTRACT

We prove the existence of fixed points of asymptotic pointwise nonexpansive mappings in modular function spaces.

1. Introduction

The purpose of this paper is to give an outline of a fixed point theory for asymptotic pointwise nonexpansive mappings defined on some subsets of modular function spaces which are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii spaces and many others. Recently, the authors presented a series of fixed point results for pointwise contractions and asymptotic pointwise contractions acting in modular functions spaces [15]. The current paper operates within the same framework of convex function modulars. Methods used in the current paper are based on notions of modular uniform convexity and hence differ from the methods in [15] which exploited the ideas of a modular version of the Opial property and of a uniform continuity of function modulars and their use for proving fixed point theorems.

The importance for applications of nonexpansive mappings in modular function spaces consists in the richness of structure of modular function spaces, that – besides being Banach spaces (or F-spaces in a more general settings) – are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

The theory of contractions and nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s (see e.g. [4,8,20,6,5]), and generalized to other metric spaces (see e.g. [7,2,14]), and modular

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function spaces (see e.g. [16,17,12]). The corresponding fixed point results were then extended to larger classes of mappings like asymptotic mappings [21,13], pointwise contractions [19] and asymptotic pointwise contractions and nonexpansive mappings [22,23,10,15].

As noted in [15], questions are sometimes asked whether the theory of modular function spaces provides general methods for the consideration of fixed point properties, similarly as this is the case in the Banach space setting. We believe that this paper, building upon [15], provides further evidence for the existence of such a general theory. Indeed, the most common approach in the Banach space fixed point theory for generalized nonexpansive mappings is to assume the uniform convexity of the norm which implies the reflexivity, and – via the Milman Theorem – guarantees the weak compactness of the closed bounded sets. In this paper we introduce and investigate a notion of a uniform convexity of function modulars, which in conjunction with the property (R) being the modular equivalent of the Banach space reflexivity [17,15], equips us with the powerful tools for proving the fixed point property for asymptotic pointwise nonexpansive (in the modular sense) mappings. Let us recall that the property (R) represents the most important, from the fixed point theory viewpoint, geometric characterization of reflexive spaces: every nonincreasing sequence of nonempty, convex, bounded sets has a nonempty intersection. The property (R) also aligns well to the metric equivalents of reflexivity defined by the notions of compact convexity structures [10]. The other building blocks of our theory are: (a) The unique best approximant property for nonempty, convex and closed (in the modular sense) sets; (b) The parallelogram property being a generalization of the norm parallelogram property in uniformly convex Banach spaces; (c) The minimizing sequence property which states that any minimizing sequence for any type function defined in a closed and bounded (in the modular sense) set, is convergent and that its modular limit is independent of the choice of a minimizing sequence. The working of our theory can be summarised as follows:

(1) The uniform convexity property implies the unique best approximant property (Theorem 3.1).
(2) The uniform convexity property via the unique best approximant property implies the property (R) (Theorem 3.3).
(3) The uniform convexity property implies the parallelogram property (Lemma 4.2).
(4) The parallelogram property implies the minimizing sequence property for type functions when the minimum is strictly positive (Lemma 4.3, Case 1).
(5) The property (R) implies the minimizing sequence property for type functions when the minimum is equal to zero (Lemma 4.3, Case 2).
(6) The minimizing sequence property for type functions implies the fixed point property for asymptotic pointwise nonexpansive mappings (Theorem 4.1); the modular limit of a minimizing sequence for a type function defined by an orbit is an obvious candidate for a fixed point. We prove that this is indeed the case.

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material and establishes the terminology and key notation conventions.
(b) Section 3 gives a brief exposition of the theory of the uniform convexity of a function modular.
(c) Section 4 presents the fixed point theory for asymptotic pointwise nonexpansive mappings acting in modular function spaces.

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leqslant |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e. $|f(\omega)| \leqslant |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leqslant \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e. $\rho(f 1_{A\cup B}) \leqslant \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;
(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g 1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \big\{ f \in \mathcal{M}_\infty;\ \big|f(\omega)\big| < \infty\ \rho\text{-a.e.} \big\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2. Let ρ be a regular function pseudomodular.

(1) We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.;
(2) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f 1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Definition 2.1.1 in [26] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [24–26], see also Musielak [30] for the basics of the general modular theory.

Remark 2.1. We limit ourselves to convex function modulars in this paper. However, omitting convexity in Definition 2.1 or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in [26].

Definition 2.3. (See [24–26].) Let ρ be a convex function modular.

(a) A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \big\{ f \in \mathcal{M};\ \rho(\lambda f) \to 0 \text{ as } \lambda \to 0 \big\}.$$

(b) The following formula defines a norm in L_ρ (frequently called the Luxemburg norm):

$$\|f\|_\rho = \inf\big\{ \alpha > 0;\ \rho(f/\alpha) \leqslant 1 \big\}.$$

In the following theorem we recall some of the properties of modular spaces that will be used later on in this paper.

Theorem 2.1. *(See [24–26].) Let $\rho \in \Re$.*

(1) *$(L_\rho, \|f\|_\rho)$ is complete and the norm $\|\cdot\|_\rho$ is monotone w.r.t. the natural order in $\mathcal{M}$.*
(2) *$\|f_n\|_\rho \to 0$ if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.*
(3) *If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ ρ-a.e.*
(4) *If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$ then $\rho(\alpha(f_n - f), E) \to 0$ for every $\alpha > 0$.*
(5) *Let $f_n \to f$ ρ-a.e. There exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every H_k (Egoroff Theorem).*
(6) *$\rho(f) \leqslant \liminf \rho(f_n)$ whenever $f_n \to f$ ρ-a.e. (Note: this property is equivalent to the Fatou property.)*
(7) *Defining $L^0_\rho = \{f \in L_\rho;\ \rho(f, \cdot) \text{ is order continuous}\}$ and $E_\rho = \{f \in L_\rho;\ \lambda f \in L^0_\rho \text{ for every } \lambda > 0\}$ we have:*
   (a) *$L_\rho \supset L^0_\rho \supset E_\rho$,*
   (b) *E_ρ has the Lebesgue property, i.e. $\rho(\alpha f, D_k) \to 0$ for $\alpha > 0$, $f \in E_\rho$ and $D_k \downarrow \emptyset$,*
   (c) *E_ρ is the closure of $\mathcal{E}$ (in the sense of $\|\cdot\|_\rho$).*

The following definition plays an important role in the theory of modular function spaces.

Definition 2.4. Let $\rho \in \Re$. We say that ρ has the Δ_2-property if $\sup_n \rho(2f_n, D_k) \to 0$ whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$.

Theorem 2.2. *Let $\rho \in \Re$. The following conditions are equivalent:*

(a) *ρ has Δ_2,*
(b) *L^0_ρ is a linear subspace of L_ρ,*
(c) *$L_\rho = L^0_\rho = E_\rho$,*
(d) *if $\rho(f_n) \to 0$ then $\rho(2f_n) \to 0$,*
(e) *if $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then $\|f_n\|_\rho \to 0$, i.e. the modular convergence is equivalent to the norm convergence.*

The following definition is crucial throughout this paper.

Definition 2.5. Let $\rho \in \Re$.

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to 0(\rho)$ if and only if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.
(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B.
(d) A set $B \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g);\ f \in B,\ g \in B\} < \infty$.
(e) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = \inf\bigl\{\rho(f - g);\ g \in C\bigr\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f$, $\lambda > 1$. Using Theorem 2.1 it is not difficult to prove the following

Proposition 2.1. *Let $\rho \in \Re$.*

(i) *L_ρ is ρ-complete,*
(ii) *ρ-balls $B_\rho(x, r) = \{y \in L_\rho;\ \rho(x - y) \leqslant r\}$ are ρ-closed.*

The following property plays in the theory of modular function spaces a role similar to the reflexivity in Banach spaces (see e.g. [17]).

Definition 2.6. We say that L_ρ has the property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.

Let us introduce a notion of a ρ-type, a powerful technical tool which will be used in the proofs of our fixed point results.

Definition 2.7. Let $K \subset L_\rho$ be convex and ρ-bounded.

(1) A function $\tau : K \to [0, \infty]$ is called a (ρ)-type (or shortly a type) if there exists a sequence $\{y_m\}$ of elements of K such that for any $z \in K$ there holds

$$\tau(z) = \limsup_{m\to\infty} \rho(y_m - z).$$

(2) Let τ be a type. A sequence $\{g_n\}$ is called a minimizing sequence of τ if

$$\lim_{n\to\infty} \tau(g_n) = \inf\bigl\{\tau(f);\ f \in K\bigr\}.$$

Note that τ is convex provided ρ is convex.

Let us finish this section with the modular definitions of asymptotic pointwise nonexpansive mappings. The definitions are straightforward generalizations of their norm and metric equivalents [21–23,10].

Definition 2.8. Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\rho\bigl(T^n(f) - T^n(g)\bigr) \leqslant \alpha_n(f)\rho(f - g) \quad \text{for any } f, g \in L_\rho.$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$, then T is called an asymptotic pointwise contraction.
(ii) If $\limsup_{n\to\infty} \alpha_n(f) \leqslant 1$ for any $f \in L_\rho$, then T is called an asymptotic pointwise nonexpansive mapping.
(iii) If $\limsup_{n\to\infty} \alpha_n(f) \leqslant k$ for any $f \in L_\rho$ with $0 < k < 1$, then T is called a strongly asymptotic pointwise contraction.

3. Uniform convexity in modular function spaces

This section is devoted to the discussion of the modular equivalents of uniform convexity of ρ. As demonstrated below, one concept of uniform convexity in normed spaces generates several different types of uniform convexity in modular function spaces.

Definition 3.1. Let $\rho \in \Re$. We define the following uniform convexity type properties of the function modular ρ:

(i) Let $r > 0$, $\varepsilon > 0$. Define

$$D_1(r,\varepsilon) = \big\{(f,g);\ f,g \in L_\rho,\ \rho(f) \leqslant r,\ \rho(g) \leqslant r,\ \rho(f-g) \geqslant \varepsilon r\big\}.$$

Let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right);\ (f,g) \in D_1(r,\varepsilon)\right\}, \quad \text{if } D_1(r,\varepsilon) \neq \emptyset,$$

and $\delta_1(r,\varepsilon) = 1$ if $D_1(r,\varepsilon) = \emptyset$. We say that ρ satisfies $(UC1)$ if for every $r > 0$, $\varepsilon > 0$, $\delta_1(r,\varepsilon) > 0$. Note, that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(ii) We say that ρ satisfies $(UUC1)$ if for every $s \geqslant 0$, $\varepsilon > 0$ there exists

$$\eta_1(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \quad \text{for } r > s.$$

(iii) Let $r > 0$, $\varepsilon > 0$. Define

$$D_2(r,\varepsilon) = \left\{(f,g);\ f,g \in L_\rho,\ \rho(f) \leqslant r,\ \rho(g) \leqslant r,\ \rho\left(\frac{f-g}{2}\right) \geqslant \varepsilon r\right\}.$$

Let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right);\ (f,g) \in D_2(r,\varepsilon)\right\}, \quad \text{if } D_2(r,\varepsilon) \neq \emptyset,$$

and $\delta_2(r,\varepsilon) = 1$ if $D_2(r,\varepsilon) = \emptyset$. We say that ρ satisfies $(UC2)$ if for every $r > 0$, $\varepsilon > 0$, $\delta_2(r,\varepsilon) > 0$. Note, that for every $r > 0$, $D_2(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(iv) We say that ρ satisfies $(UUC2)$ if for every $s \geqslant 0$, $\varepsilon > 0$ there exists

$$\eta_2(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \quad \text{for } r > s.$$

(v) We say that ρ is Strictly Convex (SC), if for every $f,g \in L_\rho$ such that $\rho(f) = \rho(g)$ and

$$\rho\left(\frac{f+g}{2}\right) = \frac{\rho(f)+\rho(g)}{2}$$

there holds $f = g$.

Remark 3.1.

(i) Let us observe that for $i = 1,2$, $\delta_i(r,0) = 0$, and $\delta_i(r,\varepsilon)$ is an increasing function of ε for every fixed r.

(ii) Note that

$$\delta_1(r,\varepsilon) = \inf\big\{\delta'(r,h);\ h \in L_\rho,\ \rho(h) \geqslant r\varepsilon\big\}, \tag{3.1}$$

$$\delta_2(r,\varepsilon) = \inf\left\{\delta'(r,h);\ h \in L_\rho,\ \rho\left(\frac{h}{2}\right) \geqslant r\varepsilon\right\}, \tag{3.2}$$

where

$$\delta'(r,h) = \inf\left\{1 - \frac{1}{r}\rho\left(f + \frac{h}{2}\right);\ f \in L_\rho,\ \rho(f) \leqslant r,\ \rho(f+h) \leqslant r\right\}. \tag{3.3}$$

Proposition 3.1. *It is easy to prove the following conditions characterizing relationship between the above defined notions*:

(1) $(UUCi) \Rightarrow (UCi)$ *for* $i = 1,2$.
(2) $\delta_1(r,\varepsilon) \leqslant \delta_2(r,\varepsilon)$.
(3) $(UC1) \Rightarrow (UC2)$.

(4) $(UUC1) \Rightarrow (UUC2)$.
(5) *If ρ is homogenous (e.g. is a norm) then all conditions* $(UC1)$, $(UC2)$, $(UUC1)$, $(UUC2)$ *are equivalent and* $\delta_1(r, 2\varepsilon) = \delta_1(1, 2\varepsilon) = \delta_2(1, \varepsilon) = \delta_2(r, \varepsilon)$.

Remark 3.2. Observe that, denoting $\rho_\alpha(u) = \alpha\rho(u)$, and the corresponding moduli of convexity by $\delta_{\rho_\alpha, i}$, where $i = 1, 2$, we have

$$\delta_{\rho_\alpha,i}(r,\varepsilon) = \delta_{\rho,i}\left(\frac{r}{\alpha}, \varepsilon\right), \tag{3.4}$$

or

$$\delta_{\rho,i}(r,\varepsilon) = \delta_{\rho_\alpha,i}(r\alpha, \varepsilon). \tag{3.5}$$

Hence, ρ is (UCx), where (UCx) is any of the conditions from Definition 3.1, if and only if there exists $\alpha > 0$ such that ρ_α is (UCx). In particular, taking $\alpha = \frac{1}{r}$, it is enough to prove any of the conditions defining (UCx) with $r = 1$.

It turns out that within the class of convex function modulars with the Δ_2 property both notions of uniform convexity coincide.

Proposition 3.2. *Let $\rho \in \Re$ satisfy Δ_2. Then ρ is $(UUC1)$ if and only if ρ is $(UUC2)$.*

Proof. In view of Proposition 3.1, it is enough to prove that $(UUC2)$ implies $(UUC1)$. We claim that to every $M_1 > 0$ there exists $M_2 > 0$ such that

$$\rho(2x) \geqslant M_1 \quad \Rightarrow \quad \rho(x) \geqslant M_2, \quad \text{where } x \in L_\rho. \tag{3.6}$$

Indeed, assume this is not the case. Hence there exists $M_1 > 0$ and a sequence $\{x_n\} \subset L_\rho$ such that $\rho(2x_n) \geqslant M_1$ while $\rho(x_n) \leqslant \frac{1}{n}$ which contradicts Δ_2. Let $r_1 > 0$ and $\varepsilon_1 > 0$ be chosen arbitrarily and let $h \in L_\rho$ be such that $\rho(h) \geqslant M_1 = r_1\varepsilon_1$. Applying (3.6) with $2x = h$ we get an $M_2 > 0$ such that $\rho(\frac{h}{2}) \geqslant M_2$. Let $r_2 > 0$, $\varepsilon_2 > 0$ be such that $M_2 = r_2\varepsilon_2$. Substituting these to (3.1) and (3.2) we get

$$\delta_1(r_1, \varepsilon_1) \geqslant \delta_2(r_2, \varepsilon_2),$$

hence ρ is $(UUC1)$ as claimed. $\square$

Remark 3.3. Note that the uniform convexity of ρ defined in [17] coincides with our $(UC2)$. In the same paper, the authors proved that in Orlicz spaces over a finite, atomless measure space, both conditions $(UC2)$ and $(UUC2)$ are equivalent.

Proposition 3.3. $(UC2) \Rightarrow (SC)$.

Proof. Let $f, g \in L_\rho$, $f \neq g$, $\rho(f) = \rho(g) = r > 0$. Let $h = f - g$. By $(UC2)$ then

$$\rho\left(\frac{f+g}{2}\right) = \rho\left(g + \frac{h}{2}\right) \leqslant \big(1 - \delta_2(r, \varepsilon)\big)r,$$

where $\varepsilon = \frac{1}{r}\rho(h/2) > 0$. Since $\delta_2(r, \varepsilon) > 0$, it follows then that

$$\rho\left(\frac{f+g}{2}\right) = \rho\left(g + \frac{h}{2}\right) < r = \frac{\rho(f) + \rho(g)}{2}. \qquad \square$$

Remark 3.4. It is known that for a wide class of modular function spaces with the Δ_2 property, the uniform convexity of the Luxemburg norm is equivalent to $(UC1)$. For example, in Orlicz spaces this result can be traced to early papers by Luxemburg [28], Milnes [29], Akimovic [1], and Kaminska [11]. It is also known that, under suitable assumptions, $(UC2)$ in Orlicz spaces is equivalent to the very convexity of the Orlicz function [17,32] and that the uniform convexity of the Orlicz function implies $(UC1)$ [11]. Typical examples of the Orlicz functions that do not satisfy the Δ_2 condition but are uniformly convex (and hence very convex) are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$ [29,27]. See also [9] for the discussion of some geometrical properties of Calderon–Lozanovskii and Orlicz–Lorentz spaces.

In the next theorem, we investigate relationship between the uniform convexity of function modulars and the unique best approximant property. This result, will be used in the proof of Theorem 3.2 to establish relationship between the modular uniform convexity and the property (R). For other results on best approximation in modular function spaces see e.g. [18].

Theorem 3.1. *Assume $\rho \in \Re$ is (UUC2). Let $C \subset L_\rho$ be nonempty, convex, and ρ-closed. Let $f \in L_\rho$ be such that $d = d_\rho(f, C) < \infty$. Then there exists a unique best ρ-approximant of f in C, i.e. a unique $g_0 \in C$ such that*

$$\rho(f - g_0) = d_\rho(f, C).$$

Proof. Uniqueness follows from the Strict Convexity (*SC*) of ρ (see Proposition 3.3). Let us prove the existence of the ρ-approximant. Since C is ρ-closed, we may assume without loss of any generality that $d = d_\rho(f, C) > 0$. Clearly there exists a sequence $\{f_n\} \in C$ such that

$$\rho(f - f_n) \leqslant d\left(1 + \frac{1}{n}\right).$$

We claim that $\{\frac{1}{2} f_n\}$ is ρ-Cauchy. Assume to the contrary that this is not the case. There exists then an $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\rho\left(\frac{f_{n_k} - f_{n_p}}{2}\right) \geqslant \varepsilon_0,$$

for any $p, k \geqslant 1$. Since ρ is $(UUC2)$, then ρ is $(UC2)$. Hence

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leqslant \left(1 - \delta_2\left(d(k, p), \frac{\varepsilon_0}{d(k, p)}\right)\right) d(k, p),$$

where $d(k, p) = (1 + \frac{1}{\min(n_p, n_k)})d$. For $p, k \geqslant 1$, we have $d(k, p) \leqslant 2d$. Hence

$$\delta_2\left(d(k, p), \frac{\varepsilon_0}{d(k, p)}\right) \geqslant \delta_2\left(d(k, p), \frac{\varepsilon_0}{2d}\right).$$

Since ρ is $(UUC2)$ then there exists $\eta > 0$ such that

$$\delta_2\left(r, \frac{\varepsilon_0}{2d}\right) \geqslant \eta,$$

for any $r > d/3$. Since $d(k, p) \geqslant d > d/3$, we get

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leqslant (1 - \eta) d(k, p),$$

for any $k, p \geqslant 1$. By the convexity of C, $\frac{f_{n_k} + f_{n_p}}{2} \in C$. Using the definition of d, we get

$$d \leqslant \rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leqslant (1 - \eta) d(k, p),$$

for any $k, p \geqslant 1$. If we let k, p go to infinity, we get $d \leqslant (1 - \eta)d$, which is impossible. Hence $\{\frac{1}{2} f_n\}$ is ρ-Cauchy. By Proposition 2.1, $\{\frac{1}{2} f_n\}$ ρ-converges to a $g \in L_\rho$. Fix $m \geqslant 1$. Since $\{\frac{f_m + f_n}{2}\} \in C$ and ρ-converges to $\frac{f_m}{2} + g$ and C is ρ-closed, then we have $\frac{f_m}{2} + g \in C$. Letting $m \to \infty$, we get $2g \in C$. By Theorem 2.1 parts (2) and (6), passing to a subsequence if necessary, we get

$$\rho(f - 2g) \leqslant \liminf_{n\to\infty} \rho\left(f - g - \frac{f_n}{2}\right) \leqslant \liminf_{n\to\infty} \liminf_{m\to\infty} \rho\left(f - \frac{f_n + f_m}{2}\right).$$

Since ρ is convex, we get

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \rho\left(f - \frac{f_n + f_m}{2}\right) \leqslant \liminf_{n\to\infty} \liminf_{m\to\infty} \frac{\rho(f - f_n) + \rho(f - f_m)}{2} \leqslant d.$$

Hence, $\rho(f - 2g) \leqslant d$. Since $2g \in C$, we get $d \leqslant \rho(f - 2g)$. Therefore, $\rho(f - 2g) = d$. In other words, $g_0 = 2g$ is the ρ-approximant of f in C. □

In our next results, we prove that the modular uniform convexity implies the property (R). As elaborated in the introductory section, this is parallel to the well-known fact that uniformly convex Banach spaces are reflexive. The property (R) will be essential for the proof of the Fixed Point Theorem (Theorem 4.1).

Theorem 3.2. *Assume $\rho \in \Re$ is (UUC2). Let $\{C_n\}$ be a nonincreasing sequence of nonempty, convex, ρ-closed subsets of L_ρ. Assume that there exists $f \in L_\rho$ such that $\sup_{n\geqslant 1} d_\rho(f, C_n) < \infty$. Then, $\bigcap_{n\geqslant 1} C_n \neq \emptyset$.*

Proof. Using the proximinality of ρ-closed convex subsets of L_ρ (Theorem 3.1), for every $n \geqslant 1$ there exists $f_n \in C_n$ such that $\rho(f - f_n) = d_\rho(f, C_n)$. It is easy to show that $\{d_\rho(f, C_n)\}$ is nondecreasing and bounded. Hence $\lim_{n\to\infty} d_\rho(f, C_n) = d$ exists. If $d = 0$, then $d_\rho(f, C_n) = 0$, for any $n \geqslant 1$. Since all sets C_n are ρ-closed, we get $f \in C_n$ for any $n \geqslant 1$, which implies $\bigcap_{n\geqslant 1} C_n \neq \emptyset$. Therefore, we can assume $d > 0$. In this case we claim that $\{\frac{1}{2} f_n\}$ is ρ-Cauchy. Indeed if we assume not, then there exists $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\rho\left(\frac{f_{n_k} - f_{n_p}}{2}\right) \geqslant \varepsilon_0,$$

for any $p, k \geqslant 1$. Since ρ is $(UUC2)$, then ρ is $(UC2)$. Hence

$$\rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leqslant \left(1 - \delta_2\left(d, \frac{\varepsilon_0}{d}\right)\right)d,$$

for any $p, k \geqslant 1$. So

$$d_\rho(f, C_{\min(n_p, n_k)}) \leqslant \rho\left(f - \frac{f_{n_k} + f_{n_p}}{2}\right) \leqslant \left(1 - \delta_2\left(d, \frac{\varepsilon_0}{d}\right)\right)d,$$

for any $p, k \geqslant 1$. If we let $p, k \to \infty$, we will get

$$d \leqslant \left(1 - \delta_2\left(d, \frac{\varepsilon_0}{d}\right)\right)d,$$

which is a contradiction because $\delta_2(d, \frac{\varepsilon_0}{d}) > 0$ by $(UC2)$. Hence $\{\frac{1}{2} f_n\}$ is ρ-Cauchy and it ρ-converges to some $g \in L_\rho$. Let us prove that $2g \in C_n$, for any $n \geqslant 1$. Indeed, we have $\frac{f_k + f_p}{2} \in C_n$, for any $p, k \geqslant n$. Fix any $k \geqslant n$. Since $\{\frac{f_k + f_p}{2}\}$ ρ-converges to $\frac{f_k}{2} + g$ as $p \to \infty$, and C_n is ρ-closed, then $\frac{f_k}{2} + g \in C_n$, for any $k \geqslant n$. If we let $k \to \infty$, we get $2g \in C_n$, for any $n \geqslant 1$. Hence, $\bigcap_{n\geqslant 1} C_n \neq \emptyset$. □

The next result describes the relationship between the modular uniform convexity and the property (R).

Theorem 3.3. *Let $\rho \in \Re$ be $(UUC2)$ then L_ρ has property (R).*

Proof. Let $\{C_n\}$ be a nonincreasing sequence of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ. According to Definition 2.6 we need to demonstrate that $\{C_n\}$ has nonempty intersection. Fix any $f \in C_1$. By the ρ-boundedness of C_1, there exists a finite constant $M > 0$ such that for any $n \geqslant 1$, $\rho(f - g) < M$ for any $g \in C_n \subset C_1$. Hence,

$$\sup_{n\geqslant 1} d_\rho(f, C_n) < \infty.$$

By Theorem 3.2 then, $\bigcap_{n\geqslant 1} C_n \neq \emptyset$. □

4. Fixed point theorem for asymptotic pointwise nonexpansive mappings

We will start with the following lemma being a modular equivalent of a norm result from [31].

Lemma 4.1. *Let $\rho \in \Re$ be $(UUC1)$. If there exists $R > 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \leqslant R, \qquad \limsup_{n\to\infty} \rho(g_n) \leqslant R,$$

and

$$\lim_{n\to\infty} \rho\left(\frac{f_n + g_n}{2}\right) = R,$$

then

$$\lim_{n\to\infty} \rho(f_n - g_n) \to 0.$$

Proof. Assume this is not the case. Let $\gamma > 0$ be arbitrarily chosen. For n sufficiently large, passing to subsequences if necessary we may assume that there exists $\varepsilon > 0$ such that $\rho(f_n) \leqslant R + \gamma$, $\rho(g_n) \leqslant R + \gamma$ and $\rho(f_n - g_n) \geqslant R\varepsilon$, for all $n \geqslant 1$. By $(UUC1)$ then

$$0 < \eta_1(R, \varepsilon) < \delta_1(R+\gamma, \varepsilon) \leqslant 1 - \frac{1}{R+\gamma}\rho\left(\frac{f_n + g_n}{2}\right) \to \frac{\gamma}{R+\gamma}.$$

Letting $\gamma \to 0$ we get the contradiction completing the proof. □

Remark 4.1. The assumption $R > 0$ is important. It is easy to prove that Lemma 4.1 is true for $R = 0$ if and only if ρ satisfies Δ_2 property.

We will establish now a modular version of the parallelogram inequality for uniformly convex modular function spaces. See the papers of Xu [33] and Beg [3] for the norm and metric versions respectively.

Lemma 4.2. *For each $0 < s < r$ and $\varepsilon > 0$ set*

$$\Psi(r, s, \varepsilon) = \inf\left\{\frac{1}{2}\rho^2(f) + \frac{1}{2}\rho^2(g) - \rho^2\left(\frac{f+g}{2}\right)\right\}, \tag{4.1}$$

where the infimum is taken over all $f, g \in L_\rho$ such that $\rho(f) \leqslant r$, $\rho(g) \leqslant r$, $\max(\rho(f), \rho(g)) \geqslant s$, and $\rho(f-g) \geqslant r\varepsilon$.

If $\rho \in \Re$ is (UUC1) then $\Psi(r, s, \varepsilon) > 0$ for any $0 < s < r$ and $\varepsilon > 0$. Moreover, for fixed $r, s > 0$, we have

(i) $\Psi(r, s, 0) = 0$;
(ii) *$\Psi(r, s, \varepsilon)$ is a nondecreasing function of ε;*
(iii) *if $\lim_{n\to\infty} \Psi(r, s, t_n) = 0$, then $\lim_{n\to\infty} t_n = 0$.*

Proof. Using the inequality $2ab \leqslant a^2 + b^2$, for any $a, b \in \mathbb{R}$, one can easily prove

$$\rho^2\left(\frac{f+g}{2}\right) \leqslant \frac{1}{2}\rho^2(f) + \frac{1}{2}\rho^2(g),$$

since ρ is convex. Hence $\Psi(r, s, \varepsilon) \geqslant 0$. Assume that ρ is $(UUC1)$ and that there exist $0 < s < r$ and $\varepsilon > 0$ such that $\Psi(r, s, \varepsilon) = 0$. Then there exist $\{f_n\}$ and $\{g_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{2}\rho^2(f_n) + \frac{1}{2}\rho^2(g_n) - \rho^2\left(\frac{f_n + g_n}{2}\right) = 0 \tag{4.2}$$

and $\rho(f_n) \leqslant r$, $\rho(g_n) \leqslant r$, $\max(\rho(f_n), \rho(g_n)) \geqslant s$, and $\rho(f_n - g_n) \geqslant r\varepsilon$. Since

$$\rho^2\left(\frac{f_n + g_n}{2}\right) \leqslant \left(\frac{\rho(f_n) + \rho(g_n)}{2}\right)^2 \leqslant \frac{\rho^2(f_n) + \rho^2(g_n)}{2},$$

we get

$$\left(\frac{\rho(f_n) - \rho(g_n)}{2}\right)^2 \leqslant \frac{1}{2}\rho^2(f_n) + \frac{1}{2}\rho^2(g_n) - \rho^2\left(\frac{f_n + g_n}{2}\right).$$

This implies $\lim_{n\to\infty}(\rho(f_n) - \rho(g_n)) = 0$. Without loss of any generality, we may assume $\lim_{n\to\infty} \rho(f_n) = R$ exists. This also implies that $\lim_{n\to\infty} \rho(g_n) = R$. By (4.2) we get then

$$\lim_{n\to\infty} \rho(g_n) = \lim_{n\to\infty} \rho\left(\frac{f_n + g_n}{2}\right) = R.$$

Observe that

$$R = \lim_{n\to\infty} \max\bigl(\rho(f_n), \rho(g_n)\bigr) \geqslant s > 0.$$

By Lemma 4.1 then $\rho(f_n - g_n) \to 0$ contradicting the fact that $\rho(f_n - g_n) \geqslant r\varepsilon > 0$. The proofs of (i), (ii) and (iii) are easy. □

The following lemma plays the crucial role in the proof of the Fixed Point Theorem for pointwise asymptotically nonexpansive mappings in modular function spaces.

Lemma 4.3. *Assume that $\rho \in \Re$ is (UUC1). Let C be a ρ-closed ρ-bounded convex nonempty subset. Let τ be a type defined on C. Then any minimizing sequence of τ is ρ-convergent. Its limit is independent of the minimizing sequence.*

Proof. Let $\{f_n\} \subset C$ be such that $\tau(f) = \limsup_{n\to\infty} \rho(f_n - f)$. Denote $\tau_0 = \inf\{\tau(h);\ h \in C\}$. Let $\{g_k\}$ be a minimizing sequence of τ. Since C is ρ-bounded, there exists $R > 0$ such that $\rho(f - g) \leqslant R$ for any $f, g \in C$. The rest of the proof is split into two cases.

Case 1: Assume that $\tau_0 > 0$. Let us choose a $\sigma > 0$ such that $\tau_0 - \sigma > 0$. Let us fix g_m *and* g_k and select a subsequence $\{h_n\}$ of $\{f_n\}$ such that

$$0 < \tau_0 \leqslant \tau\left(\frac{g_m + g_k}{2}\right) = \lim_{n\to\infty} \rho\left(\frac{g_m + g_k}{2} - h_n\right). \tag{4.3}$$

Then for n sufficiently large we have

$$0 < \tau_0 - \sigma \leqslant \rho\left(\frac{g_m + g_k}{2} - h_n\right) \leqslant \max\bigl(\rho(g_m - h_n), \rho(g_k - h_n)\bigr). \tag{4.4}$$

Using (4.1) from Lemma 4.2 then

$$\rho^2\left(\frac{g_m + g_k}{2} - h_n\right) \leqslant \frac{1}{2}\rho^2(g_m - h_n) + \frac{1}{2}\rho^2(g_k - h_n) - \Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_m - g_k)\right),$$

and passing with n to infinity we get

$$\tau^2\left(\frac{g_k + g_m}{2}\right) \leqslant \frac{1}{2}\tau^2(g_k) + \frac{1}{2}\tau^2(g_m) - \Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_k - g_m)\right).$$

Hence

$$\tau_0^2 \leqslant \frac{1}{2}\tau^2(g_k) + \frac{1}{2}\tau^2(g_m) - \Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_k - g_m)\right),$$

for any $k, m \geqslant 1$. So

$$\Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_k - g_m)\right) \leqslant \frac{1}{2}\tau^2(g_k) + \frac{1}{2}\tau^2(g_m) - \tau_0^2.$$

Hence $\lim_{k,m\to\infty} \Psi(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_k - g_m)) = 0$. The properties satisfied by Ψ imply that $\{g_k\}$ is ρ-Cauchy. Since L_ρ is ρ-complete and C is ρ-closed, then $\{g_k\}$ is ρ-convergent to some point $g \in C$. Let us prove that any other minimizing sequence also ρ-converges to g. Indeed let $\{u_n\} \in C$ be any minimizing sequence of τ. Using the same argument as previously we have

$$\tau_0^2 \leqslant \tau^2\left(\frac{g_n + u_n}{2}\right) \leqslant \frac{1}{2}\tau^2(g_n) + \frac{1}{2}\tau^2(u_n) - \Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_n - u_n)\right),$$

for some $\sigma > 0$ such that $\tau_0 - \sigma > 0$ and

$$\Psi\left(R, \tau_0 - \sigma, \frac{1}{R}\rho(g_n - u_n)\right) \leqslant \frac{1}{2}\tau^2(g_n) + \frac{1}{2}\tau^2(u_n) - \tau_0^2,$$

for any $n \geqslant 1$. As before, we get $\lim_{n\to\infty} \rho(g_n - u_n) = 0$. Since ρ is convex, we get

$$\rho\left(\frac{(u - g)}{3}\right) \leqslant \frac{1}{3}\rho(u - u_n) + \frac{1}{3}\rho(u_n - g_n) + \frac{1}{3}\rho(g_n - g),$$

where u is the ρ-limit of $\{u_n\}$. Clearly our assumptions imply that $\rho(\frac{(u-g)}{3}) = 0$ or $u = g$. This completes the proof of the lemma for Case 1.

Case 2: Assume that $\tau_0 = 0$. Let

$$K = \bigcap_{n\geqslant 1} \overline{\mathrm{conv}}_\rho\bigl(\{f_k;\ k \geqslant n\}\bigr),$$

which is nonempty in view of the property (R); recall $(UUC1)$ implies (R) by Theorem 3.3. Let $f_\infty \in K$. Let $h \in C$, $\varepsilon > 0$. By definition of τ, there exists $n_0 > 0$ such that for every $n > n_0$

$$\rho(f_n - h) \leqslant \tau(h) + \varepsilon.$$

Therefore, $f_n \in B_\rho(h, \tau(h) + \varepsilon)$ for $n > n_0$. This fact implies

$$K \subset \overline{\mathrm{conv}}_\rho\bigl(\{f_n;\ n \geqslant n_0\}\bigr) \subset B_\rho\bigl(h, \tau(h) + \varepsilon\bigr).$$

Hence $f_\infty \in B_\rho(h, \tau(h)+\varepsilon)$. Since this is true for every $\varepsilon > 0$, there holds $f_\infty \in B_\rho(h, \tau(h))$, i.e.

$$\rho(f_\infty - h) \leqslant \tau(h). \tag{4.5}$$

Let $\{g_k\}$ be a minimizing sequence of τ. Using (4.5) with $h = g_k$ we get

$$\rho(f_\infty - g_k) \leqslant \tau(g_k) \to \tau_0 = 0 \quad \text{as } k \to \infty, \tag{4.6}$$

which means that $\{g_k\}$ is ρ-convergent to f_∞. Since this limit is independent of the sequence $\{g_k\}$, the proof of Case 2 is complete and of the lemma. □

Using the above results, we are ready to prove the main result of this paper.

Theorem 4.1. *Assume $\rho \in \Re$ is (UUC1). Let C be a ρ-closed ρ-bounded convex nonempty subset. Then any $T : C \to C$ pointwise asymptotically nonexpansive has a fixed point. Moreover, the set of all fixed points $Fix(T)$ is ρ-closed.*

Proof. Let $f \in C$. Define the type

$$\tau(h) = \limsup_{n\to\infty} \rho\big(T^n(f) - h\big), \quad \text{for any } h \in C.$$

Let $\tau_0 = \inf\{\tau(h);\ h \in C\}$. Let $\{g_n\} \subset C$ be a minimizing sequence of τ and $g \in C$ its ρ-limit which exists in view of Lemma 4.3. Let us prove that g is a fixed point of T. First notice that $\tau(T^m(h)) \leqslant \alpha_m(h)\tau(h)$, for any $h \in C$ and $m \geqslant 1$. In particular, we have $\tau(T^m(g_n)) \leqslant \alpha_m(g_n)\tau(g_n)$, for any $n, m \geqslant 1$. By induction, we build an increasing sequence $\{m_k\}$ such that $\alpha_{m_k+m}(g_k) \leqslant 1 + \frac{1}{k}$, for $k, m \geqslant 1$. Indeed, since T is pointwise asymptotically nonexpansive, we have $\limsup_{m\to\infty} \alpha_m(g_1) \leqslant 1$. So there exists $m_1 \geqslant 1$ such that for any $m \geqslant m_1$ we have $\alpha_m(g_1) \leqslant 1 + \frac{1}{1}$. Since $\limsup_{m\to\infty} \alpha_m(g_2) \leqslant 1$, there exists $m_2 > m_1$ such that for any $m \geqslant m_2$, we have $\alpha_m(g_2) \leqslant 1 + \frac{1}{2}$. Assume m_k is built, then since $\limsup_{m\to\infty} \alpha_m(g_{k+1}) \leqslant 1$, there exists $m_{k+1} > m_k$ such that for any $m \geqslant m_{k+1}$, we have $\alpha_m(g_{k+1}) \leqslant 1 + \frac{1}{k+1}$, which completes our induction claim. This forces $\{T^{m_k+p}(g_k)\}$ to be a minimizing sequence of τ, for any $p \geqslant 0$. Lemma 4.3 implies $\{T^{m_k+p}(g_k)\}$ is ρ-convergent to g, for any $p \geqslant 0$. In particular, we have $\{T^{m_k+1}(g_k)\}$ is ρ-convergent to g. Since

$$\rho\big(T^{m_k+1}(g_k) - T(g)\big) \leqslant \alpha_1(g)\rho\big(T^{m_k}(g_k) - g\big),$$

we conclude that $\{T^{m_k+1}(g_k)\}$ is also ρ-convergent to $T(g)$. Since the ρ-limit of any ρ-convergent sequence is by Lemma 4.3 unique, we must have $T(g) = g$. To prove that $Fix(T)$ is ρ-closed, let $f_n \in Fix(T)$ and $\rho(f_n - f) \to 0$. Observe that

$$\begin{aligned} \rho\left(\frac{1}{3}\big(T(f) - f\big)\right) &\leqslant \rho\big(T(f) - T(f_n)\big) + \rho\big(T(f_n) - f_n\big) + \rho(f_n - f) \\ &\leqslant \alpha_1(f)\rho(f_n - f) + \rho(f_n - f) \to 0. \end{aligned}$$

Hence, $f \in Fix(T)$ proving $Fix(T)$ is ρ-closed. □

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