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# Approximating common fixed points in hyperbolic spaces

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**Abstract**
We establish strong convergence and $\Delta$-convergence theorems of an iteration scheme associated to a pair of nonexpansive mappings on a nonlinear domain. In particular we prove that such a scheme converges to a common fixed point of both mappings. Our results are a generalization of well-known similar results in the linear setting. In particular, we avoid assumptions such as smoothness of the norm, necessary in the linear case.

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## 1 Introduction

Let $C$ be a nonempty subset of a metric space $(X,d)$ and $T : C \to C$ be a mapping. Denote the set of fixed points of $T$ by $F(T)$. Then $T$ is (i) nonexpansive if $d(Tx,Ty) \le d(x,y)$ for $x,y \in C$ (ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $d(Tx,y) \le d(x,y)$ for $x \in C$ and $y \in F(T)$. For an initial value $x_1 \in C$, Das and Debata [1] studied the strong convergence of Ishikawa iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S\bigl(\beta_n T x_n + (1-\beta_n)x_n\bigr) + (1-\alpha_n)x_n \tag{1.1}$$

for two quasi-nonexpansive mappings $S$, $T$ on a nonempty closed and convex subset of a strictly convex Banach space. Takahashi and Tamura [2] proved weak convergence of (1.1) to a common fixed point of two nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable and strong convergence in a strictly convex Banach space (see also [3, 4]). Mann and Ishikawa iterative procedures are well-defined in a vector space through its built-in convexity. In the literature, several mathematicians have introduced the notion of convexity in metric spaces; for example [5–8]. In this work, we follow the original metric convexity introduced by Menger [9] and used by many authors like Kirk [5, 6] and Takahashi [8]. Note that Mann iterative procedures were also investigated in hyperbolic metric spaces [10, 11].

In this paper we investigate the results published in [2] and generalize them to uniformly convex hyperbolic spaces. A particular example of such metric spaces is the class of CAT(0)-spaces (in the sense of Gromov) and $\mathbb{R}$-trees (in the sense of Tits). Heavy use of the linear structure of Banach spaces in [2] presents some difficulties when extending these results to metric spaces. For example, a key assumption in many of their results is the smoothness of the norm which is hard to extend to metric spaces.

## 2 Menger convexity in metric spaces

Let $(X,d)$ be a metric space. Assume that for any $x$ and $y$ in $X$, there exists a unique metric segment $[x,y]$, which is an isometric copy of the real line interval $[0,d(x,y)]$. Note by $\mathcal{F}$ the family of the metric segments in $X$. For any $\beta \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = (1-\beta)d(x,y) \quad \text{and} \quad d(z,y) = \beta d(x,y).$$

Throughout this paper we will denote such point by $\beta x \oplus (1-\beta)y$. Such metric spaces are usually called *convex metric spaces* [9]. Moreover, if we have

$$d\big(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y\big) \leq \alpha d(p,q) + (1-\alpha)d(x,y)$$

for all $p$, $q$, $x$, $y$ in $X$ and $\alpha \in [0,1]$, then $X$ is said to be a *hyperbolic metric space* (see [11–13]). For $q = y$, the hyperbolic inequality reduces to the convex structure inequality [8]. Throughout this paper, we will assume

$$\alpha x \oplus (1-\alpha)y = (1-\alpha)y \oplus \alpha x$$

for any $\alpha \in [0,1]$ and any $x,y \in X$.

An example of hyperbolic spaces is the family of Banach vector spaces or any normed vector spaces. Hadamard manifolds [14], the Hilbert open unit ball equipped with the hyperbolic metric [15], and the CAT(0) spaces [6, 16–20] (see Example 2.1) are examples of nonlinear structures which play a major role in recent research in metric fixed point theory. A subset $C$ of a hyperbolic space $X$ is said to be convex if $[x,y] \subset C$, whenever $x,y \in C$ (see also [21]).

**Definition 2.1** [22, 23] Let $(X,d)$ be a hyperbolic metric space. For any $r > 0$ and $\varepsilon > 0$, set

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\right\}$$

for any $a \in X$. $X$ is said to be uniformly convex whenever $\delta(r,\varepsilon) > 0$, for any $r > 0$ and $\varepsilon > 0$.

Throughout this paper we assume that if $X$ is a uniformly convex hyperbolic space, then for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ such that

$$\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0 \quad \text{for any } r > s.$$

**Remark 2.1**

(i) We have $\delta(r,0) = 0$. Moreover, $\delta(r,\varepsilon)$ is an increasing function of $\varepsilon$.

(ii) For $r_1 \leq r_2$, we have

$$1 - \frac{r_2}{r_1}\left(1 - \delta\left(r_2, \varepsilon\frac{r_1}{r_2}\right)\right) \leq \delta(r_1,\varepsilon).$$

Next we give a very important example of uniformly convex hyperbolic metric space.

**Example 2.1** [16] Let $(X,d)$ be a metric space. A *geodesic* from $x$ to $y$ in $X$ is a mapping $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, $c$ is an isometry and $d(x,y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) *segment* joining $x$ and $y$. The space $(X,d)$ is said to be a *geodesic space* if every two points of $X$ are joined by a geodesic and $X$ is said to be *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$, which will be denoted by $[x,y]$, and called the segment joining $x$ to $y$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $X$ consists of three points $x_1$, $x_2$, $x_3$ in $X$ (the *vertices* of $\Delta$) and a geodesic segment between each pair of vertices (the *edges* of $\Delta$). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X,d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$. Such a triangle always exists (see [18]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let $\Delta$ be a geodesic triangle in $X$ and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x,y) \le d(\bar{x}, \bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [16]). If $x, y_1, y_2$ are points of a CAT(0) space, then the CAT(0) inequality implies

$$d^2\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \le \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

The above inequality is known as the (CN) inequality of Bruhat and Tits [24]. The (CN) inequality implies that CAT(0) spaces are uniformly convex with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

In a hyperbolic space $X$, (1.1) is written as

$$x_{n+1} = \alpha_n S\bigl(\beta_n T x_n \oplus (1 - \beta_n) x_n\bigr) \oplus (1 - \alpha_n) x_n, \tag{2.1}$$

where $\alpha_n, \beta_n \in [0,1]$. If $S = T$ in (2.1), it reduces to the following Ishikawa iteration process of one mapping:

$$x_{n+1} = \alpha_n T\bigl(\beta_n T x_n \oplus (1 - \beta_n) x_n, n \ge 1,\bigr) \oplus (1 - \alpha_n) x_n, \tag{2.2}$$

where $\alpha_n, \beta_n \in [0,1]$. Let $\{x_n\}$ be a bounded sequence in a metric space $X$ and $C$ be a nonempty subset. Define $r(\cdot, \{x_n\}) : C \to [0, \infty)$, by

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $\rho_C$ of $\{x_n\}$ with respect to $C$ is given by

$$\rho_C = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in C\bigr\}.$$

$\rho$ will denote the asymptotic radius of $\{x_n\}$ with respect to $X$. A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to $C$ if $r(\xi, \{x_n\}) = r(C, \{x_n\}) = \min\{r(x, \{x_n\}) : x \in C\}$. We denote with $A(C, \{x_n\})$, the set of asymptotic centers of $\{x_n\}$ with respect to $C$. When $C = X$, we call $\xi$ an asymptotic center of $\{x_n\}$ and we use the notation $A(\{x_n\})$ instead of $A(X, \{x_n\})$. In general, the set $A(C, \{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or may even contain infinitely many points. Note that in the study of the geometry of Banach spaces, the function $r(\cdot, \{x_n\})$ is also known as a type. For more on types in metric spaces, we refer to [25].

The $\Delta$-convergence, introduced independently several years ago by Kuczumow [26] and Lim [27], is shown in CAT(0) spaces to behave similarly as the weak convergence in Banach spaces.

**Definition 2.2** A bounded sequence $\{x_n\}$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \stackrel{\Delta}{\to} x$ ($\{x_n\}$ $\Delta$-converges to $x$).

In this paper, we study the iteration schemes (2.1)-(2.2) for nonexpansive mappings. We study strong convergence of these iterates in strictly convex hyperbolic spaces and prove $\Delta$-convergence results in uniformly convex hyperbolic spaces without requiring any condition similar to norm Fréchet differentiability.

In the sequel, the following results will be needed.

**Lemma 2.1** [25, 28] *Let $X$ be a hyperbolic metric spaces. Assume that $X$ is uniformly convex. Let $C$ be a nonempty, closed and convex subset of $X$. Then every bounded sequence $\{x_n\} \in X$ has a unique asymptotic center with respect to $C$.*

**Lemma 2.2** [25, 28] *Let $X$ be a hyperbolic metric spaces. Assume that $X$ is uniformly convex. Let $C$ be a nonempty, closed and convex subset of $X$. Let $C$ be a nonempty closed and convex subset of $X$, and $\{x_n\}$ be a bounded sequence in $C$ such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is a sequence in $C$ such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

The following lemma [29] will be useful in studying the sequence generated by (2.1) in uniformly convex metric spaces. Here we give a proof based on the ideas developed in [25].

**Lemma 2.3** *Let $X$ be a uniformly convex hyperbolic space. Then for arbitrary positive numbers $\varepsilon > 0$ and $r > 0$, and $\alpha \in [0,1]$, we have*

$$d\bigl(a, \alpha x \oplus (1-\alpha)y\bigr) \le r\bigl(1 - \delta\bigl(r, 2\min\{\alpha, 1-\alpha\}\varepsilon\bigr)\bigr)$$

*for all $a, x, y \in X$, such that $d(z,x) \le r$, $d(z,y) \le r$, and $d(x,y) \ge r\varepsilon$.*

*Proof* Without loss of generality, we may assume $\alpha < \frac{1}{2}$. In this case, we have $\min\{\alpha, 1 - \alpha\} = \alpha$. Let $a \in X$ be fixed and $x, y \in X$. Set $\bar{x} = 2\alpha x \oplus (1-2\alpha)y$. Since

$$d\left(\frac{1}{2}\bar{x} \oplus \frac{1}{2}y, x\right) \le (1-\alpha)d(x,y) \quad \text{and} \quad d\left(\frac{1}{2}\bar{x} \oplus \frac{1}{2}y, y\right) = \alpha d(x,y),$$

the uniform convexity of $X$ will imply $\frac{1}{2}\bar{x} \oplus \frac{1}{2}y = \alpha x \oplus (1-\alpha)y$. Using the uniform convexity of $X$, we get

$$d\left(a, \frac{1}{2}\bar{x} \oplus \frac{1}{2}y\right) \leq r\big(1 - \delta(r, 2\alpha\varepsilon)\big).$$

Hence

$$d\big(a, \alpha x \oplus (1-\alpha)y\big) \leq r\big(1 - \delta\big(r, 2\min\{\alpha, 1-\alpha\}\varepsilon\big)\big). \qquad \square$$

**Remark 2.2** If $(X, d)$ is uniformly convex, then $(X, d)$ is strictly convex, *i.e.*, whenever

$$d\big(\alpha x \oplus (1-\alpha)y, a\big) = d(x, a) = d(y, a)$$

for $\alpha \in (0, 1)$ and any $x, y, a \in X$, then we must have $x = y$.

The following result is very useful.

**Lemma 2.4** [25] *Let $(X, d)$ be a uniformly convex hyperbolic space. Let $R \in [0, +\infty)$ be such that*

$$\limsup_{n\to\infty} d(x_n, a) \leq R, \qquad \limsup_{n\to\infty} d(y_n, a) \leq R \quad \text{and} \quad \lim_{n\to\infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = R.$$

*Then we have*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

But since we use convex combinations other than the middle point, we will need the following generalization obtained by using Lemma 2.3.

**Lemma 2.5** *Let $(X, d)$ be a uniformly convex hyperbolic space. Let $R \in [0, +\infty)$ be such that $\limsup_{n\to\infty} d(x_n, a) \leq R$, $\limsup_{n\to\infty} d(y_n, a) \leq R$, and*

$$\lim_{n\to\infty} d\big(a, \alpha_n x_n \oplus (1-\alpha_n)y_n\big) = R,$$

*where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then we have*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

A subset $C$ of a metric space $X$ is Chebyshev if for every $x \in X$, there exists $c_0 \in C$ such that $d(c_0, x) < d(c, x)$ for all $c \in C$, $c \neq c_0$. In other words, for each point of the space, there is a well-defined nearest point of $C$. We can then define the nearest point projection $P : X \to C$ by sending $x$ to $c_0$. We have the following result.

**Lemma 2.6** [25] *Let $(X, d)$ be a complete uniformly convex hyperbolic space. Let $C$ be nonempty, convex and closed subset of $X$. Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of $x$ in $C$, i.e., there exists a unique $c_0 \in C$ such that*

$$d(x, c_0) = d(x, C) = \inf\big\{d(x, c); c \in C\big\},$$

*i.e., $C$ is Chebyshev.*

## 3 Convergence in strictly convex hyperbolic space

Let $(X,d)$ be a hyperbolic metric space. Let $C$ be a nonempty closed convex subset of $X$. Let $S,T:C\to C$ be two nonexpansive mappings. Throughout the paper, assume that $F=F(S)\cap F(T)$. Let $x_1\in C$ and $p\in F$ (assuming $F$ is not empty). Set $r=d(x_1,p)$. Then

$$C(x_1)=C\cap B(p,r)=\{x\in C; d(p,x)\le r\}$$

is nonempty and invariant by both $S$ and $T$. Therefore one may always assume that $C$ is bounded provided that $S$ and $T$ have a common fixed point. Moreover, if $\{x_n\}$ is the sequence generated by (2.1), then we have

$$\begin{aligned} d(x_{n+1},p) &= d\big(\alpha_n Sy_n\oplus(1-\alpha_n)x_n,p\big)\\ &\le \alpha_n d(Sy_n,p)+(1-\alpha_n)d(x_n,p)\\ &\le \alpha_n d(y_n,p)+(1-\alpha_n)d(x_n,p)\\ &= \alpha_n d\big(\beta_n Tx_n\oplus(1-\beta_n)x_n,p\big)+(1-\alpha_n)d(x_n,p)\\ &\le \alpha_n\big[\beta_n d(Tx_n,p)+(1-\beta_n)d(x_n,p)\big]+(1-\alpha_n)d(x_n,p)\\ &\le d(x_n,p), \end{aligned}$$

where $y_n=\beta_n Tx_n\oplus(1-\beta_n)x_n$. This proves that $\{d(x_n,p)\}$ is decreasing, which implies that $\lim_{n\to\infty} d(x_n,p)$ exists. Using the above inequalities, we get

$$\begin{aligned} \lim_{n\to\infty} d(x_n,p) &= \lim_{n\to\infty} d\big(\alpha_n Sy_n\oplus(1-\alpha_n)x_n,p\big)\\ &= \lim_{n\to\infty} \alpha_n d(Sy_n,p)+(1-\alpha_n)d(x_n,p)\\ &= \lim_{n\to\infty} \alpha_n d(y_n,p)+(1-\alpha_n)d(x_n,p)\\ &= \lim_{n\to\infty} \alpha_n d\big(\beta_n Tx_n\oplus(1-\beta_n)x_n,p\big)+(1-\alpha_n)d(x_n,p)\\ &= \lim_{n\to\infty} \alpha_n\big[\beta_n d(Tx_n,p)+(1-\beta_n)d(x_n,p)\big]+(1-\alpha_n)d(x_n,p). \end{aligned}$$

The first result of this work discusses the convergence behavior of the sequence generated by (2.1).

**Theorem 3.1** *Let $X$ be a strictly convex hyperbolic space. Let $C$ be a nonempty bounded, closed and convex subset of $X$. Let $S,T:C\to C$ be two nonexpansive mappings. Assume that $F\neq\emptyset$. Let $x_1\in C$ and $\{x_n\}$ be given by* (2.1). *Then the following hold*:

(i) *if $\alpha_n\in[a,b]$ and $\beta_n\in[0,b]$, with $0<a\le b<1$, then $x_{n_i}\to y$ implies $y\in F(S)$;*
(ii) *if $\alpha_n\in[a,1]$ and $\beta_n\in[a,b]$, with $0<a\le b<1$, then $x_{n_i}\to y$ implies $y\in F(T)$;*
(iii) *if $\alpha_n,\beta_n\in[a,b]$, with $0<a\le b<1$, then $x_{n_i}\to y$ implies $y\in F$. In this case, we have $x_n\to y$.*

*Proof* Assume that $x_{n_i}\to y$. Let $p\in F$. Without loss of generality, we may assume $\lim_{n\to\infty}\alpha_{n_i}=\alpha$ and $\lim_{n\to\infty}\beta_{n_i}=\beta$. Since $\{d(x_n,p)\}$ is decreasing, we get

$$\lim_{n_i\to\infty} d(x_n,p)=\lim_{n_i\to\infty} d(x_{n_i},p)=d(y,p).$$

The above inequalities imply

$$\begin{aligned} d(y,p) &= d\big(\alpha S\big(\beta Ty \oplus (1-\beta)y\big) \oplus (1-\alpha)y, p\big) \\ &= \alpha d\big(S\big(\beta Ty \oplus (1-\beta)y\big), p\big) + (1-\alpha)d(y,p) \\ &= \alpha d\big(\beta Ty \oplus (1-\beta)y, p\big) + (1-\alpha)d(y,p) \\ &= \alpha\big[\beta d(Ty,p) + (1-\beta)d(y,p)\big] + (1-\alpha)d(y,p). \end{aligned}$$

Set $r = d(y,p)$. Without loss of generality we may assume $r > 0$ otherwise most of the conclusions in the theorem are trivial. Assume that $\liminf_{n\to\infty} \alpha_n > 0$. Then $\alpha \neq 0$. Hence

$$d\big(S\big(\beta Ty \oplus (1-\beta)y\big), p\big) = d\big(\beta Ty \oplus (1-\beta)y, p\big) = \beta d(Ty,p) + (1-\beta)r = r,$$

which implies $\beta d(Ty,p) = \beta r$. If we assume that $\liminf_{n\to\infty} \beta_n > 0$, then $\beta \neq 0$, which implies $d(Ty,p) = r$.

(1) If $\alpha \in (0,1)$ and $\beta > 0$, then

$$d\big(p, S\big(\beta Ty \oplus (1-\beta)y\big)\big) = d\big(\alpha S\big(\beta Ty \oplus (1-\beta)y\big) \oplus (1-\alpha)y, p\big) = r.$$

The strict convexity of $X$ will imply $S(\beta Ty \oplus (1-\beta)y) = y$.

(2) If $\alpha \in (0,1)$ and $\beta = 0$, then

$$d(p,y) = d\big(p, S(y)\big) = d\big(\alpha S(y) \oplus (1-\alpha)y, p\big).$$

The strict convexity of $X$ will imply $S(y) = y$.

(3) If $\beta \in (0,1)$ and $\alpha > 0$, then

$$d(p,y) = d\big(p, T(y)\big) = d\big(p, \beta Ty \oplus (1-\beta)y\big).$$

The strict convexity of $X$ will imply $T(y) = y$.

(4) If $\alpha, \beta \in (0,1)$, then $T(y) = y$ and $S(\beta Ty \oplus (1-\beta)y) = y$. Hence $S(y) = y$.

Let us finish the proof of Theorem 3.1. Note that (i) implies $\alpha \in [a,b]$ and $\beta \in [0,b]$. If $\beta = 0$, then the conclusion (2) above implies $y \in F(S)$. Otherwise the conclusion (4) will imply $y \in F$. This proves (i).

For (ii), notice that $\alpha \in [a,1]$ and $\beta \in [a,b]$. Hence the conclusion (3) will imply $y \in F(T)$ which proves (ii).

For (iii), notice that $\alpha, \beta \in [a,b]$. Hence the conclusion (4) will imply $y \in F(T)$. Since

$$\lim_{n\to\infty} d(y_n, y) = \lim_{n\to\infty} d(y_{n_i}, y) = 0,$$

we get $x_n \to y$, which completes the proof of (iii). □

If we assume compactness, Theorem 3.1 will imply the following result.

**Theorem 3.2** *Let $X$ be a strictly convex hyperbolic space. Let $C$ be a nonempty bounded, closed and convex subset of $X$. Let $S, T : C \to C$ be two nonexpansive mappings. Assume* *that $F \neq \emptyset$. Fix $x_1 \in C$. Assume that $\overline{\text{co}}\{\{x_1\} \cup S(C) \cup T(C)\}$ is a compact subset of $C$. Define $\{x_n\}$ as in* (2.1) *where $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, and $x_1$ is the initial element of the sequence. Then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.*

*Proof* We have $x_n \in \overline{\text{co}}\{\{x_1\} \cup S(C) \cup T(C)\}$. Since $\overline{\text{co}}\{\{x_1\} \cup S(C) \cup T(C)\}$ is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$, *i.e.*, $x_{n_i} \to z$. By Theorem 3.1, we have $z \in F$ and $x_n \to z$. □

The existence of a common fixed point $T$ and $S$ is crucial. If one assumes that $T$ and $S$ commute, *i.e.*, $S \circ T = T \circ S$, then a common fixed point exists under the assumptions of Theorem 3.2. Indeed, fix $x_0 \in C$ and define

$$T_n x = \frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) Tx$$

for $x \in C$ and $n \geq 1$. Then

$$\begin{aligned} d(T_n x, T_n y) &= d\left(\frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) Tx, \frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) Ty\right) \\ &\leq \left(1 - \frac{1}{n}\right) d(Tx, Ty) \\ &\leq \left(1 - \frac{1}{n}\right) d(x, y). \end{aligned}$$

That is, $T_n$ is a contraction. By the Banach contraction principle (BCP), $T_n$ has a unique fixed point $u_n$ in $C$. Since the closure of $T(C)$ is compact, there exists a subsequence $\{Tu_{n_i}\}$ of $\{Tu_n\}$ such that $Tu_{n_i} \to u$. Since $T(C)$ is bounded and

$$d(u_n, Tu_n) = d\left(\frac{1}{n} x_0 \oplus \left(1 - \frac{1}{n}\right) Tu_n, Tu_n\right) \leq \frac{1}{n} d(x_0, Tu_n),$$

we have $d(u_n, Tu_n) \to 0$. In particular, we have $u_{n_i} \to u$. Continuity of $T$ implies $Tu = u$. Since $X$ is strictly convex, then $F(T)$ is a nonempty convex subset of $X$. Since $T$ and $S$ commute, we have $S(F(T)) \subset F(T)$. Moreover, since the closure of $T(C)$ is compact, we see that $F(T)$ is compact. The above proof shows that $S$ has a fixed point in $F(T)$, *i.e.*, $F = F(S) \cap F(T) \neq \emptyset$.

The case $S = T$ gives the following result.

**Theorem 3.3** *Let $C$ be a nonempty closed and convex subset of a complete strictly convex hyperbolic space $X$. Let $T : C \to C$ be a nonexpansive mapping such that $\overline{\text{co}}\{\{c_0\} \cup T(C)\}$ is a compact subset of $C$, where $c_0 \in C$. Define $\{x_n\}$ by* (2.2), *where $x_1 = c_0$, $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof* We saw that in this case, we have $F(T) \neq \emptyset$. Since $x_n \in \overline{\text{co}}\{\{x_1\} \cup T(C)\}$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in C$. By Theorem 3.1, we have $Tz = z$ and $x_n \to z$. □

## 4 Convergence in uniformly convex hyperbolic spaces

The following result is similar to the well-known demi-closedness principle discovered by Göhde in uniformly convex Banach spaces [30].

**Lemma 4.1** *Let $C$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $X$. Let $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\} \in C$ be an approximate fixed point sequence of $T$, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. If $x \in C$ is the asymptotic center of $\{x_n\}$ with respect to $C$, then $x$ is a fixed point of $T$, i.e., $x \in F(T)$. In particular, if $\{x_n\} \in C$ is an approximate fixed point sequence of $T$, such that $x_n \overset{\Delta}{\to} x$, then $x \in F(T)$.*

*Proof* Let $\{x_n\}$ be an approximate fixed point sequence of $T$. Let $x \in C$ be the unique asymptotic center of $\{x_n\}$ with respect to $C$. Since

$$d(Tx, x_n) \leq d(Tx, Tx_n) + d(Tx_n, x_n) \leq d(x, x_n) + d(Tx_n, x_n),$$

we get

$$\begin{aligned} r\bigl(Tx, \{x_n\}\bigr) &= \limsup_{n\to\infty} d(Tx, x_n) \\ &\leq \limsup_{n\to\infty} \bigl[d(x, x_n) + d(Tx_n, x_n)\bigr] = r\bigl(x, \{x_n\}\bigr). \end{aligned}$$

By the uniqueness of the asymptotic center, we get $Tx = x$. □

The following theorem is necessary to discuss the behavior of the iterates in (2.1).

**Theorem 4.1** *Let $C$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $X$. Let $S, T : C \to C$ be nonexpansive mappings such that $F \neq \emptyset$. Fix $x_1 \in C$ and generate $\{x_n\}$ by (2.1). Set*

$$y_n = \beta_n Tx_n \oplus (1 - \beta_n)x_n$$

*for any $n \geq 1$.*

(i) *If $\alpha_n \in [a, b]$, where $0 < a \leq b < 1$, then*

$$\lim_{n\to\infty} d(x_n, Sy_n) = 0.$$

(ii) *If $\liminf_{n\to\infty} \alpha_n > 0$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

(iii) *If $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, then*

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

*Proof* Let $p \in F$. Then the sequence $\{d(x_n, p)\}$ is decreasing. Set $c = \lim_{n\to\infty} d(x_n, p)$. If $c = 0$, then all the conclusions are trivial. Therefore we will assume that $c > 0$. Note that we have

$$d(x_{n+1},p) \le \alpha_n d(Sy_n,p) + (1-\alpha_n)d(x_n,p) \tag{4.1}$$

and

$$d(Sy_n,p) \le d(y_n,p) \le \beta_n d(Tx_n,p) + (1-\beta_n)d(x_n,p) \le d(x_n,p) \tag{4.2}$$

for any $n \ge 1$. In order to prove (i), assume that $\alpha_n \in [a,b]$, where $0 < a \le b < 1$. From the inequalities (4.1) and (4.2), we get

$$d(x_{n+1},p) = d\big(\alpha_n Sy_n \oplus (1-\alpha_n)x_n, p\big) \le \alpha_n d(Sy_n,p) + (1-\alpha_n)d(x_n,p) \le d(x_n,p),$$

which implies $\lim_{n\to\infty} d(Sy_n,p) = c$. Indeed, let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Then we have $\lim_{\mathcal{U}} \alpha_n = \alpha \in [a,b]$ and $\lim_{\mathcal{U}} d(x_n,p) = \lim_{\mathcal{U}} d(x_{n+1},p) = c$. Hence

$$c \le \alpha \lim_{\mathcal{U}} d(Sy_n,p) + (1-\alpha)c \le c.$$

Since $\alpha \neq 0$, we get $\lim_{\mathcal{U}} d(Sy_n,p) = c$. Since $\mathcal{U}$ was an arbitrary ultrafilter, we get $\lim_{n\to\infty} d(Sy_n,p) = c$ as claimed. Therefore we have

$$\lim_{n\to\infty} d(x_n,p) = \lim_{n\to\infty} d(Sy_n,p) = \lim_{n\to\infty} d\big(\alpha_n Sy_n \oplus (1-\alpha_n)x_n, p\big) = c.$$

Using Lemma 2.5, we get $\lim_{n\to\infty} d(Sy_n,x_n) = 0$.

Next we prove (ii). Assume $\liminf_{n\to\infty} \alpha_n > 0$ and $\beta_n \in [a,b]$, with $0 < a \le b < 1$. First note that from (4.1) and (4.2), we get

$$d(x_{n+1},p) \le \alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p) \le d(x_n,p),$$

which implies $\lim_{n\to\infty} \alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p) = c$. Since $\liminf_{n\to\infty} \alpha_n > 0$, we conclude that $\lim_{n\to\infty} d(y_n,p) = c$. Since $\beta_n \ge a > 0$, we get in a similar fashion $\lim_{n\to\infty} d(Tx_n,p) = c$. Therefore we have

$$\lim_{n\to\infty} d(x_n,p) = \lim_{n\to\infty} d(Tx_n,p) = \lim_{n\to\infty} d\big(\beta_n Tx_n \oplus (1-\beta_n)x_n, p\big) = c.$$

Using Lemma 2.5, we get $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.

Finally let us prove (iii). Assume that $\alpha_n, \beta_n \in [a,b]$, with $0 < a \le b < 1$. Then from (i) and (ii), we know that

$$\lim_{n\to\infty} d(x_n,Sy_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

Since

$$\begin{aligned} d(x_n,Sx_n) &\le d(x_n,Sy_n) + d(Sy_n,Sx_n) \\ &\le d(x_n,Sy_n) + d(y_n,x_n) \end{aligned}$$

$$= d(x_n, Sy_n) + \beta_n d(Tx_n, x_n)$$
$$\leq d(x_n, Sy_n) + d(Tx_n, x_n),$$

we conclude that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$. □

The conclusion of Theorem 4.1(iii) is amazing because the sequence generated by (2.1) gives an approximate fixed point sequence for both $S$ and $T$ without assuming that these mappings commute.

**Remark 4.1** If we assume that $0 \leq \beta_n \leq b < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$, then

$$\lim_{n\to\infty} \beta_n d(x_n, Tx_n) = 0.$$

Indeed, if we assume this not to be so, then there exists a subsequence $\{\beta_{n'}\}$ and $\delta > 0$ such that

$$\beta_{n'} d(x_{n'}, Tx_{n'}) \geq \delta$$

for any $n \geq 1$. In particular, it is clear, since $\{d(x_n, Tx_n)\}$ is bounded, that $\lim_{n\to\infty} \beta_{n'} \neq 0$. Without loss of generality, we may assume that $\beta_{n'} \geq a > 0$, for $n \geq 1$. The proof of (ii) will imply

$$\lim_{n\to\infty} d(x_{n'}, Tx_{n'}) = 0,$$

which is a contradiction since $\{\beta_n\}$ is a bounded sequence. Therefore we must have

$$\lim_{n\to\infty} \beta_n d(x_n, Tx_n) = 0.$$

In particular, if we assume $\alpha_n \in [a, b]$, then we get

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

As a direct consequence to Theorem 4.1 and Remark 4.1, we get the following result which discusses the $\Delta$-convergence of the iterative sequence defined by (2.1).

**Theorem 4.2** *Let $C$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $X$. Let $S, T : C \to C$ be two nonexpansive mappings such that $F \neq \emptyset$. Fix $x_1 \in C$ and generate $\{x_n\}$ by* (2.1). *Then*

(i) *if $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$, with $0 < a \leq b < 1$, then $x_n \stackrel{\Delta}{\to} y$ and $y \in F(S)$;*
(ii) *if $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_n \stackrel{\Delta}{\to} y$ and $y \in F(T)$;*
(iii) *if $\alpha_n, \beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_n \stackrel{\Delta}{\to} y$ and $y \in F$.*

*Proof* Let us prove (i). Assume $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$, with $0 < a \leq b < 1$. Theorem 4.1 and Remark 4.1 imply that $\{x_n\}$ is an approximate fixed point sequence of $S$, *i.e.*,

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Let $y$ be the unique asymptotic center of $\{x_n\}$. Then Lemma 4.1 implies that $y \in F(S)$. Let us prove that in fact $\{x_n\}$ $\Delta$-converges to $y$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Let $z$ be the unique asymptotic center of $\{x_{n_i}\}$. Again since $\{x_{n_i}\}$ is an approximate fixed point sequence of $S$, we get $z \in F(S)$. Hence

$$\limsup_{n_i \to \infty} d(x_{n_i}, z) \le \limsup_{n_i \to \infty} d(x_{n_i}, y).$$

Since $y, z \in F(S)$, we get

$$\limsup_{n_i \to \infty} d(x_{n_i}, z) = \lim_{n \to \infty} d(x_n, z) \quad \text{and} \quad \limsup_{n_i \to \infty} d(x_{n_i}, y) = \lim_{n \to \infty} d(x_n, y).$$

Since $y$ is the unique asymptotic center of $\{x_n\}$, we get $y = z$. This proves that $\{x_n\}$ $\Delta$-converges to $y$.

Next we prove (ii). Assume $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \le b < 1$. Then Theorem 4.1 implies that $\{x_n\}$ is an approximate fixed point sequence of $T$, *i.e.*,

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

Following the same proof as given above for (i), we get $\{x_n\}$ $\Delta$-converges to its unique asymptotic center which is a fixed point of $T$.

The conclusion (iii) follows easily from (i) and (ii). □

As a corollary to Theorem 4.2, we get the following result when $S = T$.

**Corollary 4.1** *Let $C$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $X$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point. Suppose that $\{x_n\}$ is given by* (2.2), *where $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$, with $0 < a \le b < 1$. Then $x_n \stackrel{\Delta}{\to} p$, with $p \in F(T)$.*

Using the concept of near point projection, we establish the following amazing convergence result.

**Theorem 4.3** *Let $C$ be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space $X$. Let $S, T : C \to C$ be nonexpansive mappings such that $F \neq \emptyset$. Let $P$ be the nearest point projection of $C$ onto $F$. For an initial value $x_1 \in C$, define $\{x_n\}$ as given in* (2.1), *where $\alpha_n, \beta_n \in [a, b]$, with $0 < a \le b < 1$. Then $\{Px_n\}$ converges strongly to the asymptotic center of $\{x_n\}$.*

*Proof* First, we claim that

$$d(Px_n, x_{n+m}) \le d(Px_n, x_n) \quad \text{for } m \ge 1, n \ge 1. \tag{4.3}$$

We prove (4.3) by induction on $m \ge 1$. For $m = 1$, we have

$$\begin{aligned} d(Px_n, x_{n+1}) &= d\big(Px_n, \alpha_n Sy_n \oplus (1 - \alpha_n)x_n\big) \\ &\le \alpha_n d(Px_n, Sy_n) + (1 - \alpha_n) d(Px_n, x_n) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n d(Px_n, y_n) + (1-\alpha_n)d(Px_n, x_n) \\
&= \alpha_n d\bigl(Px_n, \beta_n Tx_n \oplus (1-\beta_n)x_n\bigr) + (1-\alpha_n)d(Px_n, x_n) \\
&\leq \alpha_n\bigl[\beta_n d(Px_n, Tx_n) + (1-\beta_n)d(Px_n, x_n)\bigr] + (1-\alpha_n)d(Px_n, x_n) \\
&\leq \alpha_n\bigl[\beta_n d(Px_n, x_n) + (1-\beta_n)d(Px_n, x_n)\bigr] + (1-\alpha_n)d(Px_n, x_n) \\
&= d(Px_n, x_n).
\end{aligned}$$

That is,

$$d(Px_n, x_{n+1}) \leq d(Px_n, x_n)$$

for $n \geq 1$. Assume that (4.3) is true for $m = k$. That is,

$$d(Px_n, x_{n+k}) \leq d(Px_n, x_n)$$

for $n \geq 1$. Hence

$$\begin{aligned}
d(Px_n, x_{n+k+1}) &= d\bigl(Px_n, \alpha_{n+k} Sy_{n+k} \oplus (1-\alpha_{n+k})x_{n+k}\bigr) \\
&\leq \alpha_{n+k} d(Px_n, Sy_{n+k}) + (1-\alpha_{n+k})d(Px_n, x_{n+k}) \\
&\leq \alpha_{n+k} d(Px_n, y_{n+k}) + (1-\alpha_{n+k})d(Px_n, x_{n+k}) \\
&= \alpha_{n+k} d\bigl(Px_n, \beta_{n+k} Tx_{n+k} \oplus (1-\beta_{n+k})x_{n+k}\bigr) \\
&\quad + (1-\alpha_{n+k})d(Px_n, x_{n+k}) \\
&\leq \alpha_{n+k}\bigl[\beta_{n+k} d(Px_n, Tx_{n+k}) + (1-\beta_{n+k})d(Px_n, x_{n+k})\bigr] \\
&\quad + (1-\alpha_{n+k})d(Px_n, x_{n+k}) \\
&\leq \alpha_{n+k}\bigl[\beta_{n+k} d(Px_n, x_{n+k}) + (1-\beta_{n+k})d(Px_n, x_{n+k})\bigr] \\
&\quad + (1-\alpha_{n+k})d(Px_n, x_{n+k}) \\
&= d(Px_n, x_{n+k}) \\
&\leq d(Px_n, x_n).
\end{aligned}$$

This completes the proof of (4.3). Let us complete the proof of Theorem 4.3. We know from Theorem 4.2(iii) that $\{x_n\}$ $\Delta$-converges to its unique asymptotic center $y$, which is in $F$. Let us prove that $\{Px_n\}$ converges strongly to $y$. Assume not, *i.e.*, there exist $\varepsilon > 0$ and a subsequence $\{Px_{n_i}\}$ such that $d(Px_{n_i}, y) \geq \varepsilon$, for any $n_i \geq 1$. It is clear that we must have $R = d(x_1, y) > 0$, otherwise $\{x_n\}$ is a constant sequence. Since

$$\begin{cases}
d(x_{n_i}, y) \leq d(x_{n_i}, y), \\
d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y), \\
d(Px_{n_i}, y) \geq \varepsilon = d(x_{n_i}, y)\frac{\varepsilon}{d(x_{n_i}, y)} \geq d(x_{n_i}, y)\frac{\varepsilon}{R}
\end{cases}$$

we get

$$d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y)\left(1 - \delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right)\right)$$

for any $n_i \geq 1$. Using the properties of the modulus of uniform convexity, there exists $\eta > 0$ such that

$$\delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right) \geq \eta$$

for any $n_i \geq 1$. Hence

$$d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y)(1-\eta)$$

for any $n_i \geq 1$. Using the definition of the nearest point projection $P$, we get

$$d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y)(1-\eta)$$

for any $n_i \geq 1$. Using the inequality (4.3) above, we get

$$d(x_{n_i+m}, Px_{n_i}) \leq d(x_{n_i}, y)(1-\eta)$$

for any $n_i \geq 1$ and $m \geq 1$. Since $Px_{n_i} \in F$, we know that $\{d(x_n, Px_{n_i})\}$ is decreasing (in $n$ and fixed $n_i$). Hence

$$\limsup_{m\to\infty} d(x_{n_i+m}, Px_{n_i}) = \lim_{n\to\infty} d(x_n, Px_{n_i}) \leq d(x_{n_i}, y)(1-\eta)$$

for any $n_i \geq 1$. Since $y$ is the asymptotic center of $\{x_n\}$, we get

$$\lim_{n\to\infty} d(x_n, y) \leq \lim_{n\to\infty} d(x_n, Px_{n_i}) \leq d(x_{n_i}, y)(1-\eta)$$

for any $n_i \geq 1$. Finally since $y \in F$, if we let $n_i \to \infty$, we get

$$\lim_{n\to\infty} d(x_n, y) \leq \lim_{n\to\infty} d(x_n, y)(1-\eta).$$

Since $\varepsilon \leq d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y)$, we conclude that $\varepsilon \leq \lim_{n\to\infty} d(x_n, y)$, which implies $1 \leq 1-\eta$ which is our desired contradiction. Therefore $\{Px_n\}$ converges strongly to $y$. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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