Introduction To Metric Fixed Point Theory

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Chapter 1

Introduction to Metric Fixed Point Theory

The fixed point problem (at the basis of the Fixed Point Theory) may be stated as:
Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $f : A \to B$ be a map. When does a point $x \in A$ such that $f(x) = x$, also called a fixed point of f ?

A multivalued fixed point problem may be stated but in these lectures we will mainly focus on the single valued functions.

Fixed Point Theory is divided into three major areas:

1. Topological Fixed Point Theory
2. Metric Fixed Point Theory
3. Discrete Fixed Point Theory

Historically the boundary lines between the three areas was defined by the discovery of three major theorems:

1. Brouwer's Fixed Point Theorem
2. Banach's Fixed Point Theorem
3. Tarski's Fixed Point Theorem

In these lectures, we will focus mainly on the second area though from time to time we may say a word on the other areas.

1.1 Metric Fixed Point Theory

In 1922 Banach published his fixed point theorem also known as **Banach's Contraction Principle** uses the concept of Lipschitz mappings.

Definition. Let (M, d) be a metric space. The map $T : M \to M$ is said to be **lipschitzian** if there exists a constant $k > 0$ (called lipschitz constant) such that

$$d\Big(T(x), T(y)\Big) \le k\, d(x, y)$$

for all $x, y \in M$.

A lipschitzian mapping with a lipschitz constant k less than 1, i.e. $k < 1$, is called **contraction**.

Theorem. (**Banach's Contraction Principle**) Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping. Then T has a unique fixed point x_0, and for each $x \in M$, we have

$$\lim_{n\to\infty} T^n(x) = x_0$$

Moreover,for each $x \in M$, we have

$$d\Big(T^n(x), x_0\Big) \le \frac{k^n}{1-k}\, d\Big(T(x), x\Big)\ .$$

Remark. Another proof, due to Caristi, is not very popular though very powerful and uses the function $\varphi : M \to \mathbf{R}^+$ defined by

$$\varphi(x) = \frac{1}{1-k}\, d\Big(T(x), x\Big)\ .$$

Note that we have

$$d\Big(x, T(x)\Big) \le \varphi(x) - \varphi(T(x)), \qquad x \in M.$$

Theorem. Suppose (M, d) is a complete metric space and suppose $T : M \to M$ is a mapping for which T^N is a contraction mapping for some positive integer N. Then T has a unique fixed point.

1.1.1 Caristi-Ekeland Principle

This is one of the most interesting extensions of Banach's Contraction Principle.

Let $\varphi : M \to \mathbf{R}$ and assume that φ is *lower semicontinuous* (l.s.c.), i.e. for any sequence $\{x_n\} \subset M$, if $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \varphi(x_n) = r$, then $\varphi(x) \leq r$.

Theorem. (Ekeland, 1974) Let (M, d) be a complete metric space and $\varphi : M \to \mathbf{R}^+$ l.s.c. Define:

$$x \leq y \Leftrightarrow d(x, y) \leq \varphi(x) - \varphi(y),\ x, y \in M.$$

Then $(M, \leq)$ has a maximal element.
Theorem. (Caristi, 1975) Let M and φ be as above. Suppose $g : M \to M$ satisfies:

$$d(x, g(x)) \leq \varphi(x) - \varphi(g(x)),\ x \in M.$$

Then g has a fixed point.

The proof of is based on discrete technique: Zorn's lemma and Axiom of Choice applied to Bronsted partial order. There are some trials of finding a pure metric proof of Carisit's theorem (without success so far).

1.1.2 The Converse Problem

In other words, what kind of set M will have the conclusion of Banach Contraction Principle? The most elegant result in this direction is due to Bessaga.

Theorem. Suppose that M is an arbitrary nonempty set and suppose that $T : M \to M$ has the property that T and each of its iterates T^n has a unique fixed point. Then for each $\lambda \in (0, 1)$ there exists a metric d_λ on M such that (M, d_λ) is complete and for which

$$d_\lambda\Big(T(x), T(y)\Big) \leq \lambda d_\lambda(x, y) \quad \text{for each } x, y \in M\ .$$

1.1.3 Ultrametric Spaces

Though the definition of ultrametric spaces is too strong, it is a very natural concept used in logic programming for example. Recall that a metric space (M, d) is ultrametric if and only if for each $x, y, z \in M$, we have

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

It is immediate from the above definition that if $d(x,y) \neq d(y,z)$, then we have $d(x,z) = \max\{d(x,y), d(y,z)\}$, i.e. each three points are vertices of an isoscele triangle. This leads to:
If $B(a,r_1)$ and $B(b,r_2)$ are two closed balls in an ultrametric space, with $r_1 \leq r_2$, then either $B(a,r_1) \cap B(b,r_2) = \emptyset$ or $B(a,r_1) \subset B(b,r_2)$.

An ultrametric space M is said to be spherically complete if every descending sequence of closed balls in M has nonempty intersection. Thus a spherically complete ultrametric space is complete. Some nice and immediate consequences of isoscele property are:

1. If $\mathcal{F}$ is a family of closed balls in a spherically complete ultrametric space such that each two members of $\mathcal{F}$ intersect, then the family $\mathcal{F}$ has a nonempty intersection, i.e. $\cap\mathcal{F} \neq \emptyset$.
2. If $B(a,r_1)$ and $B(b,r_2)$ are two closed balls in an ultrametric space, such that $B(a,r_1) \subset B(b,r_2)$, and if $b \notin B(a,r_1)$, then $d(b,a) = d(b,z)$ for any $z \in B(a,r_1)$.

Priess-Crampe proved an analogue to Banach's Contraction Principle in ultrametric spaces.

Theorem. An ultrametric space M is spherically complete if and only if every strictly contractive mapping $T : M \to M$ has a (unique) fixed point.

Recall that $T : M \to M$ is strictly contractive if

$$d\Big(T(x), T(y)\Big) < d\Big(x, y\Big), \quad \text{for any } x, y \in M.$$

1.1.4 Some Examples.

Many examples are known to use Banach's Contraction Principle. Here we will discuss two of them.

ODE and Integral Equations

Consider the integral equation

$$f(x) = \lambda \int_a^x K(x,t) f(t) dt + \phi(x)$$

for a fixed real number λ, where $K(x,t)$ is continuous on $[a,b] \times [a,b]$. Consider the metric space $\mathcal{C}[a,b]$ of continuous real-valued functions defined on $[a,b]$. Consider the map $T : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by

$$\Big(T(f)\Big)(x) = \lambda \int_a^x K(x,t) f(t) dt + \phi(x)\,.$$

For $f_1, f_2 \in \mathcal{C}[a,b]$, we have

$$d\Big(T^n(f_1), T^n(f_2)\Big) \leq |\lambda|^n M^n \frac{(b-a)^n}{n!} d(f_1, f_2)$$

where

$$M = \max\{|K(x,t)|;\ (x,t) \in [a,b] \times [a,b]\}.$$

which implies that the above equation has a unique solution $f(x)$.

Recall

$$d(f_1, f_2) = \max\{|f_1(x) - f_2(x)|;\ x \in [a,b]\}.$$

for $f_1, f_2 \in \mathcal{C}[a,b]$.

Note: The map T is not a contraction on $[a,b]$. Bielecki discovered another way to remedy to this "problem". Indeed, for $\lambda > 0$, set

$$\|f\|_\lambda = \max_{a \leq x \leq b} \{e^{-\lambda\,(x-a)} |f(x)|\},$$

it is now possible to prove that for any $f_1, f_2 \in \mathcal{C}[a,b]$, we have

$$d_\lambda\Big(T(f_1), T(f_2)\Big) = \|T(f_1) - T(f_2)\|_\lambda \leq \frac{M}{\lambda}\ \|f_1 - f_2\|_\lambda$$

where $M = \max\limits_{a \leq x,y \leq b} |K(x,y)|$ is as before. It is then clear that for λ sufficiently large T is a contraction for the new distance d_λ.

Cantor and Fractal sets

Let (M,d) be a complete metric space, let $\mathcal{M}$ denote the family of all nonempty bounded closed subsets of M, and let $\mathcal{C}$ denote the subfamily of $\mathcal{M}$ consisting of all compact sets. For $A \in \mathcal{M}$ and $\varepsilon > 0$ define the ε-neighborhood of A to be the set

$$N_\varepsilon(A) = \{x \in M : dist(x, A) < \varepsilon\}$$

where $dist(x, A) = \inf\limits_{y \in A} d(x,y)$. Now for $A, B \in \mathcal{M}$, set

$$H(A,B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \ \text{ and } \ B \subseteq N_\varepsilon(A)\}.$$

Then $(\mathcal{M}, H)$ is a metric space, and H is called the Hausdorff metric on $\mathcal{M}$. Notice that if (M,d) is complete, then $(\mathcal{M}, H)$ is also complete. Let $T_i : M \to M$, $i = 1, .., n$, be a family of contractions. Define the map $T^* : \mathcal{C} \to \mathcal{C}$ by

$$T^*(X) = \bigcup_{i=1}^{n} T_i(X).$$

Then T^* is a contraction and its lipschitz constant is smaller than the maximum of all lipschitz constants of the mappings T_i, $i = 1, .., n$. Then Banach's Contraction Principle implies the existence of a unique nonempty compact subset X of M such that

$$X = \bigcup_{i=1}^{n} T_i(X).$$

As an application of this, consider the real line $\mathbf{R}$ and the two contractions

$$T_1(x) = \frac{1}{3}x; \;\; T_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then the compact X which satisfies $X = T_1(X) \cup T_2(X)$ is the well-known Cantor set.

1.1.5 Historical Note

We all have learned that the origins of the metric contraction principles and, ergo, metric fixed point theory itself, rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of such celebrated nineteenth century mathematicians as Cauchy, Liouville, Lipschitz, Peano, and specially Picard. In fact the iterative process used in the proof of the Contraction theorem bears the name of Picard iterates. It is quite sad to see that in 1429 Al-Kashani already published a book entitled:"The Calculator's Key", where he used Picard iterates. In fact, Al-Kashani set the stage to the so-called numerical techniques some 600 years ago. He was keen to develop ideas with practical matters, like the approximate values of $\sin(1^o)$ which enabled muslim scientists to come up with a very good approximations to the circumference of the earth. It is amazing, when reading his ideas, to see that most of the ingredients in Banach's Contraction Principle were already known to him.

Chapter 2

Metric Fixed Point Theory in Banach Spaces

The formal definition of Banach spaces is due to Banach himself. But examples like the finite dimensional vector space $\mathbf{R}^n$ was studied prior to Banach's formal definition of Banach spaces. In 1912, Brouwer proved the following:

Theorem. Let B be a closed ball in $\mathbf{R}^n$. Then any continuous mapping $T : B \to B$ has at least one fixed point.

This theorem has a long history. The ideas used in its proof were known to Poincare as early as 1886. In 1909, Brouwer proved the theorem when $n = 3$. And in 1910 Hadamard gave the first proof for arbitrary n, and Brouwer gave another proof in 1912. All of these are older results than the Banach Contraction Principle. Though in nature the two theorems are different, they bare some similarities. A combination of the two led to the so-called metric fixed point theorem in Banach spaces. Indeed, in Brouwer's theorem the convexity, compactness, and the continuity of T are crucial, while the lipshitz behavior of the contraction and completeness are crucial in Banach's fixed point theorem. In infinite normed linear vector spaces, we lose the compactness of the bounded closed convex sets (like closed balls). So if we assume completeness we get Banach spaces. On these, we have another natural topology (other than the norm topology), that is the weak-topology. So a weakly-compact convex set needs not to be compact for the norm. The best example here is the Hilbert space. In it any bounded closed convex set is weakly-compact. The lipschitz condition to be considered is when the lipschitz constant is equal to 1. Such mappings are called **nonexpansive**. In other words, if (M, d) is a metric space, then $T : M \to M$ is nonexpansive if

$$d\Big(T(x), T(y)\Big) \leq d(x, y) \quad \text{for any } x, y \in M.$$

The metric fixed point problem in Banach spaces becomes:

Let X be a Banach space, and C a nonempty bounded closed convex subset of X. When does any nonexpansive mapping $T : C \to C$ have at least one fixed point?

Other interesting problems closely related to this one are:

1. The structure of the fixed points sets.

2. The approximation of fixed points.

3. Abstract Metric Theory.

Recognition of fixed point theory for nonexpansive mappings as a noteworthy avenue of research almost surely dates from the 1965 publication of

1. **Browder-Gohde Theorem.** If K is a bounded closed convex subset of a uniformly convex Banach space X and if $T : K \to K$ is nonexpansive, then T has a fixed point. Moreover the fixed point set of T is a closed convex subset of K.

2. **Kirk Theorem.**If K is a weakly-compact convex subset of a Banach space X. Assume that K has the normal structure property, then any nonexpansive mapping $T : K \to K$ has a fixed point.

2.0.6 Classical Existence Results

While studying a paper by Brodskii and Milman (1948), Kirk (1965) was able to discover the stated result. So the normal structure property is an old concept not directly related to nonexpansive mappings. Hilbert space and uniformly convex Banach spaces have the normal structure property. In fact, the proofs given independently by Browder and Gohde of their results do not use this property.

Let K be a bounded closed convex subset of a Banach space X and $T : K \to K$ be a nonexpansive mapping. Let $\varepsilon \in (0, 1)$ and $x_0 \in K$. Define T_ε by

$$T_\varepsilon(x) = \varepsilon x_0 + (1 - \varepsilon)T(x)\ , \quad \text{for any } x \in K.$$

It is easy to check that $T_\varepsilon(K) \subset K$ (since K is convex) and it is a contraction. The Banach's Contraction Principal implies the existence of a unique point $x_\varepsilon \in K$ such that

$$x_\varepsilon = \varepsilon x_0 + (1 - \varepsilon)T(x_\varepsilon)\ .$$

which implies

$$\|x_\varepsilon - T(x_\varepsilon)\| = \varepsilon\ \|x_0 - T(x_\varepsilon)\| \leq \varepsilon\ \mathrm{diam}(K)\ .$$

Therefore we have

$$\inf_{x\in K} \|x - T(x)\| = 0 \ .$$

This property is known as the approximate fixed point property. So any nonexpansive mapping defined on a bounded closed convex subset of Banach space has this property. By taking a sequence $\{\varepsilon_n\}$ which goes to 0, we generate a sequence of points $\{x_n\}$ from K such that

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0 \ .$$

Such sequence is called an **approximate fixed point sequence** (a.f.p.s.).

In order to grasp the ideas behind the proofs of the existence of fixed point for nonexpansive mappings in the Hilbert and uniformly convex spaces, we need the concept of **asymptotic center** of a sequence discovered by Edelstein. This concept is very useful whenever one deals with sequential approximations in Banach spaces.

Let $\{x_n\}$ be a bounded sequence in a Banach space X, and let C be a closed convex subset of X. Consider the functional $f : C \to [0, \infty)$ defined by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\| \ .$$

Usually we use the notation $f(x) = r(x, \{x_n\})$. The infimum of $f(x)$ over C is called the asymptotic radius of $\{x_n\}$ and denoted by $r(C, \{x_n\})$,i.e.

$$r(C, \{x_n\}) = \inf_{x\in C} r(x, \{x_n\}) = \inf_{x\in C} \left(\limsup_{n\to\infty} \|x_n - x\| \right).$$

The set

$$A(C, \{x_n\}) = \{x \in C;\ r(x, \{x_n\}) = r(C, \{x_n\})\}$$

is the set of all asymptotic centers of $\{x_n\}$. If $\{x_n\}$ converges to $x \in C$, then $A(C, \{x_n\}) = \{x\}$. In general the set $A(C, \{x_n\})$ is not reduced to one point.

Theorem. Every bounded sequence in a uniformly convex Banach space X has a unique asymptotic center with respect to any closed convex subset of X.

In the Hilbert case, we get more:

Theorem. In a Hilbert space H, the weak limit of a weakly convergent sequence coincides with its asymptotic center with respect to H.

Note that in any Hilbert space H, the asymptotic center of a bounded sequence $\{x_n\}$ with respect to H belongs to the closed convex hull of $\{x_n\}$. But this does not necessarily hold in (even uniformly convex) Banach spaces.

Let C be a bounded closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be nonexpansive.

1. Let $\{x_n\}$ be an a.f.p.s. of T in C. Let $z \in C$ be its asymptotic center with respect to C, then it is quite easy to check that $T(z)$ is also an asymptotic center of $\{x_n\}$ with respect to C. By the uniqueness of the asymptotic center, we get $T(z) = z$.

2. Let $x \in X$ and consider the orbit of x under T, i.e. $\{T^n(x)\}$. Let $z \in C$ be its asymptotic center with respect to C, then again it is quite easy to check that $T(z)$ is also an asymptotic center of $\{T^n(x)\}$ with respect to C. By the uniqueness of the asymptotic center, we get $T(z) = z$.

Remark. In 1965, Browder in fact discovered something truly amazing. Let C be a bounded closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be nonexpansive. If $\{x_n\} \subset C$ converges weakly to x and $\{x_n - T(x_n)\}$ converges strongly (with respect to the norm) to 0, then we have

$$x - T(x) = 0 \ .$$

This is known as the **Demiclosedness principle**.

As we mentioned before, Kirk's theorem is based on the normal structure property discovered by Brodskii and Milman.

2.0.7 The Normal Structure Property

The reason behind separating it from the other results is the important role it played during the first 20 years (since 1965) of the theory. In order to appreciate this property, let us give more information on nonexpansive mappings in Banach spaces. Indeed, let C be a weakly compact convex subset of a Banach space X. Let $T : C \to C$ be nonexpansive. The first pioneers of the metric fixed point theory were mostly concerned about the existence of fixed points. So assume that T fails to have a fixed point in C. Since C is weakly compact, there exists (by Zorn) a minimal nonempty closed convex subset K of C invariant under T, i.e. $T(K) \subset K$. The central research of the metric fixed point theory in Banach spaces always centered around the discovery of new properties of such minimal convex sets. The first property is due to Kirk.

Theorem. Under the above notations, we have

1. $\overline{conv}(T(K)) = K$.

2. $\sup\limits_{x \in K} \|z - x\| = \mathrm{diam}(K) > 0$, for any $z \in K$.

The normal structure property forbids the conclusion 2 to hold.

Definition. A closed convex subset C of a Banach space X is said to have the **normal structure property** if any bounded convex subset K of C which contains more than one point, contains a **nondiametral point**, i.e. there exists a point $x_0 \in K$ such that

$$\sup_{x \in K} \|x_0 - x\| < \operatorname{diam}(K) .$$

We will also say that X has the normal structure property if any bounded closed convex subset has the normal structure property.

Throughout this section we will use the following notations:

$$\operatorname{diam}(C) = \sup_{x,y \in C} \|x - y\|;$$

$$r_z(C) = \sup_{x \in C} \|z - x\|;$$

$$r(C) = \inf_{x \in C} r_x(C);$$

$$\mathcal{C}(C) = \{z \in C;\ r_z(C) = r(C)\}.$$

The number $r(C)$ and the set $\mathcal{C}(C)$ are called the **Chebyshev radius** and **Chebyshev center** of C, respectively.

Now we are ready to state Kirk's theorem.

Theorem. Let X be a Banach space and suppose that C is a nonempty weakly compact convex subset of X which has the normal structure property. Then any nonexpansive mapping $T : C \to C$ has a fixed point.

For the following ten years most of the results were solely concentrated on the study of the normal structure property. For example, most of the classical known Banach spaces have normal structure property. So most of the fixed point existence theorems were in fact theorems about normal structure property. It got so bad that some thought that any reflexive (or superreflexive) Banach spaces have the normal structure property. It was not before 1972 (1968??) that James renormed the Hilbert space l_2 to get rid of this property.

For any $\beta > 0$, define the new norm $\|.\|_\beta$ on l_2 by

$$\|x\|_\beta = \max\{\|x\|_2, \beta\, \|x\|_\infty\} .$$

Set $X_\beta = (l_2, \|.\|_\beta)$. James showed that the supereflexive Banach space $X_{\sqrt{2}}$ fails the normal structure property. This answered the question asked by the people working on the fixed point property in Banach spaces. Right after James published his result the natural question of whether $X_{\sqrt{2}}$ has the fixed point property arose. It was Karlovitz (in 1975) who settled this question to the affirmative. We will discuss more Karlovitz ideas later on. Hence we see that normal structure property (which is a geometric property) implies the fixed point property but they are not equivalent. Then people started to dissociate themselves from the normal structure property. In fact, Baillon and Schoneberg (1981) studied the space $X_{\sqrt{2}}$ and proved that this space has a geometric, known as **asymptotic normal structure property** which implies the fixed point property. In fact, they proved that X_β has this geometric property for $\beta < 2$. Then it was asked what happened to X_β for $\beta \geq 2$. No definitive answer was given to a point that some though again that may be these spaces may lead an example of a superreflexive space which fails the fixed point property. No luck again. This time it was Lin (1985) who settled this question by proving that X_β has the fixed point property for any $\beta > 0$. After this, and other results some went too far to claim that no geometric property is equivalent to the fixed point property. This question is still open.

2.0.8 More on Normal Structure Property

Since Kirk publication of his paper, people intensified their investigation of this property. For example it was proved that uniformly convex Banach spaces have this property. In particular, it was shown that if X is a Banach space set

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}$$

defined for $\varepsilon \in [0,2]$. The characteristic (or coefficient) of convexity of the Banach space X is the number

$$\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta(\varepsilon) = 0\}.$$

X is uniformly convex if and only if $\varepsilon_0(X) = 0$, i.e. $\delta(\varepsilon) > 0$, for any $\varepsilon \in [0,2]$. And X is said to be **uniformly non-square** if and only if $\varepsilon_0(X) < 2$. James studied these spaces extensively. Assume that $\varepsilon_0(X) < 1$, and let $\varepsilon \in (\varepsilon_0(X), 1)$. Then for any nonempty bounded closed convex subset of X, there exists a point $x \in C$ such that

$$r(x, C) \leq \Big(1 - \delta(\varepsilon)\Big)\ \mathrm{diam}(C)\ .$$

So we have

$$N(X) = \sup\left\{\frac{r(C)}{\mathrm{diam}(C)}\right\} \leq \Big(1 - \delta(\varepsilon_0(X))\Big)$$

where the supremum is taken over all nonempty bounded closed convex subset of X with more than one point. The number $N(X)$ is known as the **coefficient of uniform normal structure property**. So we have the normal structure property and even more: uniform normal structure property, i.e. $N(X) < 1$. So uniformly convex Banach spaces have a stronger geometric property. In fact, a weaker version was discovered to still imply the normal structure property. It is known as **uniform convexity in every direction** (U.C.E.D.). Indeed, let $z \in X$ be a unit vector ($\|z\| = 1$). The modulus of uniform convexity $\delta_X(\varepsilon, z)$ of X in the direction z is defined by

$$\inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1,\ x - y = \alpha z\ , \|x - y\| \geq \varepsilon \right\} .$$

It is not hard to show that if X is U.C.E.D., then for any nonempty bounded closed convex subset K of X, the Chebishev center $\mathcal{C}(K)$ of K has at most one point. So clearly if X is U.C.E.D., then it has the normal structure property. Spaces which are U.C.E.D. or equivalently renormed to be U.C.E.D. were extensively studied by Zizler (1971).

Remark. There is a sequential characterization of the normal structure property discovered by Brodskii and Milman in their original work, which played a key role in studying this property.

A bounded sequence $\{x_n\}$ in a Banach space is said to be a **diametral sequence** if

$$\lim_{n \to \infty} dist(x_{n+1}, \overline{conv}\{x_1, x_2, \cdots, x_n\}) = \text{diam}\{x_1, x_2, \cdots\}.$$

We have

Theorem. A bounded convex subset K of a Banach space has normal structure if and only if it does not contain a diametral sequence.

2.0.9 Normal Structure and Smoothness

As with convexity, it is possible to scale Banach spaces with respect to their smoothness.

Let X be a Banach space. Define the modulus of smoothness ρ_X of X by

$$\rho(\eta) = \sup \left\{ \left\| \frac{x + \eta y}{2} \right\| + \left\| \frac{x - \eta y}{2} \right\| - 1\ ; \|x\| \leq 1,\ \|y\| \leq 1 \right\}$$

for any $\eta > 0$. X is said to be uniformly smooth (U.S.) if

$$\lim_{\eta \to 0} \frac{\rho_X(\eta)}{\eta} = 0\ .$$

Theorem. For any Banach space X, we have

1. $\rho_{X^*}(\eta) = \sup\{\eta\varepsilon/2 - \delta_X(\varepsilon); 0 \leq \varepsilon \leq 2\}$, for any $\eta > 0$;

2. $\lim_{\eta \to 0} \frac{\rho_{X^*}(\eta)}{\eta} = \frac{\varepsilon_0(X)}{2}$;

3. X is uniformly convex if and only if X^* is uniformly smooth.

Note that one may reverse the roles of X and X^* to obtain for example

$$\rho'_X(0) = \lim_{\eta \to 0} \frac{\rho_X(\eta)}{\eta} = \frac{\varepsilon_0(X^*)}{2} \ .$$

Using James non-squareness result, we see that X is superreflexive if and only if $\rho'_X(0) < \frac{1}{2}$.

In 1978-79 Baillon proved that any uniformly smooth space X has the fixed point property. In fact, he precisely proved this conclusion provided $\rho'_X(0) < \frac{1}{2}$.

In 1982, Turett published a paper where he proved that the condition $\rho'_X(0) < \frac{1}{2}$ implies the normal structure property. Unfortunately Turett's proof is an exact reproduction of Baillon's original proof. Later on this conclusion was strengthened by Prus (1988) and Khamsi (1987) to show that the condition $\rho'_X(0) < \frac{1}{2}$ implies the super-normal structure property (and hence the uniform normal structure property) for both X and its dual X^*. The proof is based on nonstandard techniques.

2.0.10 Karlovitz-Goebel Lemma

As mentioned before, if K is a minimal convex associated to a nonexpansive mapping, then little is known about its properties except the two properties discovered by Kirk. In 1975, independently Karlovitz and Goebel proved the following:

Theorem. Let K be a subset of a Banach space X which is minimal with respect to being nonempty, weakly compact, convex, and T-invariant for some nonexpansive mapping T, and suppose $\{x_n\} \subseteq K$ is an a.f.p.s., i.e.

$$\lim_{n \to \infty} \|x_n - T(x_n)\| = 0.$$

Then for each $x \in K$,

$$\lim_{n \to \infty} \|x - x_n\| = \operatorname{diam}(K).$$

Recall again that it was this property that allowed Karlovitz to prove that the space $X_{\sqrt{2}}$ has the fixed point property. In this theorem a.f.p.s. are important. It is this that pushed Maurey to use ultrapowers.

From 1965 to 1980 most of the classical Banach spaces were investigated and proved to have the fixed point property. Missing to the call was c_0 and (general) lattice Banach spaces. During that period of time, it was strongly believed that any weakly compact convex subset of any Banach space have the fixed point property. It was Alspach (1980-81) who disproved this claim by constructing an isometry on a weakly compact convex subset of $L^1[0,1]$ without a fixed point. This example set the stage to the second revolution in the theory (after the first one in 1965). Indeed, almost right after Alspach's example was made public, Maurey (1980) published his famous results (using nonstandard techniques).

The Open Problem. It is still unknown whether reflexive (or superreflexive) Banach spaces have the fixed point property.

2.1 Non-Standard Techniques

In order to appreciate Maurey's ideas, we need to define the concept of ultrapower of a Banach space. Throughout this section X denotes a Banach space and $\mathcal{U}$ a nontrivial ultrafilter over the positive integers.

Consider the vector space

$$\ell_\infty(X) = \{(x_n) \subset X : \sup_{1 \leq n < \infty} \|x_n\| < \infty\}.$$

It is known (and easily checked) that $\ell_\infty(X)$ is a Banach space with the norm defined by

$$\|(x_n)\|_\infty = \sup_{1 \leq n < \infty} \|x_n\| \ , \quad \text{for } (x_n) \in \ell_\infty(X).$$

Set

$$\mathcal{N} = \{(x_n) \in \ell_\infty(X);\ \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

The Banach space **ultrapower** $\tilde{X}$ of X (relative to the ultrafilter $\mathcal{U}$ is the quotient space $\ell_\infty(X)/\mathcal{N}$. Thus the elements of $\tilde{X}$ are equivalence classes $[(x_n)]$ of bounded sequences $(x_n) \subset X$, where one agrees that two such sequences (x_n) and (y_n) are equivalent if and only if

$$\lim_{\mathcal{U}} \|x_n - y_n\| = 0.$$

The norm $\|\cdot\|_{\mathcal{U}}$ in $\tilde{X}$ is the usual quotient norm. Thus for $\tilde{x} = [(x_n)] \in \tilde{X}$,

$$\|\tilde{x}\|_{\mathcal{U}} = \inf\{\|(x_n + y_n)\|_\infty : (y_n) \in \mathcal{N}\}.$$

We remark that there is another approach which leads to the same thing. Notice that since $\{\|x_n\|\}$ is bounded it lies in a compact subset of $\mathbf{R}$. Therefore $\lim_{\mathcal{U}} \|x_n\|$ always exists. In fact one can prove that for $\tilde{x} = [(x_n)] \in \tilde{X}$,

$$\|\tilde{x}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_n\| \ .$$

For each $x \in X$ let (x_n) denote the sequence for which $x_n = x$, and let $\dot{x} = [(x_n)] \in \tilde{X}$. Then the subspace $\dot{X} = \{\dot{x} : x \in X\}$ is isometric to X via the mapping $x \to \dot{x}$ since

$$\|\dot{x}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_n\| = \lim_{\mathcal{U}} \|x\| = \|x\|.$$

Definition. We will say that the Banach space X has the property super-$\mathcal{P}$ if and only if its ultrapower $\tilde{X}$ has the property $\mathcal{P}$.

A precise definition using finite representability may be used but here we tried a simple approach.

For example, we can talk of: superreflexive, super normal structure property, super fixed point property, etc...

Extending Mappings to Ultrapowers

Let $K \subseteq X$ be bounded closed and convex, and suppose $T : K \to K$. Letting

$$\tilde{K} = \{[(x_n)] \in \tilde{X};\ x_n \in K \text{ for each } n\},$$

there is a canonical way to extend T to a mapping $\tilde{T} : \tilde{K} \to \tilde{K}$ by setting for $\tilde{x} = [(x_n)] \in \tilde{K}$,

$$\tilde{T}(\tilde{x}) = [(T(x_n))].$$

It is immediate that $\tilde{K}$ is a bounded closed and convex subset of $\tilde{X}$. It is quite straightforward to prove that if T is nonexpansive, then $\tilde{T}$ is nonexpansive as well. Moreover, since K is bounded and T is nonexpansive there always exist a.f.p. s. $(x_n) \subset K$, i.e.

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Set $\tilde{x} = [(x_n)]$. Then $\tilde{T}(\tilde{x}) = \tilde{x}$, i.e. $Fix(\tilde{T}) \neq \emptyset$. Not only does the mapping $\tilde{T}$ always have fixed points, but in fact we can prove that $Fix(\tilde{T})$ is metrically convex (Maurey 1980). When dealing with minimal invariant convex sets, more can be said. Indeed, let X be a Banach space and C be a nonempty, convex, weakly compact subset of X. Assume that there exists a nonexpansive mapping $T : C \to C$ whose fixed point set $Fix(T)$ is empty. Since C is weakly compact, there exists K a closed convex subset of C, which is T-invariant and minimal. Before, we have seen some of the properties of this minimal set. Define $\tilde{K}$ as usual to be

$$\tilde{K} = \Big\{[(x_n)] \in \tilde{X} : x_n \in K \text{ for each } n\Big\}.$$

Maurey proved some more properties of $\tilde{K}$:

1. $\operatorname{diam}(K) = \operatorname{diam}(\tilde{K}) = \operatorname{diam}(Fix(\tilde{T}))$;
2. For any $x \in K$, $dist(\dot{x}, Fix(\tilde{T})) = \operatorname{diam}(K)$;

3. For any $\tilde{x} \in Fix(\tilde{T})$, $dist(\tilde{x}, \dot{K}) = \mathrm{diam}(K)$;

Using these properties Maurey proved the follwoing:

Theorem. Any reflexive subspace of $L^1[0,1])$ has the fixed point property. As well as the Hardy space H^1.

Theorem. Let K be a weakly compact, convex subset of a superreflexive space X. Then any isometric map $T : K \to K$ has a fixed point.

Recall that up to 1980, it was still unknown whether the classical space c_0 has the fixed point property. Maurey, through his new ideas, was able to prove it by using the basic lattice properties of c_0. In fact, Borwein and Sims (1984) did rewrite these ideas in general lattice Banach spaces to obtain similar conclusions.

Refining some of the properties satisfied by the minimal invariant sets (in the ultrapower language), Lin (1985) proved the following result:

Theorem. Let $\{\tilde{w}_n\}$ be an approximate fixed point sequence (a.f.p.s.) for $\tilde{T}$ in $\tilde{K}$, i.e. $\|\tilde{T}(\tilde{w}_n) - \tilde{w}_n\|_{\mathcal{U}} \to 0$ as n goes to ∞. Then for any $x \in K$, we have

$$\lim_{n\to\infty} \|\tilde{w}_n - \dot{x}\|_{\mathcal{U}} = \mathrm{diam}(K).$$

This enabled him to prove one of the most elegant applications of Maurey's ideas:

Theorem. Let X be a Banach space with an unconditional Schauder basis. Assume that its constant of unconditionality λ satisfies

$$\lambda < \frac{\sqrt{33}-3}{2} \approx 1.37$$

Then X has the fixed point property.

Note that the canonical basis of the spaces X_β are 1-unconditional, then X_β has the fixed point property for any $\beta > 0$. This settled the question was for all.

Chapter 3

Metric Fixed Point Theory in Hyperconvex Spaces

3.1 Preface

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi (1956) who discovered it when investigating an extension of Hahn-Banach theorem in metric spaces. The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin. The nonlinear theory is still developing. The recent interest into these spaces goes back to the results of Sine and Soardi (1979) who proved independently that fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces.

Recall also that Jawhari, Misane and Pouzet (1986) were able to show that Sine and Soardi's fixed point theorem is equivalent to the classical Tarski's fixed point theorem in complete ordered sets (via generalized metric spaces). Therefore, the notion of hyperconvexity should be understood and appreciated in a more abstract formulation.

3.2 Introduction and basic definitions

Recall the Hahn-Banach theorem:

Theorem 3.2.1. *Let X be a real vector space, Y be a linear subspace of X, and ρ a seminorm on X. Let f be a linear functional defined on Y such that $f(y) \leq \rho(y)$, for all $y \in Y$. Then there exists a linear functional g defined on X, which is an extension of f (i.e. $g(y) = f(y)$, for all $y \in Y$), which satisfies $g(x) \leq \rho(x)$, for all $x \in X$.*

The proof is based on the following well-known fundamental property of the real line $\mathbb{R}$:

"If $\{I_\alpha\}_{\alpha\in\Gamma}$ is a collection of intervals such that $I_\alpha \cap I_\beta \neq \emptyset$, for any $\alpha, \beta \in \Gamma$, then we have $\bigcap\limits_{\alpha\in\Gamma} I_\alpha \neq \emptyset$".

It is this property that is at the heart of the new concept discovered by Aronszajn and Panitchpakdi. Note that an interval may also be seen on the real line as a closed ball. Indeed, the interval $[a, b]$ is also the closed ball centered at $(a + b)/2$ with radius $r = (b - a)/2$, i.e.

$$[a, b] = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

So the above intersection property may also be seen as a ball intersection property.

Definition 3.2.2. *(Menger convexity) Let M be a metric space. We say that M is metrically convex if for any points $x, y \in M$ and positive numbers α and β such that $d(x, y) \leq \alpha+\beta$, there exists $z \in M$ such that $d(x, z) \leq \alpha$ and $d(z, y) \leq \beta$, or equivalently $z \in B(x, \alpha) \cap B(y, \beta)$.*

So M is metrically convex if

$$B(x, \alpha) \cap B(y, \beta) \neq \emptyset \text{ iff } d(x, y) \leq \alpha + \beta$$

for any points $x, y \in M$ and positive numbers α and β.

Therefore, the Hahn-Banach extension theorem is closely related to an intersection property of the closed balls combined with some kind of metric convexity. Hyperconvexity captures these ideas.

Definition 3.2.3. *The metric space M is said to be hyperconvex if for any collection of points $\{x_\alpha\}_{\alpha\in\Gamma}$ in M and positive numbers $\{r_\alpha\}_{\alpha\in\Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α and β in Γ, we must have*

$$\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset\,.$$

3.3 Some basic properties of Hyperconvex spaces

Clearly from the previous section, the real line $\mathbb{R}$ is hyperconvex. In fact, we can easily prove that the infinite dimensional Banach space l_∞ is hyperconvex.

Theorem 3.3.1. *Let* $(M_\alpha, d_\alpha)_{\alpha\in\Gamma}$ *be a collection of hyperconvex metric spaces. Consider the product space* $\mathcal{M} = \prod_{\alpha\in\Gamma} M_\alpha$*. Fix* $a = (a_\alpha) \in \mathcal{M}$ *and consider the subset* M *of* $\mathcal{M}$ *defined by*

$$M = \left\{ (x_\alpha) \in \mathcal{M};\ \sup_{\alpha\in\Gamma} d_\alpha(x_\alpha, a_\alpha) < \infty \right\}.$$

Then (M, d_∞) *is a hyperconvex metric space where* d_∞ *is defined by*

$$d_\infty\Big((x_\alpha), (y_\alpha)\Big) = \sup_{\alpha\in\Gamma} d_\alpha(x_\alpha, y_\alpha)$$

for any $(x_\alpha), (y_\alpha) \in M$*.*

Its proof is an easy consequence from the fact:

$$B((x_\alpha), r) = \prod_{\alpha\in\Gamma} B_\alpha(x_\alpha, r)\ .$$

Note that any metric space M may be embedded isometrically into a hyperconvex metric space. Indeed, let (M, d) be a metric space. Set

$$l_\infty(M) = \{(x_m)_{m\in M} \in \mathbb{R}^M;\ \sup_{m\in M} |x_m| < \infty\}.$$

To see that M embeds isometrically into $l_\infty(M)$, fix $a \in M$ and consider the map $I : M \to l_\infty(M)$ defined by

$$I(b) = \Big(d(b,m) - d(a,m)\Big)_{m\in M}, \quad \text{for any } b \in M.$$

It is easy to check that $d_\infty\Big(I(b), I(c)\Big) = d(b,c)$ for any $b, c \in M$.

In order to prove that hyperconvex spaces are complete, it is enough to have a weaker version of the intersection of balls. Indeed, we say that M has the **ball intersection property** if $\bigcap_{\alpha\in\Gamma} B_\alpha \neq \emptyset$ for any collection of balls $(B_\alpha)_{\alpha\in\Gamma}$ such that $\bigcap_{\alpha\in\Gamma_f} B_\alpha \neq \emptyset$, for any finite subset $\Gamma_f \subset \Gamma$.

Proposition 3.3.2. *Any metric space* M *which has the ball intersection property is complete. In particular any hyperconvex metric space is complete.*

At this point we introduce some notation which will be used throughout the remainder of this work. For a subset A of a metric space M, set:

$$\begin{array}{lcl}
r_x(A) & = & \sup\{d(x,y) : y \in A\},\ \ x \in M; \\
r(A) & = & \inf\{r_x(A) : x \in M\}; \\
R(A) & = & \inf\{r_x(A) : x \in A\}; \\
\mathrm{diam}(A) & = & \sup\{d(x,y) : x, y \in A\}; \\
C(A) & = & \{x \in M : r_x(A) = r(A)\}; \\
C_A(A) & = & \{x \in A : r_x(A) = r(A)\}; \\
\mathrm{cov}(A) & = & \bigcap\{B : B \text{ is a ball and } B \supseteq A\}.
\end{array}$$

$r(A)$ is called the *radius* of A (relative to M), diam(A) is called the *diameter* of A, $R(A)$ is called the *Chebyshev radius* of A, $C(A)$ is called the *center* of A (in M), $C_A(A)$ is called the *Chebyshev center* of A, and cov(A) is called the *cover* of A.

We have the following:

Theorem 3.3.3. *Suppose A is a bounded subset of a hyperconvex metric space M. Then:*
1. $cov(A) = \bigcap\{B(x, r_x(A)) : x \in M\}$.
2. $r_x(cov(A)) = r_x(A)$, for any $x \in M$.
3. $r(cov(A)) = r(A)$.
4. $r(A) = \frac{1}{2} diam(A)$.
5. $diam(cov(A)) = diam(A)$.
6. If $A = cov(A)$, then $r(A) = R(A)$. In particular we have $R(A) = \frac{1}{2} diam(A)$.

Definition 3.3.4. *Let M be a metric space. By $\mathcal{A}(M)$ we denote the collection of all subsets of M which are intersection of balls, i.e. $\mathcal{A}(M) = \{A \subset M; A = cov(A)\}$. The elements of $\mathcal{A}(M)$ are called **admissible** subsets of M.*

It is clear that $\mathcal{A}(M)$ contains all the closed balls of M and is stable by intersection, i.e. the intersection of any collection of elements from $\mathcal{A}(M)$ is also in $\mathcal{A}(M)$. From the above results, for any $A \in \mathcal{A}(M)$, we have

$$C(A) = \bigcap_{a \in A} B\Big(a, R(A)\Big) \bigcap A \in \mathcal{A}(M).$$

Moreover, diam$(C(A)) \leq$ diam$(A)/2$. So we have $A = C(A)$ if and only if $A \in \mathcal{A}(M)$ and diam$(A) = 0$, i.e. A is reduced to one point.

Additionally, we will denote the class of hyperconvex subsets of a metric space M as $\mathcal{H}(M)$.

Definition 3.3.5. *A subset E of a metric space M is said to be proximinal (with respect to M) if the intersection $E \cap B(x, dist(x, E))$ is nonempty for each $x \in M$.*

It is not hard to show that admissible subsets of a hyperconvex metric space M are proximinal in M.

3.4 Hyperconvexity, Injectivity and Retraction

In this section, we will discuss Aronszajn and Panitchpakdi ideas on how hyperconvexity captures Hahn-Banach extension theorem in metric spaces.

A metric space M is said to be *injective* if it has the following extension property: Whenever Y is a subspace of X and $f : Y \to M$ is nonexpansive, then f has a nonexpansive extension $\tilde{f} : X \to M$. This fact has several nice consequences.

Theorem 3.4.1. *Let H be a metric space. The following statements are equivalent:*

(i) H is hyperconvex;

(ii) H is injective.

A similar result to Aronszajn and Panitchpakdi's main theorem may be stated in terms of retractions as follows.

Theorem 3.4.2. *Let H be a metric space. The following statements are equivalent:*

(i) H is hyperconvex;

(ii) for every metric space M which contains H metrically, there exists a nonexpansive retraction $R : M \to H$;

(iii) for any point ω not in H, there exists a nonexpansive retraction $R : H \cup \{\omega\} \to H$.

Remark 3.4.3. *Note that statement (ii) is also known as an absolute retract property. This is why hyperconvex metric spaces are also called absolute nonexpansive retract (or in short ANR).*

Using the statement (iii), Khamsi (1986) introduced a new concept called **1-local retract**.

Definition 3.4.4. *Let M be a metric space. A subset N is called a 1-local retract of M if for any point $x \in M \setminus N$, there exists a nonexpansive retraction $R : M \cup \{x\} \to N$.*

If we take in this definition the set M to be any metric space which contains N metrically, the 1-local retract property becomes *absolute 1-local retract* property. Note that absolute 1-local retracts are absolute nonexpansive retract, i.e. hyperconvex.

Notice that statement (ii), in the above theorem, has a nice extension. Indeed, we have the following result:

Corollary 3.4.5. *Let H be a metric space. The following statements are equivalent:*

(i) H is hyperconvex;

(ii) there exist a hyperconvex metric space H^* which contains H metrically and a nonexpansive retraction $R : H^* \to H$;

In other words, a nonexpansive retract of a hyperconvex metric space is also hyperconvex.

Remark 3.4.6. *This result was used by Khamsi to prove that hyperconvex metric spaces enjoy a better convexity property than the Menger one. Indeed, let H be a hyperconvex metric space. We have seen that H embeds isometrically into $l_\infty(H)$. Hence there exists a nonexpansive retraction $R : l_\infty(H) \to H$. Let $x_1, ..., x_n$ be in H and positive numbers $\alpha_1, ..., \alpha_n$ such that $\sum\limits_{i=1}^{i=n} \alpha_i = 1$. In $l_\infty(H)$, consider the convex linear combination $\sum\limits_{i=1}^{i=n} \alpha_i x_i$. Set $\bigoplus\limits_{1 \le i \le n} \alpha_i x_i = R\left(\sum\limits_{i=1}^{i=n} \alpha_i x_i\right) \in H$. Then, for any $\omega \in H$, we have*

$$d\Big(\bigoplus_{1 \le i \le n} \alpha_i x_i, \omega\Big) = d\Big(R\Big(\sum_{i=1}^{i=n} \alpha_i x_i\Big), R(\omega)\Big) \le \left\|\sum_{i=1}^{i=n} \alpha_i x_i - \omega\right\|_{l_\infty(H)}$$

which implies

$$d\Big(\bigoplus_{1 \le i \le n} \alpha_i x_i, \omega\Big) \le \sum_{i=1}^{i=n} \alpha_i d(x_i, \omega).$$

We will rewrite the above inequality as

$$d\Big(\bigoplus_{1 \le i \le n} \alpha_i x_i, \cdot\Big) \le \sum_{i=1}^{i=n} \alpha_i d(x_i, \cdot).$$

In fact we have a more general formula

$$d\Big(\bigoplus_{1 \le i \le n} \alpha_i x_i, \bigoplus_{1 \le i \le n} \alpha_i y_i\Big) \le \sum_{1 \le i \le n} \alpha_i d(x_i, y_i).$$

One may argue that the choice of our convex combination in H depends on the retraction R and the choice of the isometric embedding, the answer is yes. Therefore depending on the problem, one may have to be careful about this choice.

Sine (1989) began the study of the retraction property in hyperconvex spaces in a little more detailed way than what we have stated so far. His results are crucial in investigating nonexpansive mappings defined on hyperconvex metric spaces. We will begin the final part of this section focusing on admissible subsets, for which Sine proved the following relevant fact. For any positive real number r and any set A in a metric space M, we define the *r-parallel* set of A as

$$A + r = \bigcup \Big\{ B(a, r);\ a \in A \Big\}.$$

We have the following result.

Theorem 3.4.7. *Let H be a hyperconvex metric space. Let J be an admissible subset of H. Set $J = \cap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha)$. Then for any $r \geq 0$, we have*

$$J + r = \bigcap_{\alpha\in\Gamma} B\Big(x_\alpha, r_\alpha + r\Big).$$

In other words, the r-parallel of an admissible subset of a hyperconvex metric space is also an admissible set (this is not a common property of metric spaces).

Remark 3.4.8. *For any positive real number r and any set A in a metric space M, we define the* r-neighborhood

$$N_r(A) = \{x \in M; dist(x, A) \leq r\}.$$

If A is proximinal, i.e. for every $x \in M$ there exists $a \in A$ such that $dist(x, A) = d(x, a)$, then the r-parallel set $A + r$ and the r-neighborhood $N_r(A)$ are identical.

Before we state Sine's result recall that a map T defined on a metric space M is said to be ε*-constant* if $d(x, Tx) \leq \varepsilon$ for all $x \in M$. We have the following result:

Theorem 3.4.9. *Let H be a hyperconvex metric space. Let J be a nonempty admissible subset of H. Then for any $\varepsilon > 0$ there exists an ε-constant nonexpansive retraction of the parallel set $J_\varepsilon = J + \varepsilon$ onto J.*

3.5 More on Hyperconvex spaces

Hyperconvexity, as we mentioned before, is an intersection property. In other words, if a metric space is hyperconvex then the family of admissible sets has some kind of compactness behavior.

Early investigators of hyperconvexity wondered whether the compactness of the family of admissible sets holds also for the family of hyperconvex sets. Notice that this family is not stable under intersection. Therefore one may ask whether any descending chain of nonempty hyperconvex sets has a nonempty intersection. This question was answered by Baillon (1988) in a highly technical proof. It is not known to us if a simple proof of this result exists.

Theorem 3.5.1. *Let M be a bounded metric space. Let $(H_\beta)_{\beta\in\Gamma}$ be a decreasing family of nonempty hyperconvex subsets of M, where Γ is totally ordered. Then $\bigcap\limits_{\beta\in\Gamma} H_\beta$ is not empty and is hyperconvex.*

One of the implications of Baillon's Theorem is the existence of hyperconvex closures. Indeed, let M be a metric space and consider the family $\mathcal{H}(M) = \{H; H \text{ is hyperconvex and } M \subset H\}$. In view of what we said previously, the family $\mathcal{H}(M)$ is not empty. Using Baillon's result, any descending chain of elements of $\mathcal{H}(M)$ has a nonempty intersection. Therefore one may use Zorn's lemma which will insure us of the existence of minimal elements. These minimal hyperconvex sets are called *hyperconvex hulls.* Isbell (1964) was among the first to investigate the properties of the hyperconvex hulls. In fact he was the first one to give a concrete construction of a hyperconvex hull.

It is clear that hyperconvex hulls are not unique. But they do enjoy some kind of uniqueness. Indeed, we have:

Proposition 3.5.2. *Let M be a metric space. Assume that H_1 and H_2 are two hyperconvex hulls of M. Then H_1 and H_2 are isometric.*

Remark 3.5.3. *Though hyperconvex hulls are not unique, the previous proposition shows that up to an isometry they are indeed unique. It is quite an amazing result. From now on, we will denote the hyperconvex hull of M by $h(M)$. Recall that if M is a subset of a hyperconvex set H, then there exists a hyperconvex hull $h(M)$ such that $M \subset h(M) \subset H$.*

Isbell, in his study of the hyperconvex hulls, showed that if M is compact then $h(M)$ is also compact. A generalization of this result was discovered by Espinola and Lopez (1996).

Definition 3.5.4. *Let M be a metric space and let $\mathcal{B}(M)$ denote the collection of nonempty, bounded subsets of M. Then:*

(i) The Kuratowski measure of noncompactness $\alpha : \mathcal{B}(M) \to [0,\infty)$ is defined by

$$\alpha(A) = \inf\left\{\varepsilon > 0;\ A \subset \bigcup_{i=1}^{i=n} A_i \ \text{ with } A_i \in \mathcal{B}(M) \text{ and } diam(A_i) \le \varepsilon\right\}.$$

(ii) The Hausdorff (or ball) measure of noncompactness $\chi : \mathcal{B}(M) \to [0,\infty)$ is defined by

$$\chi(A) = \inf\left\{r > 0;\ A \subset \bigcup_{i=1}^{i=n} B(x_i, r) \ \text{ with } x_i \in M\right\}.$$

These two measures are very much related to each other and to compactness. Indeed, the following classical properties are well known.

(1) For any $A \in \mathcal{B}(M)$, we have $0 \leq \alpha(A) \leq \delta(A) = \operatorname{diam}(A)$.

(2) For any $A \in \mathcal{B}(M)$, we have $\alpha(A) = 0$ if and only if A is precompact.

(3) For any $A \in \mathcal{B}(M)$ and $B \in \mathcal{B}(M)$, we have $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(4) For any $A \in \mathcal{B}(M)$, we have $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.

(5) If $(A_i)_{i\in I}$ is a decreasing chain of closed bounded sets such that $\inf_{i\in I}\ \alpha(A_i) = 0$, then $\bigcap\limits_{i\in I} A_i$ is not empty and is compact, i.e. $\alpha\Big(\bigcap\limits_{i\in I} A_i\Big) = 0$.

In hyperconvex metric spaces, the two measures behave nicely. Indeed, we have:

Proposition 3.5.5. *Let H be a hyperconvex metric space and A be a bounded subset of H. Then we have $\alpha(A) = 2\chi(A)$.*

Theorem 3.5.6. *Let M be any bounded metric space and $h(M)$ its hyperconvex hull. Then we have*

$$\chi\big(h(M)\big) = \chi(M) \ \ \textit{and} \ \ \alpha\big(h(M)\big) = \alpha(M).$$

3.6 Fixed point property and Hyperconvexity

Sine and Soardi results are at the origin of the recent interest to hyperconvex metric spaces. Both Sine and Soardi showed that nonexpansive mappings defined on a bounded hyperconvex metric space have fixed points. Their results were stated in different context but the underlying spaces are simply hyperconvex spaces.

Theorem 3.6.1. *Let H be a bounded hyperconvex metric space. Any nonexpansive map $T : H \to H$ has a fixed point. Moreover, the fixed point set of T, $Fix(T)$, is hyperconvex.*

Note that since $\mathrm{Fix}(T)$ is hyperconvex, then any commuting nonexpansive maps T_i, $i = 1, 2, ..., n$, defined on a bounded hyperconvex set H, have a common fixed point. Moreover their common fixed point set $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \cdots \cap \mathrm{Fix}(T_n)$ is hyperconvex.

Combining these results with Baillon's theorem, we get the following:

Theorem 3.6.2. *Let H be a bounded hyperconvex metric space. Any commuting family of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : H \to H$, has a common fixed point. Moreover, the common fixed point set $\bigcap\limits_{i\in I} Fix(T_i)$ is hyperconvex.*

Remark 3.6.3. *Baillon asked whether boundedness may be relaxed. He precisely asked whether the conclusion holds if the nonexpansive map has a bounded orbit. In the classical Kirk's fixed point theorem, having a bounded orbit implies the existence of a fixed point. Prus answered this question in the negative. Indeed, consider the hyperconvex Banach space $H = l_\infty$ and the map $T : H \to H$ defined by*

$$T\Big((x_n)\Big) = (1 + \lim_{\mathcal{U}} x_n, x_1, x_2, ...)$$

where $\mathcal{U}$ is a nontrivial ultrafilter on the set of positive integers. We may also take a Banach limit instead of a limit over an ultrafilter. The map T is an isometry and has no fixed point. On the other hand, we have

$$T^n(0) = (1, 1, ..., 1, 0, 0, ...)$$

where the first block of length n has all its entries equal to 1 and 0 after that. So T has bounded orbits.

Recently, we wondered whether Sine and Soardi's theorem holds for asymptotically nonexpansive mappings. Recall that a map T is said to be **asymptotically nonexpansive** if

$$d(T^n(x), T^n(y)) \leq \lambda_n d(x, y)$$

and $\lim_n \lambda_n = 1$. This question is till unknown. But a partial positive answer is known for approximate fixed points. Before we state this result, recall that if $T : H \to H$ is a map, then $x \in H$ is an ε**-fixed point** if $d(x, T(x)) \leq \varepsilon$ where $\varepsilon \geq 0$. The set of ε-fixed points of T is denoted by $\mathrm{Fix}_\varepsilon(T)$. Sine obtained the following wonderful result:

Theorem 3.6.4. *Let H be a bounded hyperconvex metric space and $T : H \to H$ a nonexpansive map. For any $\varepsilon > 0$, $Fix_\varepsilon(T)$ is not empty and is hyperconvex.*

Now we are ready to state the following unpublished result.

Theorem 3.6.5. *Let H be a bounded hyperconvex metric space and $T : H \to H$ be asymptotically nonexpansive. For any $\varepsilon > 0$, $Fix_\varepsilon(T)$ is not empty, in other words we have*

$$\inf_{x \in H} d(x, T(x)) = 0.$$

3.7 Set-valued mappings in Hyperconvex spaces

Recall that $\mathcal{A}(M)$ denotes the family of all nonempty admissible subsets of a metric space M. In both instances endowed with the usual Hausdorff metric d_H. Recall that the distance between two closed subsets A, B of a metric space in the Hausdorff sense is given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$$

where $N_\varepsilon(A)$ denotes the closed ε-neighborhood of A.

Theorem 3.7.1. *Let H be hyperconvex, and let $T^* : H \to \mathcal{A}(H)$. Then there exists a mapping $T : H \to H$ for which $T(x) \in T^*(x)$ for each $x \in H$ and for which $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ for each $x, y \in H$.*

As a corollary of this we have:

Corollary 3.7.2. *Let H be bounded and hyperconvex, and suppose $T^* : H \to \mathcal{A}(H)$ is nonexpansive. Then T^* has a fixed point, that is, there exists $x \in H$ such that $x \in T^*(x)$.*

The method of Theorem **??** also gives the following. In this theorem $\text{Fix}(T^*) = \{x \in H : x \in T^*(x)\}$.

Theorem 3.7.3. *Let H be hyperconvex, let $T^* : H \to \mathcal{A}(H)$ be nonexpansive and suppose Fix$(T^*) \neq \emptyset$. Then there exists a nonexpansive mapping $T : H \to H$ with $T(x) \in T^*(x)$ for each $x \in H$ and for which Fix(T) = Fix(T^*). In particular, Fix(T^*) is hyperconvex.*

Remark 3.7.4. *Due to its importance in different branches of mathematics, selection problems have been widely studied along the last fifty years. The problem is usually as follows: given a certain multivalued mapping to be able to find an univalued selection of it with certain properties as, for instance, continuity or measurability. Theorem* **??** *is very surprising since it is not common at all to be able to guarantee that a nonexpansive multivalued mapping admits a nonexpansive selection, in fact this seems to be a quite characteristic fact from the hyperconvex geometry. The reader will find a large collection of results and references on multivalued selection problems in the recent book:*

D. Repovs, and P.V. Semenov, ***Continuous Selections of Multivalued Mappings****, Kluwer Academic Publishers, Dordrecht, 1998.*

We finish this section with two applications of Theorem **??**. First we show that the family of all bounded λ-lipschitzian functions of a hyperconvex space M into itself is itself hyperconvex and second we will study a best approximation problem in hyperconvex spaces.

Let f and g be two bounded λ-lipschitzian functions of a hyperconvex space M into itself, we define the distance between them in the usual way, that is, if $f, g : M \to M$, set

$$d(f, g) = \sup_{x \in M} d(f(x), g(x)).$$

Theorem 3.7.5. *Let M be hyperconvex and for $\lambda > 0$ let $\mathfrak{F}_\lambda$ denote the family of all bounded λ-lipschitzian functions of M into M. Then $\mathfrak{F}_\lambda$ is itself a hyperconvex space.*

This leads to the following.

Corollary 3.7.6. *Let M be a bounded hyperconvex metric space and let $f \in \mathcal{F}_1$. Then the family*

$$R = \{r \in \mathcal{F}_1 : r(M) \subset Fix(f)\}$$

is a nonexpansive retract of $\mathcal{F}_1$.

One of the most important concepts in approximation theory is that one of *metric projection.* In this last part of the section we study the problem of finding nonexpansive selections of the metric projection. Let us introduce some definitions first.

Definition 3.7.7. *Let M be a metric space and A a proximinal subset of M, then the mapping $R : M \to 2^A$ defined as*

$$R(x) = B(x, dist(x, A)) \cap A$$

for every $x \in M$ is called the metric projection on A (relative to M).

Notice that the proximinality of A guarantees that $R(x) \neq \emptyset$ for all $x \in M$. We will also deal with the following concept.

Definition 3.7.8. *A subset A of a metric space M is said to be a proximinal nonexpansive retract of M if there exists a nonexpansive selection of the metric projection on A, i.e. if there exists a nonexpansive retraction $r : M \to A$ such that $r(x) \in B(x, dist(x, A)) \cap A$ for each $x \in M$.*

The first one in taking up the problem of characterizing proximinal nonexpansive retracts in hyperconvex metric spaces was Sine.

Theorem 3.7.9. *Let E be an admissible subset of a hyperconvex metric space H. Then E is a proximinal nonexpansive retract of H.*

Remark 3.7.10. *Most of the theorem cited here were proven for a larger class than admissible subsets: externally hyperconvex, weak-externally hyperconvex.*

3.8 Lambda-Hyperconvexity

Since the beginning it was known that the Hilbert space ℓ_2 fails hyperconvexity. By studying this case closely, Khamsi and al. (2000) introduced a property very similar to hyperconvexity, called λ-hyperconvexity. The idea is to expand the radius of the given balls by a uniform factor. For example, every pairwise intersecting collection of balls in ℓ_2 has non-empty intersection if the radius of the balls are increased by the factor $\sqrt{2}$. In light of this, the following definition becomes natural.

Definition 3.8.1. *Let M be a metric space and let $\lambda \geq 1$. We say that the metric space M is λ-hyperconvex if for every non-empty admissible set $A \in \mathcal{A}(M)$, for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$, centered at $x_\alpha \in A$ for $\alpha \in \Lambda$, the condition*

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \text{ for every } \alpha, \beta \in \Lambda,$$

implies

$$A \cap \Big(\bigcap_{\alpha\in\Lambda} B(x_\alpha, \lambda r_\alpha) \Big) \neq \emptyset.$$

Let $\lambda(M)$ be the infimum of all constants λ such that M is λ-hyperconvex, and say that $\lambda(M)$ is exact if M is $\lambda(M)$-hyperconvex.

Grünbaum (1920) and other authors have studied a similar property not involving the underlying admissible set A. But its introduction becomes essential when we try to connect this concept to the fixed point property via the normal structure property.

Let us recall Grünbaum's definition: For a metric space M, let the *expansion constant* $E(M)$ be the infimum of all constants μ such that the following holds: Whenever a collection $\{B(x_\alpha, r_\alpha) : \alpha \in \Lambda\}$ intersects pairwise, then

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha, \mu \cdot r_\alpha) \neq \emptyset.$$

We say $E(M)$ is *exact*, if the condition is even satisfied for $\mu = E(M)$.

Trivially, $E(M) \leq \Lambda(M)$ holds in metrically convex spaces (see Definition **??**). On the other hand, if M is a two element metric space, then $E(M) = 1$, while $\Lambda(M) = 2$, so both concepts do not coincide in general.

Let us first summarize some basic properties of λ-hyperconvex metric spaces, some of which are trivial, while the others can be easily derived from corresponding results about expansion constants:

Theorem 3.8.2. *Let M be a metric space.*

1. *M is hyperconvex if and only if it is 1-hyperconvex.*
2. *Every λ-hyperconvex metric space is complete.*
3. *Reflexive Banach spaces and dual Banach spaces are 2-hyperconvex.*
4. *There is a subspace X of ℓ_1 which fails to be 2-hyperconvex.*
5. *Hilbert space is $\sqrt{2}$-hyperconvex.*

We will finish this section, and hence this chapter on hyperconvexity, by studying the connection between λ-hyperconvexity and the fixed point property. The theorem we will finish with is based on a metric generalization of Kirk's fixed point theorem established by Khamsi (1989). In order to state this generalization we need to know what a *uniform normal structure* on a metric space is.

Let M be a metric space and $\mathcal{F}$ a family of subsets of M. Then we say that $\mathcal{F}$ defines a *convexity structure* on M if it contains the closed balls and is stable by intersection. For instance $\mathcal{A}(M)$, the class of the admissible subsets of M, defines a convexity structure on any metric space M. We say that $\mathcal{F}$ is a *uniform normal structure* on M if there exists $c < 1$ such that $R(A) \leq c \cdot \mathrm{diam}(A)$ for every $A \in \mathcal{F}$ with $\mathrm{diam}(A) > 0$, where $R(A)$ and $\mathrm{diam}(A)$ are, respectively, the Chebyshev radius and diameter of A defined in Section 3.

Now we may state the generalization of Kirk's fixed point theorem.

Theorem 3.8.3. *Let M be a bounded complete metric space. If M has a uniform normal structure then it has the fixed point property for nonexpansive mappings.*

The connection between λ-hyperconvexity and the fixed point property is giving by the following theorem.

Theorem 3.8.4. *Let M be a bounded λ-hyperconvex space. If $\lambda < 2$, then any nonexpansive mapping $T : M \to M$ has a fixed point.*

Chapter 4

Metric Fixed Point Theory in Metric Spaces

4.1 The Metric Convexity

The Banach Contraction Principle theorem is a metric result and does not depend on any linear structure. But Kirk's fixed point theorem is strongly connected to the linear convexity structure of linear spaces. As early as 1965, many have tried to weaken this tiding. Takahashi (1970) was may be the first one to give a metric analogue to Kirk's theorem. His approach was based on defining a convexity in metric spaces extremely similar to the linear convexity.

<u>Definition.</u> Let (M, d) be a metric space with $I = [0, 1]$. A mapping $W : M \times M \times I \to M$ is said to be a convex structure on M if for each $(x, y, \lambda) \in M \times M \times I$ and $z \in M$, we have

$$d\Big(z, W(x, y, \lambda)\Big) \leq \lambda d(z, x) + (1 - \lambda) d(z, y)$$

Sometimes the point $W(x, y, \lambda)$ is denoted $\lambda x \oplus (1 - \lambda) y$ whenever the choice of the convexity mapping W is irrelevant. Using the convexity structure W, one will easily define a convex subset of M and prove similar properties of convex sets in the linear case. It is not hard to check that balls are convex sets. A very nice example of such situation is given by:

Let B be the open unit ball of the infinite Hilbert space H. On B, we consider the Poincare hyperbolic metric ρ:

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \alpha(\gamma(t), \gamma'(t)) dt$$

where the infimum is taken over all piece-wise differentiable curves such that

$\gamma(0) = x$ and $\gamma(1) = y$, and where

$$\alpha(x, v) = \sup_{f \in \mathcal{F}} \|Df(x)(v)\|$$

with $\mathcal{F} = \{f : B \to B; \text{holomorphic}\}$. (B, ρ) is a complete metric space (unbounded). Using the Mobius transformations, one can prove that

$$\rho(x, y) = \text{Argth}\Big(1 - \sigma(x, y)\Big)^{1/2} \ \text{ with } \ \sigma(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - < x, y > |^2} \ .$$

Using some complicated computations, one may prove that for any x, y in B, and $\lambda \in [0, 1]$, there exists a unique $z \in B$ such that

$$\rho(z, w) \leq \lambda \rho(x, w) + (1 - \lambda)\rho(y, w)$$

for any $w \in B$. In other words, we have $z = \lambda x \oplus (1 - \lambda)y$. In fact, the metric space (B, ρ) enjoys some geometric properties similar to uniformly convex Banach spaces. For example, (B, ρ) has a kind of weak compactness, that is the family of convex subsets of B (in the abstract sense) has the finite infinite intersection property. Moreover, we have

$$\begin{cases} \rho(a, x) \leq r \\ \rho(a, y) \leq r \\ \rho(x, y) \geq r\varepsilon \end{cases} \implies \rho\Big(a, \frac{1}{2}x \oplus \frac{1}{2}y\Big) \leq r(1 - \delta(r, \varepsilon)) \ ,$$

for any a, x, y in B and any positive r and ε, where

$$\delta(r, \varepsilon) = 1 - \frac{1}{r}\text{Argth}\left(\frac{\sinh(r(1 + \varepsilon/2))\sinh(r(1 - \varepsilon/2))}{\cosh(r)}\right)^{1/2} .$$

It is easy to check that for $r > 0$ and $\varepsilon > 0$, we have $\delta(r, \varepsilon) > 0$. This is the analogue of the uniform convexity in the nonlinear case. It is not hard to check that convex subsets enjoy the normal structure property and that the Chebishev center is reduced to one point. Note that holomorphic mappings are nonexpansive mappings for the Poincare distance ρ.

4.2 The Convexity Structures

In 1977, Penot was successful in giving an abstract formulation of Kirk's theorem via the **Convexity Structures**:

Let M be an abstract set. A family Σ of subsets of M is called a convexity structure if

(i) the empty set $\emptyset \in \Sigma$;

(ii) $M \in \Sigma$;

(iii) Σ *is closed under arbitrary intersections.*

The convex subsets of M are the elements of Σ. If M is a metric space, we will always assume that closed balls are convex. The smallest convexity structure which contains the closed balls is $\mathcal{A}(M)$ the family of admissible subsets of M. Recall that A is an admissible subset of M if it is an intersection of closed balls. Since Kirk's theorem involves some kind of compactness and the normal structure property, it was of no surprise that the generalized attempts did define these two concepts in metric spaces. Takahashi in his attempt considered compact metric spaces, which was very restrictive. Penot on the other hand, defined compactness for convexity structures which leads to weak-compactness in the linear case. Indeed, a convexity structure Σ is said to be compact if and only if every family of subsets of Σ which has the finite intersection property has a nonempty intersection, i.e. if $(A_i)_{i\in I}$, with $A_i \in \Sigma$, then

$$\bigcap_{i\in I} A_i \neq \emptyset$$

provided $\bigcap\limits_{i\in I_f} A_i \neq \emptyset$ for any finite subset I_f of I. Since the normal structure property is a metric notion, then it was not that difficult to extend it to convexity structures. Indeed, the convexity structure Σ is said to be normal if and only if for any nonempty and bounded $A \in \Sigma$ not reduced to one point, there exists $a \in A$ such that

$$\sup\{d(a,x);\ x \in A\} < \sup\{d(x,y);\ x,y \in A\} = diam(A)\ .$$

Penot's formulation becomes

Theorem. If (M,d) is a nonempty bounded metric space which possesses a convexity structure which is compact and normal, then every nonexpansive mapping $T : M \to M$ has a fixed point.

Remark. In the general Kirk's theorem, the Banach space is supposed to have normal structure which means that the family of all convex sets is normal. But This family is a large one and contains the admissible sets. For example, the Banach space l^∞ is a wonderful example which illustrates the power behind Penot's formulation. Indeed, l^∞ fails to satisfy the normal structure property but $\mathcal{A}(l^\infty)$ is compact and normal which implies the following theorem discovered separately by Sine and Soardi in 1979:

Theorem. If A is a nonempty admissible subset of l^∞, then every nonexpansive mapping $T : A \to A$ has a fixed point.

Remark. In the original proof of Kirk's theorem, the weak compactness is used to prove the existence of a minimal invariant set via Zorn's lemma. Gillespie and Williams (1979) showed that a constructive proof may be found which uses only countable compactness. In other words, the convexity structure Σ is assumed to satisfy a countable intersection property, i.e. for any $(A_n)_{1\leq n}$, with $A_n \in \Sigma$, then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

provided $\bigcap_{n=1}^{m} A_n \neq \emptyset$ for any $m \geq 1$. This weakening is very important since in many practical cases, we do not have a compactness generated by a topology but a compactness defined sequentially. The latest usually generates some kind of countable compactness. Note also that if the convexity structure Σ is uniformly normal then it is countably compact. This is an amazing metric translation of a well known similar result in Banach spaces due to Maluta. It is natural to ask whether a convexity structure which is countably compact is basically compact. The answer is yes if we are dealing with uniform normal structure. In fact, if $\mathcal{A}(M)$ is countably compact and normal, then $\mathcal{A}(M)$ is compact.

4.3 Generalized Metric Spaces

In the next example, we discuss another amazing translation of Kirk's theorem. Indeed, traditionally discrete techniques are used in metric spaces in our day-to-day research. The other way is not that common. But let us generalize the common distance into the discrete case. For that let M be an arbitrary set and $\mathcal{V}$ a set with a binary operation which will be denoted $\oplus$. We assume that $\oplus$ enjoys most of the properties that the classical addition does. In particular, we have a zero element $0 \in \mathcal{V}$ which satisfies $u \oplus 0 = 0 \oplus u = u$, for any $u \in \mathcal{V}$. We also assume that $\mathcal{V}$ is ordered by $\leq$ such that $0 \leq u$ and $u \oplus v \leq u' \oplus v'$ whenever $u \leq u'$ and $v \leq v'$. Note that $\mathcal{V}$ is not totally behaving like the set of positive numbers since we do not have a multiplication operation on $\mathcal{V}$.

Definition. Let $\mathcal{V}$ be a set as described above and M be an arbitrary set. The mapping $d : M \times M \to \mathcal{V}$ is called a generalized distance if

(i) $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = \tau\Big(d(y, x)\Big)$, where τ is an involution, that is $\tau(\tau(x)) = x$;

(iii) $d(x, y) \leq d(x, z) \oplus d(z, y)$

To appreciate the power behind the above new concept, let us show how any ordered set is a generalized metric space. Let $(\mathcal{M}, \prec)$ be an ordered set. Set $\mathcal{V} = \{0, \alpha, \beta, 1\}$ ordered as: $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and α and β are not comparable. Define $\oplus$ as the max operation, i.e. $u \oplus v = \max(u, v)$. Define the generalized distance d by

(i) $d(x, x) = 0$;

(ii) $d(x, y) = \alpha$, if $x \prec y$;

(iii) $d(x, y) = \beta$, if $y \prec x$;

(iv) $d(x, y) = 1$, if x and y are not comparable;

(v) $\tau(0) = 0$, $\tau(\alpha) = \beta$, $\tau(\beta) = \alpha$, and $\tau(1) = 1$.

With d, $\mathcal{M}$ becomes a generalized metric space. Armed with this distance, we can talk about balls (left and right to be precise) and admissible sets. It is quite amazing to show that if $(\mathcal{M}, \prec)$ is a complete lattice, then $\mathcal{A}(\mathcal{M})$ is compact and normal. Also it is an amazing fact to see that nonexpansive maps are exactly the monotone increasing maps. The famous Tarski theorem becomes Kirk theorem. More can be said and little is known....

4.4 Structure of the Fixed Point Set

In the previous discussion, we looked at the fundamental problem of whether a fixed point exists. But once this problem is resolved, it is quite natural to wonder about the structure of the fixed point set. In particular, if we want to prove the existence of a common fixed point for two commuting mappings, we will need to find out more about the fixed point set. In particular, whether the main fixed point existence result is valid for such sets. Properties like retractions help solve this question. May be the first interesting result in this direction goes back to Bruck in the linear case. Inspired by hyperconvexity, we introduce the following

Definition. Let M be a metric space. A subset A of M is said to be 1-local retract of M if for any family $(B_i)_{i\in I}$ of closed balls centered in A for which

$$\bigcap_{i\in I} B_i \neq \emptyset$$

it is the case that

$$A \bigcap \Big(\bigcap_{i\in I} B_i \Big) \neq \emptyset\ .$$

One of the most fundamental properties of 1-local retracts is

Proposition. Let M be a metric space and suppose that $\mathcal{A}(M)$ is compact and normal. If N is a nonempty 1-local retract subset of M, then $\mathcal{A}(N)$ is compact and normal.

The proof is technical but follows the same ideas used in the hyperconvex case. In order to connect this concept with our previous discussion, we have

Theorem. Let M be a metric space and suppose that $\mathcal{A}(M)$ is compact and normal. If $T : M \to M$ is nonexpansive, then the fixed point set $Fix(T)$ of T is a nonempty 1-local retract subset of M.

This conclusion may be easily extended to finite commutative families:

Theorem. Let M be a metric space and suppose that $\mathcal{A}(M)$ is compact and normal. Then every finite family $\mathcal{F}$ of commuting nonexpansive self-mappings of M has a nonempty common fixed point set $Fix(\mathcal{F})$. Moreover, $Fix(\mathcal{F})$ is a nonempty 1-local retract subset of M.

The conclusion of this theorem is still valid for infinite families but the proof is extremely difficult and not intuitive. Again it uses ideas developed by Baillon in the hyperconvex case.