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Delay differential equations: a partially ordered sets approach in vectorial metric spaces

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Abstract

In this paper we develop a fixed point theorem in the partially ordered vector metric space $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ by using vectorial norm. Then we use it to prove the existence of periodic solutions to nonlinear delay differential equations.

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1 Introduction

In this paper we investigate the existence of periodic solutions to delay differential equations in partially ordered metric spaces. More precisely, we consider the following equations:

$$\frac{dx(t)}{dt} = f(t,x_t), \quad t \in I, \tag{1.1}$$

$$x(0) = \varphi(0) = x(\varphi)(\omega), \tag{1.2}$$

$$x(\theta) = \varphi(\theta), \quad \theta \in [-\tau,0], \tag{1.3}$$

where $I = [0,\omega]$, $\omega > 0$, and $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $x(t) \in \mathbb{R}^n$ and $x_t(\theta) = x(t+\theta)$, where $-\tau \leq \theta \leq 0$, $\tau > 0$. A solution of (1.1) is a function $x(\cdot) \in \mathcal{C}([-\tau,\omega];\mathbb{R}^n)$ which is absolutely continuous differential on every compact interval $[0,\omega]$, $\omega > 0$ and satisfies (1.1) for almost all $t \geq 0$. It is well known from the theory of delay differential equations (see, for example, [1]) that (1.1) admits a unique solution $x(t) = x(t;\varphi)$ for every continuous and Lipschitz function f and every initial condition

$$x(0) = \varphi(0), \qquad x(\theta) = \varphi(\theta), \quad -\tau \leq \theta \leq 0,$$

where $(\varphi(0),\varphi) \in M^2 = \mathbb{R}^n \times \mathcal{C}$, where $\mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is the space of a continuous function from $[-\tau,0]$ into $\mathbb{R}^n$. Moreover, on compact intervals, $x(\cdot,\varphi)$ depends continuously on φ and f. The existence, uniqueness and continuous dependency results for delay differential equations motivate the definition of the state of the system to the pair

$z(t) = (x(t), x_t) \in M^2$, which completely describes the past history of the solution at time $t \geq 0$.

This type of problems was already investigated for delay or ordinary differential equations in real-valued spaces by Drici *et al.* and also Nieto and Rodríguez-López [2–4]; see also [5–10]. The papers cited therein provide additional reading on this topic. Recently, many authors such as Dhage [11] started to investigate the Krasnosel'skii theorem in partially ordered spaces with applications to nonlinear fractional differential equations in real-valued spaces.

The aim of this paper is to extend the fixed point results of contraction mappings in vectorial partially ordered sets by using vectorial norms introduced by Agarwal [12, 13] which will allow us to investigate the existence of periodic solutions of vectorial delay differential equations. For more on fixed point theory, the reader may consult the books [14, 15]. To be able to prove such results, we will need some monotonicity result of the flow. The main difficulty encountered comes from the fact that we are not working on a classical normed vector space. In fact, consider the example of the evolution of a burning zone. The velocity of its evolution is not the same if the burning area is narrow or if it is wide. In this case we cannot study the trajectory of one point without taking into account the others. Therefore we need to consider mappings defined on the subsets of $\mathbb{R}^n$ which have nice properties with respect to a metric distance other than the regular norm.

In this paper we will use heavily the partial order in $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ defined by

$$\varphi \geq \phi \quad \text{if and only if} \quad \varphi_i(\theta) \geq \phi_i(\theta)$$

for any $\theta \in [-\tau, 0]$ and $\varphi = (\varphi_i)_{1 \leq i \leq n}, \phi = (\phi_i)_{1 \leq i \leq n} \in \mathcal{C}$.

In Section 3, we present an existence theorem and some examples of periodic solutions of delay differential equations. Throughout we will make the following vectorial assumption: There exist $\lambda > 0$, $\mu > 0$, with $\mu < \lambda$, such that

$$0 \leq f(t, \varphi) + \lambda\varphi(0) - \bigl(f(t, \phi) + \lambda\phi(0)\bigr) \leq \mu(\varphi - \phi) \tag{1.4}$$

for any $\varphi, \phi \in \mathcal{C}$ such that $\varphi \geq \phi$.

2 Fixed point theorems in $\mathcal{C}([a, b], \mathbb{R}^n)$

The first version of the classical Banach contraction principle in partially ordered metric spaces was given by Ran and Reurings [16]. Their result was used in applications to linear and nonlinear matrix equations.

Theorem 2.1 [16] *Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$ such that every pair of elements of X has an upper and a lower bound. Let $f : X \to X$ be monotone. Assume there exists $k \in [0, 1)$ such that*

$$x \preceq y \quad \Longrightarrow \quad d\bigl(f(x), f(y)\bigr) \leq kd(x, y)$$

for any $x, y \in X$. If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$, then for any $x \in X$, the sequence of iterates $\{f^n(x)\}$ is convergent and its limit is independent of x.

Nieto and Rodríguez-López [3] extended the above theorem to obtain the following.

Theorem 2.2 [3] *Let (X,d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ be order-preserving (or monotone). Assume that there exists $k \in [0,1)$ such that*

$$x \preceq y \quad \Longrightarrow \quad d\big(f(x),f(y)\big) \le kd(x,y)$$

for any $x,y \in X$. Assume that one of the following conditions holds:

(A) *f is continuous and there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$;*
(B) *$(X,d,\preceq)$ is such that for any convergent nondecreasing sequence $\{x_n\}$, we have $x_n \preceq \lim_{n\to\infty} x_n$ for any $n \ge 1$;*
(C) *$(X,d,\preceq)$ is such that for any convergent nonincreasing sequence $\{x_n\}$, we have $\lim_{n\to\infty} x_n \preceq x_n$ for any $n \ge 1$.*

Then f has a fixed point. Moreover, if every pair of elements of X has an upper or a lower bound, then f has a unique fixed point and for any $x \in X$, the orbit $\{f^n(x)\}$ converges to the fixed point of f.

Clearly, the vector linear space $\mathcal{C}([a,b],\mathbb{R}^n)$ is partially ordered. Indeed, for any $f,g \in \mathcal{C}([a,b],\mathbb{R}^n)$, we have $f = (f_i)$ and $g = (g_i)$ with $f_i, g_i \in \mathcal{C}([a,b],\mathbb{R})$ for $i = 1,\dots,n$. We have

$$f \preceq g \quad \Longleftrightarrow \quad f_i \le g_i, \quad i = 1,\dots,n.$$

Let us summarize some basic properties of the partially ordered set $(\mathcal{C}([a,b],\mathbb{R}^n),\preceq)$ that will be needed throughout this work. First if $f,g \in \mathcal{C}([a,b],\mathbb{R}^n)$, then

$$f \vee g = \big(\max(f_i,g_i)\big) \quad \text{and} \quad f \wedge g = \big(\min(f_i,g_i)\big)$$

are in $\mathcal{C}([a,b],\mathbb{R}^n)$. Obviously, we have

$$f \wedge g \preceq f \preceq f \vee g \quad \text{and} \quad f \wedge g \preceq g \preceq f \vee g.$$

Moreover, if $\{f_k\}$ is any nondecreasing (resp. nonincreasing) convergent sequence in $\mathcal{C}([a,b],\mathbb{R}^n)$, we have

$$f_m \preceq \lim_{k\to\infty} f_k \qquad \Big(\text{resp. } \lim_{k\to\infty} f_k \preceq f_m\Big)$$

for any $m \ge 1$.

Clearly, one may define a norm on $\mathcal{C}([a,b],\mathbb{R}^n)$ to make it a normed vector space. But it seems that a vector-valued norm will be useful in our applications to delay differential equations. Indeed, consider the vector-valued function $\|\cdot\| : \mathcal{C}([a,b],\mathbb{R}^n) \to \mathbb{R}^n_+$ defined by

$$\|f\| = \Big(\sup_{x\in[a,b]} \big|f_i(x)\big|\Big)$$

for any $f \in \mathcal{C}([a,b],\mathbb{R}^n)$. Then we have

(1) $\|f\| = 0_{\mathbb{R}^n}$ if and only if $f = 0$;
(2) $\|\lambda f\| = |\lambda|\|f\|$ for any $\lambda \in \mathbb{R}$;

(3) $\|f+g\| \preceq \|f\| + \|g\|$ for any $f,g \in \mathcal{C}([a,b],\mathbb{R}^n)$, where the order in $\mathbb{R}^n$ is the usual pointwise order inherited from $\mathbb{R}$.

Using the vector-valued norm $\|\cdot\|$, one may define the vector-valued distance

$$d(f,g) = \|f-g\|$$

for any $f,g \in \mathcal{C}([a,b],\mathbb{R}^n)$. This concept is not new. For example, Agarwal [12] (see also [13]) showed that the use of vector-valued distances may give better information about solutions of the systems of some differential equations. Note that $(\mathcal{C}([a,b],\mathbb{R}^n),d)$ is a complete metric space. Indeed, let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}([a,b],\mathbb{R}^n)$, *i.e.*,

$$\lim_{m,s\to\infty} d(f_m,f_s) = 0_{\mathbb{R}^n}.$$

Then the sequence $\{(f_i)_m\}$ is a Cauchy sequence in $\mathcal{C}([a,b],\mathbb{R})$ for any $i=1,\ldots,n$. Since $\mathcal{C}([a,b],\mathbb{R})$ is a complete metric space, there exist $f_i \in \mathcal{C}([a,b],\mathbb{R})$ such that $\lim_{m\to\infty}(f_i)_m = f_i$ for $i=1,\ldots,n$. It is easy to check that $\{f_m\}$ converges to (f_i) in $\mathcal{C}([a,b],\mathbb{R}^n)$, *i.e.*,

$$\lim_{m\to\infty} d\bigl(f_m,(f_i)\bigr) = 0_{\mathbb{R}^n}.$$

The above fixed point theorems may be stated in terms of vector-valued distances. To the best of our knowledge, this is the first attempt to do such generalization.

Theorem 2.3 *Let $T : \mathcal{C}([a,b],\mathbb{R}^n) \to \mathcal{C}([a,b],\mathbb{R}^n)$ be order-preserving* (*or monotone*). *Assume that there exists $k \in [0,1)$ such that*

$$f \preceq g \quad \Longrightarrow \quad d\bigl(T(f),T(g)\bigr) \preceq kd(f,g)$$

for any $f,g \in \mathcal{C}([a,b],\mathbb{R}^n)$. Assume that there exists $f_0 \in \mathcal{C}([a,b],\mathbb{R}^n)$ with $f_0 \preceq T(f_0)$ or $T(f_0) \preceq f_0$. Then T has a unique fixed point $h \in \mathcal{C}([a,b],\mathbb{R}^n)$, and for any $f \in \mathcal{C}([a,b],\mathbb{R}^n)$, the sequence of iterates $\{T^m(f)\}$ converges to h.

Proof First let us prove that $\{T^m(f_0)\}$ is a Cauchy sequence in $\mathcal{C}([a,b],\mathbb{R}^n)$. Without loss of generality, let us assume that $f_0 \preceq T(f_0)$. Since T is order-preserving, then we have $T^m(f_0) \preceq T^{m+1}(f_0)$, which implies

$$d\bigl(T^m(f_0),T^{m+1}(f_0)\bigr) \preceq kd\bigl(T^{m-1}(f_0),T^m(f_0)\bigr) \preceq k^m d\bigl(f_0,T(f_0)\bigr)$$

for any $m \geq 1$. Since $k<1$, we get

$$d\bigl(T^m(f_0),T^{m+s}(f_0)\bigr) \preceq \frac{k^s}{1-k} d\bigl(f_0,T(f_0)\bigr).$$

Hence $\{T^m(f_0)\}$ is a Cauchy sequence. Since $\mathcal{C}([a,b],\mathbb{R}^n)$ is complete, there exists $h \in \mathcal{C}([a,b],\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} T^m(f_0) = h.$$

Since $\{T^m(f_0)\}$ is nondecreasing, then we must have $T^m(f_0) \preceq h$ for any $m \geq 1$. The fundamental property satisfied by T will then imply

$$d\bigl(T^{m+1}(f_0), T(h)\bigr) \leq kd\bigl(T^m(f_0), h\bigr)$$

for any $m \geq 1$. Since $\{T^{m+1}(f_0)\}$ converges to h, we get $T(h) = h$ by the uniqueness of the limit. Let $f \in \mathcal{C}([a,b], \mathbb{R}^n)$ be such that $f_0 \preceq f$. Then we have $T^m(f_0) \preceq T^m(f)$ for any $m \geq 1$. So

$$d\bigl(T^m(f_0), T^m(f)\bigr) \leq kd\bigl(T^{m-1}(f_0), T^{m-1}(f)\bigr) \leq k^m d(f_0, f)$$

for any $m \geq 1$. Since $k < 1$, we get

$$\lim_{m\to\infty} d\bigl(T^m(f_0), T^m(f)\bigr) = 0_{\mathbb{R}^n}.$$

Hence $\lim_{m\to\infty} T^m(f) = h$. In order to finish the proof of Theorem 2.3, let $f \in \mathcal{C}([a,b], \mathbb{R}^n)$. Set $g = f \vee f_0$. Since $f_0 \preceq g$, we get $\lim_{m\to\infty} T^m(g) = h$. Using the same argument, we get

$$\lim_{m\to\infty} T^m(f) = \lim_{m\to\infty} T^m(g) = h.$$

The uniqueness of the fixed point follows from the above conclusion. □

3 Application to delay differential equations

Let $\lambda > 0$. Consider the following equivalent problem to (1.1)-(1.2):

$$\frac{dx(t)}{dt} + \lambda x(t) = f(t, x_t) + \lambda x(t), \quad t \in I, \tag{3.1}$$

$$x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \tag{3.2}$$

$$\varphi = x_\omega(\varphi). \tag{3.3}$$

The aim of this section is to give an existence result of periodic solutions of the delay differential equations (1.1)-(1.3). The following lemma will be needed.

Lemma 3.1 *Problem* (3.1)-(3.3) *is equivalent to the integral equation*:

$$x_t(\varphi)(\theta) = \begin{cases} e^{-\lambda\theta} \int_0^\omega G(t,s)(f(s,x_s) + \lambda x(s))\,ds, & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0, \end{cases} \tag{3.4}$$

where

$$G(t,s) = \begin{cases} \frac{e^{-\lambda(t-s)}}{1-e^{-\lambda\omega}}, & \text{if } 0 \leq s \leq t+\theta, \\ \frac{e^{-\lambda(t+\omega-s)}}{1-e^{-\lambda\omega}}, & \text{if } t+\theta < s \leq \omega. \end{cases} \tag{3.5}$$

Proof Let us consider equation (3.1), then we have

$$\frac{d(e^{\lambda t}x(t))}{dt} = \bigl(f(t,x_t) + \lambda x(t)\bigr)e^{\lambda t}, \quad t \in I = [0, \omega].$$

Let us define

$$F(t,x(t),x_t) = f(t,x_t) + \lambda x(t), \quad t \in I = [0,\omega].$$

Hence

$$x_t(\varphi)(\theta) = \begin{cases} \varphi(0)e^{-\lambda(t+\theta)} + \int_0^{t+\theta} e^{-\lambda(t+\theta-s)}F(s,x(s),x_s)\,ds, & \text{if } 0 \le t+\theta \le \omega, \\ \varphi(t+\theta), & \text{if } -\tau \le t+\theta \le 0. \end{cases}$$

If $t+\theta = \omega$, then we have

$$x(\varphi)(\omega) = \varphi(0)e^{-\lambda\omega} + \int_0^{\omega} e^{-\lambda(\omega-s)}F(s,x(s),x_s)\,ds.$$

By using the periodic condition (3.3), we will have

$$x(\varphi)(\omega) = \varphi(0) = \int_0^{\omega} \frac{e^{-\lambda(\omega-s)}}{1-e^{-\lambda\omega}}F(s,x(s),x_s)\,ds.$$

Then, if $0 \le t+\theta \le \omega$, we get

$$x_t(\varphi)(\theta) = \int_0^{\omega} \frac{e^{-\lambda(t+\theta+\omega-s)}}{1-e^{-\lambda\omega}}F(t,x(s),x_s)\,ds + \int_0^{t+\theta} e^{-\lambda(t+\theta-s)}F(t,x(s),x_s)\,ds,$$

which implies that

$$x_t(\varphi)(\theta) = \begin{cases} e^{-\lambda\theta}\int_0^{\omega} G(t,s)F(t,x(s),x_s)\,ds, & \text{if } 0 \le t+\theta \le \omega, \\ \varphi(t+\theta), & \text{if } -\tau \le t+\theta \le 0, \end{cases}$$

where $G(t,s)$ is defined by (3.5). □

Before stating the main result of this section, we will need the following definition.

Definition 3.1 A lower function for (1.1) is a function $\underline{x}(t) \in C([-r,0],\mathbb{R}^n)$ such that

$$\frac{d\underline{x}(t)}{dt} \le f(t,\underline{x}_t), \quad t \in I, \tag{3.6}$$

$$\underline{x}(0) \le \underline{x}(\varphi)(\omega), \tag{3.7}$$

$$\underline{x}(\theta) = \varphi(\theta), \quad \theta \in [-\tau,0]. \tag{3.8}$$

An upper solution for (1.1) satisfies the reversed inequalities.

The following theorem gives a sufficient condition for the existence of periodic solutions of a delay differential equation assuming the existence of a lower or an upper solution.

Theorem 3.1 *Consider problem* (1.1)-(1.2) *with f continuous and satisfying* (1.4). *Then the existence of a lower solution or an upper solution for* (1.1)-(1.2) *provides the existence of a unique solution of* (1.1)-(1.2).

Proof Let

$$S = \{y \in \mathcal{C}([0,\omega],\mathbb{R}^n) \text{ such that } y(0) = \varphi(0) = y(\omega), \varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)\}.$$

We have

$$x(t) = e^{-\lambda\theta} \int_0^\omega G(t-\theta,s)\big(f(s,x_s) + \lambda x(s)\big)\,ds, \tag{3.9}$$

where $G(t-\theta,s)$ is defined by (3.5). For each $y \in S$, we set $\tilde{f}(y)(t) = f(t,x_t)$, where

$$x_t(\theta) = \begin{cases} y(t+\theta), & \text{if } 0 \le t+\theta \le \omega, \\ \varphi(t+\theta), & \text{if } -\tau \le t+\theta \le 0. \end{cases}$$

Let us consider the following operator for each $y \in S$:

$$J(y)(t) = e^{-\lambda\theta} \int_0^\omega G(t-\theta,s)\big(\tilde{f}(y)(s) + \lambda y(s)\big)\,ds. \tag{3.10}$$

Let us now consider the vector-valued metric on S introduced in the previous section:

$$d(w,z) = \|w-z\| = \Big(\sup_{s\in[0,\omega]} \big|w_i(s) - z_i(s)\big|\Big). \tag{3.11}$$

We have from (1.4), for each $z(t)$, $w(t)$ in S such that $z \le w$,

$$0 \le J(w)(t) - J(z)(t) \le \mu e^{-\lambda\theta} \int_0^\omega G(t-\theta,s)\big(w(s) - z(s)\big)\,ds.$$

We consider now the integral

$$\begin{aligned} \int_0^\omega G(t-\theta,s)\big(w(s) - z(s)\big)\,ds &\le \int_0^\omega G(t-\theta,s) d(w,z)\,ds \\ &= \left(\int_0^\omega G(t-\theta,s)\,ds\right) d(w,z). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} e^{-\lambda\theta} \int_0^\omega G(t-\theta,s)\,ds &= e^{-\lambda\theta} \int_0^{t+\theta} G(t-\theta,s)\,ds + e^{-\lambda\theta} \int_{t+\theta}^\omega G(t-\theta,s)\,ds \\ &= \int_0^{t+\theta} \frac{e^{-\lambda(t-s)}}{1-e^{-\lambda\omega}}\,ds + \int_{t+\theta}^\omega \frac{e^{-\lambda(t+\omega-s)}}{1-e^{-\lambda\omega}}\,ds \\ &= \frac{1-e^{-(t+\theta)\lambda}}{\lambda(1-e^{-\lambda\omega})} + \frac{e^{-(t+\theta)\lambda} - e^{-\lambda\omega}}{\lambda(1-e^{-\lambda\omega})} \\ &= \frac{1}{\lambda}. \end{aligned}$$

Then we get

$$0 \le J(w)(t) - J(z)(t) \le \frac{\mu}{\lambda} d(w,z).$$

Therefore we must have

$$d\big(J(w),J(z)\big) \le \frac{\mu}{\lambda} d(w,z).$$

Let us consider the lower solution $u(t)$ of (1.1)-(1.2), then we will have

$$u'(t) + \lambda u(t) \le f(t,u_t) + \lambda u(t), \quad t \in I.$$

Hence

$$u(t) \le e^{-\lambda\theta} \int_0^\omega G(t-\theta,s)\big(f(s,u_s) + \lambda u(s)\big)\,ds = J(u)(t). \tag{3.12}$$

Otherwise, if $u(t)$ is an upper solution, we will get $J(u)(t) \le u(t)$ for any $t \in I$. Theorem 2.3 will enable us to prove the existence and uniqueness of the periodic solution. □

Example 3.1 Let us consider the example of the degenerated linear differential equation [17]:

$$\frac{du}{dt}(t) = Au(t-1), \quad t \ge 0, \tag{3.13}$$

where the $n \times n$-matrix A satisfies $A \neq 0$ and $A^2 = 0$. Choose

$$\phi(\theta) = \frac{1}{2}Ab + b\theta, \quad -1 \le \theta \le 0,$$

where $Ab \neq 0$. Then we will have $\phi(\theta) \neq 0$ for $-1 \le \theta \le 0$. Indeed, if there exists θ_0 such that $\phi(\theta_0) = 0$, then we will have $Ab = -2\theta_0 b$. Since $Ab \neq 0$, then we have $b \neq 0$ and $\theta_0 \neq 0$. This will generate a contradiction because A does not have nonzero eigenvalues. By using a simple integration over the interval $[0,1]$, one can get

$$u(t) = \phi(0) + \int_0^t A\phi(s-1)\,ds = \frac{1}{2}Ab(t-1)^2, \quad 0 \le t \le 1.$$

This implies that $u(t) \neq 0$ on $[0,1)$. And on $[1,2]$, one can get

$$\begin{aligned} u(t) &= u(1) + \int_1^t Ax(s-1)\,ds \\ &= \frac{1}{2}A^2 b \int_1^t (s-2)^2\,ds = 0. \end{aligned}$$

It is clear that $u(t)$ is a periodic solution. If we check condition (1.4) for $t \ge 1$, and for any $\lambda > 0$, $u(t-1) = \frac{1}{2}Ab_1(t-2)^2$ and $v(t-1) = \frac{1}{2}Ab_2(t-2)^2$ such that $Ab_2 < Ab_1$, we will have

$$Au_t(-1) - Av_t(-1) + \lambda\big(u(t) - v(t)\big) = 0.$$

Also we can find $\mu > 0$ such that $u(t-1) \le v(t-1)$ for all $t \ge 1$, and

$$0 \le \mu\big(u(t-1) - v(t-1)\big).$$

Then our vectorial condition (1.4) is also satisfied for the degenerated delay differential equation. Moreover, if we choose a constant vector v such that $Av \leq \frac{1}{2}Ab + b\theta$, then $\underline{u} = Av$ can be considered as a lower solution of the delay differential equation (3.13). This will guarantee the existence of periodic solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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