Uniform Kadec-Klee Property in Banach lattices

F. Chaatit and M. A. Khamsi

Abstract

We prove that a Banach lattice X which does not contain the l_∞^n-uniformly has an equivalent norm which is uniformly Kadec-Klee for a natural topology τ on X. In case the Banach lattice is purely atomic, the topology τ is the coordinatewise convergence topology.

1 Introduction

The uniform Kadec-Klee (in short UKK) property was introduced by R. Huff in [?]. Banach spaces with Schur property, as well as uniformly convex spaces, enjoy the UKK property. P. Enflo characterized Banach spaces which have an equivalent uniformly convex norm as those which are super-reflexive. Similarly R. Huff raised the following question :

Under what conditions does a Banach space possess an equivalent norm which is UKK ?

Spaces with such a property are called UKK-able. Known examples of UKK-able spaces are the Hardy space H_1 of analytic functions on the ball or on the polydisc [?], the Lorentz spaces $L_{p.1}(\mu)$ [?], the trace class C_1 [?], Gower's space which does not contain c_0, l_1, or any reflexive subspace [?]. Using Prus ideas [?], UKK-ability was intensively studied in [?] for Banach spaces with a basis.

R. Huff associated an index to UKK-ability and proved that UKK-ability implies that the index is finite. He asked whether the converse is true. To our knowledge, it is still unknown in the general case. G. Lancien [?] proved that the converse is true for the space of Bochner integrable functions $L^p(X)$. More recently, H. Knaust and E. Odell proved that the converse is true for that Banach lattices with a lower p-estimate for $1 \leq p < \infty$ are τ-UKK-able. In particular, given that the τ topology and the topology of convergence in measure are the same in L_1, we get the following result of [**?**] that L_1 is UKK for the convergence in measure eventhough it fails to be UKK-able for the weak-topology [**?**]. Let us also mention that c_0 is not τ UKK-able since it has an unconditional basis and it is known that it is not UKK-able for the weak topology, its Huff index being infinite.

We also investigate the relationship between super-reflexivity and super-UKK-ability and prove that these notions are equivalent in the lattice case.

2 Basic Definitions and Properties.

A sequence (x_n) in a Banach space X is said to be ε-separated (with $\varepsilon > 0$) if

$$\inf\{||x_n - x_m||;\ n \neq m\} \geq \varepsilon \ .$$

Let θ be any linear topology on X. We say that X has the θ-uniform Kadec-Klee (in short θ-UKK) property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every ε-separated sequence (x_n) in the unit ball of X which is θ-convergent to x, we have that $||x|| \leq 1 - \delta$.

Let X be a Banach lattice. Let u and x be elements of X. define

$$S_u(x) = x^+ \wedge |u| - x^- \wedge |u|$$

where x^+ and x^- are respectively the positive and negative parts of x under the lattice structure of X.

Let X be a Banach lattice. The topology τ on X is the coarsest linear topology for which the maps $x \to |||x| \wedge |u|||$ are continuous at 0 for every $u \in X$.

Thus a basis of neighborhoods of a point $x \in X$ is formed by the sets: $\{y \in X s.t. |||y - x| \wedge u|| \leq \}$ where $u \in X^+$ and > 0. The topology τ was introduced by M.A. Khamsi and Ph. Turpin in [?] to generalize a fixed point theorem of P.K. Lin [?] to the setting of Banach lattices. It was then observed that in the spaces of measurable functions with order continuous norm, τ is the topology of convergence in measure on every set with finite measure. On the other hand, for Banach spaces with an unconditional basis, τ is the topology of coordinatewise convergence. Among the nice properties of this topology is that it is a Hausdorff, linear topology, coarser than the norm; and if K is τ-compact then every sequence (x_n) of points of K has a τ-convergent subsequence [?].

Let X be a Banach lattice. For any x and y in X, $x - S_y(x)$ and $y - S_x(y)$ are disjoints.

We have

$$y - S_x(y) = y^+ - y^- - \Big[y^+ \wedge |x| - y^- \wedge |x|\Big] \ .$$

Therefore, we get

$$(1) \quad \begin{cases} y - S_x(y) &= (y^+ - y^+ \wedge |x|) - (y^- - y^- \wedge |x|) \\ x - S_y(x) &= (x^+ - x^+ \wedge |y|) - (x^- - x^- \wedge |y|) \end{cases}$$

Note that $y^+ - y^+ \wedge |x| \geq 0$ and $y^- - y^- \wedge |x| \geq 0$. The same is true if we exchange x with y. Since

$$y^+ \geq y^+ - y^+ \wedge |x| \quad \text{and} \quad y^- \geq y^- - y^- \wedge |x| \ ,$$

we have

$$0 = y^+ \wedge y^- \geq \Big(y^+ - y^+ \wedge |x|\Big) \wedge \Big(y^- - y^- \wedge |x|\Big) \ .$$

The same is true if we exchange y with x. Thus the decomposition in (1) is a disjoint one in both equations. In order to complete the proof, we need to show that

$$(i) \quad \Big(y^+ - y^+ \wedge |x|\Big) \wedge \Big(x^+ - x^+ \wedge |y|\Big) = 0$$

and

$$(ii) \quad \Big(y^+ - y^+ \wedge |x|\Big) \wedge \Big(x^- - x^- \wedge |y|\Big) = 0 \ .$$

Put $a = \Big(y^+ - y^+ \wedge |x|\Big) \wedge \Big(x^+ - x^+ \wedge |y|\Big)$. It is clear that $0 \leq a \leq y^+ - y^+ \wedge |x|$; therefore,

$$a + y^+ \wedge x^+ \leq a + y^+ \wedge |x| \leq y^+$$

and

$$a + y^+ \wedge x^+ \leq a + |y| \wedge x^+ \leq x^+ \ .$$

Thus, $a + y^+ \wedge x^+ \leq y^+ \wedge x^+$, which implies that $a \leq 0$. This shows that we must have $a = 0$. The proof of (ii) is done similarly.

Let (x_n) be a sequence τ-convergent to x. Then we have

$$\lim_{n\to\infty} ||S_u(x_n - x)|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||S_{x_n - x}(u)|| = 0$$

for every u in X.

By the triangle inequality we have

$$||S_u(x_n - x)|| \leq ||(x_n - x)^+ \wedge |u||| + ||(x_n - x)^- \wedge |u||| \ .$$

Since X is a lattice, there exists $K > 0$ such that $|x| \leq |y|$ implies $||x|| \leq K||y||$ for every x and y in X. We then get

$$||S_u(x_n - x)|| \leq 2K|||x_n - x| \wedge |u|||$$

which implies that

$$\lim_{n\to\infty} ||S_u(x_n - x)|| = 0 \ .$$

On the other hand, since

$$||S_{x_n - x}(u)|| = |||x_n - x| \wedge u^+ - |x_n - x| \wedge u^-|| \leq |||x_n - x| \wedge u^+|| + |||x_n - x| \wedge u^-||$$

which clearly implies that

$$\lim_{n\to\infty} ||S_{x_n - x}(u)|| = 0 \ .$$

We now recall the definition of a lower p-estimate

Let $1 \leq p < \infty$. A Banach lattice X is said to satisfy a lower p-estimate (for disjoint elements) if there exists a constant $C < \infty$ such that, for every choice of pairwise disjoint elements $(x_i)_{1 \leq i \leq n}$ in X, we have

$$\left(\sum_i ||x_i||^p \right)^{1/p} \leq C|| \sum_i x_i|| \ .$$

1: Let X be a Banach space with an unconditional basis. If X is τ-UKK-able then X is UKK-able.

Indeed, as observed in [**?**], in such a situation the τ topology and the coordinatwise topology are the same. Given that a weakly convergent sequence (x_n) to an element $x \in X$ is coordinatwise convergent to x, the τ UKK-ability is clearly seen to be stronger than the UKK-ability.

2: Let X be a Banach lattice. Then X admits an equivalent norm $||.||_0$ which satisfies

$$(*) \quad ||x||_0 + ||y||_0 \leq 2||x + y||_0$$

for any disjoint $x, y \in X$.

Indeed, set

$$||x||_0 = \max\{||x^+||, ||x^-||\} \quad \forall x \in X \ .$$

Then we have for any disjoint elements x, y in X,

$$||x||_0 \leq ||x + y||_0 \quad \text{and} \quad ||y||_0 \leq ||x + y||_0$$

which implies $||x||_0 + ||y||_0 \leq 2||x + y||_0$.

We easily check that $\|.\|_0$ is an equivalent norm, since $\|x\| = \|x^+ + x^-\| \leq$

$\|x^+\| + \|x^-\| \leq 2\|x\|_0$ and $\|x\|_0 \leq ||x^+|| + ||x^-|| \leq 2||x||$.

Note that every Banach lattice has an equivalent norm which satisfies a lower 1-estimate for any two disjoint elements with a constant $c \geq 2$. We don't have to consider the case $c = 2$ since c_0 satisfies such an estimate with $c = 2$ but is not UKK-able, and therefore is not τ-UKK-able by our first remark. But for the case $c < 2$, we do have a positive answer to the UKK-ability problem. This will be our next theorem.

We now want to establish a generalization of a theorem by Kutzarova and Zachariades [**?**] and we will use techniques of Pisier [**?**]. In the case of Banach spaces with basis this version was used by S. Dilworth, M. Girardi and D. Kutzarova in [**?**] as a criterion to show that Gower's space, that does not contain c_0, l_1 or any reflexive subspace, is UKK-able. The proof is classical and we include it here for the sake of completeness.

Let X be a Banach lattice. Suppose that there exists a constant c, with $0 < c < 2$, such that

$$(***) \quad c\,\|x + y\| \geq (\|x\| + \|y\|)$$

for all disjoint x and y in X. Then X satisfies a lower p-estimate for some $1 \leq p < \infty$.

Suppose that there exists such a constant such that (***) is satisfied for any disjoint x_1 and x_2 in X. Then $\inf\left(||x_1||, ||x_2||\right) \leq (c/2)||x_1 + x_2||$.

Similarly, $\inf\Big(||x_1||, ||x_2||, ||x_3||, ||x_4||\Big) \leq (c/2)^2||\sum_{i=1}^4 x_i||$ for all $(x_i)_{1\leq i\leq 4}$ that are disjoint.

So

$$\inf_{1\leq i\leq 2^n}\Big(||x_i||\Big) \leq (c/2)^n||\sum_{i=1}^{2^n} x_i||$$

for all $(x_i)_{1\leq i\leq 2^n}$ that are disjoint.

Now for a general m, there exists a k such that $2^k \leq m \leq 2^{k+1}$. Then

$$\begin{aligned}
||\sum_{i=1}^{m} x_i|| &= ||\sum_{i=1}^{2^k} x_i + \sum_{i=2^k+1}^{m} x_i|| \\
&\geq 1/c\Big(||\sum_{i=1}^{2^k} x_i|| + ||\sum_{i=2^k+1}^{m} x_i||\Big) \\
&\geq 1/c\ ||\sum_{i=1}^{2^k} x_i|| \\
&\geq 1/c\ (2/c)^k \inf_{1\leq i\leq 2^k} ||x_i|| \\
&\geq 1/c\ (2/c)^k \inf_{1\leq i\leq m} ||x_i||
\end{aligned}$$

Thus

$$\inf_{1\leq i\leq m} ||x_i|| \leq \frac{c^{k+1}}{2^k}||\sum_{i=1}^{m} x_i||$$

for all finite disjoint sequences in X.

Claim 1: *There exists a constant c and a real number $p < \infty$ such that*

$$\inf_{1\leq i\leq m} ||x_i|| \leq \frac{c}{m^{1/p}}||\sum_{i=1}^{m} x_i||$$

for all m and disjoint $(x_i)_{1\leq i\leq m}$ in X.

Proof of Claim 1: It suffices then to prove that $(c/2)^k \leq \frac{1}{m^{1/p}}$, which reduces to find a real number p such that $\frac{\log m}{k \log(2/c)} \leq p$. But $2^k \leq m < 2^{k+1}$, so $2\log 2 \geq (\frac{k+1}{k}) \log 2 \geq \frac{\log m}{k}$. Therefore $p = 2\frac{\log 2}{\log(2/c)}$ works, and the proof of Claim 1 is complete.

Claim 2: *For all reals $r > p$, there exists a constant K_r so that for all (x_i) that are disjoint in X, we have*

$$\Big(\sum_{i=1}^{n} ||x_i||^r\Big)^{1/r} \leq K_r || \sum_{i=1}^{n} x_i||.$$

Proof of Claim 2: Suppose that $||x_1|| \leq ||x_2|| \leq .... \leq ||x_n||$.

Then $||x_1|| \leq \frac{c}{n^{1/p}} || \sum_{i=1}^{n} x_i||$, by Claim 1.

Similarly, $||x_2|| \leq \frac{c}{(n-1)^{1/p}} || \sum_{i=2}^{n} x_i|| \leq c \cdot \frac{c}{(n-1)^{1/p}} || \sum_{i=1}^{n} x_i||$.

So $||x_3|| \leq \frac{c}{(n-2)^{1/p}} || \sum_{i=3}^{n} x_i|| \leq c \cdot \frac{c}{(n-2)^{1/p}} || \sum_{i=1}^{n} x_i||$,

and $||x_n|| \leq \frac{c}{(n-(n-1))^{1/p}} ||x_n|| \leq c \cdot \frac{c}{(n-2)^{1/p}} || \sum_{i=1}^{n} x_i||$.

Combining all inequalities together we get:

$$\sum_{i=1}^{n} ||x_i||^r \leq \Big(\sum_{i=1}^{n} \frac{c^{2r}}{i^{r/p}}\Big) || \sum_{i=1}^{n} x_i||^r.$$

So it suffices to let $K_r = \Big(\sum_{i=1}^{\infty} \frac{c^{2r}}{i^{r/p}}\Big)^{1/r}$. The proof of Claim 2 is complete, and the theorem is proved.

3 Main result.

As in [?], assume that X is a Banach lattice satisfying a lower p-estimate $(1 \leq p < \infty)$, and set

$$||x||_0 = \sup \left(\sum_n ||x_n||^p \right)^{1/p} , \tag{1}$$

where the supremum is taken over all pairwise disjoint elements (x_i) such that $x = \sum_i x_i$. It is easy to see that $||.||_0$ is a norm on X for which $||.|| \leq ||.||_0 \leq C||.||$, i.e. the norm $||.||_0$ is equivalent to $||.||$. Moreover we have

$$||x||_0^p + ||y||_0^p \leq ||x + y||_0^p$$

for any disjoint elements x and y in X.

Let X be a Banach lattice with a p-lower estimate for $1 \leq p < \infty$. Then the norm $||.||_0$ (defined by (1)) is τ-UKK.

Let $\varepsilon > 0$ and (x_n) be ε-separated elements in the unit ball of X such that $\tau - \lim x_n = x$. Since for every integers $n \neq m$ we have

$$\varepsilon \leq ||x_n - x_m||_0 \leq ||x_n - x||_0 + ||x_m - x||_0$$

we get

$$\frac{\varepsilon}{2} \leq \liminf_{n \to \infty} ||x_n - x||_0 \ .$$

Since $x_n - x - S_{|x|}(x_n - x)$ and $x - S_{|x_n - x|}(x)$ are disjoints (by Proposition 1), then we have

$$||x_n - x - S_{|x|}(x_n - x)||_0^p + ||x - S_{|x_n - x|}(x)||_0^p \leq ||x_n - x - S_{|x|}(x_n - x) + x - S_{|x_n - x|}(x)||_0^p \ .$$

But $x_n - x - S_{|x|}(x_n - x) + x - S_{|x_n - x|}(x) = x_n - S_{|x|}(x_n - x) - S_{|x_n-x|}(x)$, and since

$$\lim_{n\to\infty} ||S_{|x|}(x_n - x)||_0 = 0 = \lim_{n\to\infty} ||S_{|x_n-x|}(x)||_0,$$

then we have

$$\begin{aligned}\liminf_{n\to\infty} ||x_n - x||_0^p + ||x||_0^p &= \liminf_{n\to\infty} ||x_n - x - S_{|x|}(x_n - x)||_0^p + \liminf_{n\to\infty} ||x - S_{|x_n-x|}(x)||_0^p \\ &\leq \liminf_{n\to\infty} ||x_n||_0^p \leq 1.\end{aligned}$$

Therefore, we obtain

$$||x||_0^p \leq 1 - \liminf_{n\to\infty} ||x_n - x||_0^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p$$

which implies that $||x||_0 \leq 1 - \delta$, with

$$\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} .$$

This completes the proof of Theorem 2.

The proof of the previous Theorem actually shows the following result: Let X be a Banach lattice with a norm $||.||_0$ satisfying the inequality

$$||x||_0^p + ||y||_0^p \leq ||x + y||_0^p$$

for any disjoint elements x and y in X. Then the norm $||.||_0$ (defined by (1)) is τ-UKK.

We then get the following

Let X be a Banach lattice that does not contain l_∞^n uniformly for every n. Then X is τ-UKK-able.

Let X be a Banach lattice which does not contain l_∞^n uniformly for all n. By the result of [?] and [?] X has a lower p-estimate for some $1 \leq p < \infty$. We then use the previous result.

4 Super-UKK-ability in Banach lattices

In [?] the following question is raised: Is super-reflexivity and super-UKK-ability equivalent in any Banach space? In the next proposition, we give a positive answer in the case of Banach lattices. To our knowledge the general question is still open.

Let X be a Banach lattice. X is super-reflexive if and only if X is super-UKK-able.

If X is super-reflexive then X is clearly super-UKK-able. Suppose now that X is super-UKK-able. Take any ultrapower $\Pi X/\mathcal{U}$. This is still a Banach lattice. So it either contain c_0, or l_1, or is reflexive. It can't contain c_0 since $\Pi X/\mathcal{U}$ is UKK-able. If $\Pi X/\mathcal{U}$ contains l_1 then its ultrapower $\Pi\left(\Pi X/\mathcal{U}\right)$ would contain L_1 which is not UKK-able. Therefore $\Pi X/\mathcal{U}$ is reflexive, i.e. X is super-reflexive.

As it was pointed out to us by D. Leung, the proof of the previous proposition actually shows that if X is a super-UKK-able Banach space, then X can-not contain l_∞^n or l_1^n uniformly in n. However, it is well known that in a general Banach space, the latter condition is not sufficient for super-reflexivity. In fact G. Pisier and Q. Xu constructed for every $q > 2$ a non-reflexive space of type 2 and cotype q.

After this work was finished, we learned that P.G. Dodds, T.K. Dodds, P.N. Dowling, C.J. Lennard and F.A. Sukochev [**?**] proved a uniform Kadec Klee result for local convergence in Banach lattices with a lower p-estimate.

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Department of Mathematical Sciences
The University of Missouri at Columbia
65211 Columbia, Missouri.
&
Department of Mathematical Sciences
The University of Texas at El Paso
El Paso, Texas 79968-0514.