# Reflexive Metric Spaces
# and
# The Fixed Point Property

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## 1 Introduction

As for the linear case, compactness for the strong topology is very restrictive. Since the beginning of the fixed point theory, weak-compactness offered an acceptable alternative in Banach spaces. But when we deal with metric spaces, this natural extension is no longer easy to implement. One has to go back to the linear case and investigate the weak-topology with a new eye. First it is quite striking that convex subsets are closely related to the weak-topology. So it is natural to consider such concept in metric spaces. Two main directions have attracted most of the attention: the Menger convexity [**?**], and convexity structures [**?**]. Depending on the metric space at hand, one of the two concepts rise higher. For example, in hyperconvex metric spaces [**?**] the concept of convexity structure (introduced by Penot [**?**] in metric fixed point theory) was very successful and allowed for some beautiful results. We tried for years to use it to define some kind of weak-topology but were not successful. For example Lim and Xu [**?**] tried to extend some classical known fixed point results for uniformly Lipschitzian mappings to metric spaces using some kind of weak-convergence. In particular they introduced a property (P) which enabled them to get some kind of lower-semi-continuity of the distance for this new weak-topology. A similar property was recently given in CAT($\kappa$) metric spaces by Dhompongsa, Kirk, and Sims [**?**]. Note that when talking about a weak-topology, the world topology is a little bit abused. Usually we only focus on sequences. The investigation of these ideas in the general framework of topology will be the subject of a future research project.

## 2 Convexity in Metric Spaces

When we think of linearity, we think of a vector space and more generally a normed vector space. But many of the linear concepts may be obtained by defining convexity instead. Since these concepts are based not on the linear structure but on the convexity, then it is natural to extend them to nonlinear spaces which still carries some kind of convexity. Historically two successful routs emerged when dealing with this problem: metric convexity (Menger) and convexity structures. The first one is based on the convex combination of points while the second one follows a set theoretic approach. It is worth to mention that though the second one is more general, it is also very restrictive.

### 2.1 Menger Convexity

**Definition.** Let $(M, d)$ be a metric space with $I = [0, 1]$. A mapping $W : M \times M \times I \rightarrow M$ is said to be a convex structure on $M$ if for each $(x, y, \lambda) \in M \times M \times I$ and $z \in M$, we have

$$d\Big(z, W(x, y, \lambda)\Big) \leq \lambda d(z, x) + (1 - \lambda) d(z, y)$$

Sometimes the point $W(x, y, \lambda)$ is denoted $\lambda x \oplus (1-\lambda)y$ whenever the choice of the convexity mapping $W$ is irrelevant. Using the convexity structure $W$, one will easily define a convex subset of $M$ and prove similar properties of convex sets in the linear case. It is not hard to check that balls are convex sets.

**A Wonderful Example [?].** Let $B$ be the open unit ball of the infinite Hilbert space $H$. On $B$, we consider the Poincare hyperbolic metric $\rho$:

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \alpha(\gamma(t), \gamma'(t)) dt$$

where the infimum is taken over all piece-wise differentiable curves such that $\gamma(0) = x$ and $\gamma(1) = y$, and where

$$\alpha(x, v) = \sup_{f \in \mathcal{F}} \|Df(x)(v)\|$$

with $\mathcal{F} = \{f : B \rightarrow B; \text{holomorphic}\}$. $(B, \rho)$ is a complete metric space (unbounded). Using the Mobius transformations, one can prove that

$$\rho(x, y) = \text{Argth} \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - < x, y > |^2}\right)^{1/2} .$$

Using some complicated computations, one may prove that for any $x, y$ in $B$, and $\lambda \in [0, 1]$, there exists a unique $z \in B$ such that

$$\rho(z, w) \leq \lambda \rho(x, w) + (1 - \lambda) \rho(y, w)$$

for any $w \in B$. In other words, we have $z = \lambda x \oplus (1-\lambda)y$. In fact, the metric space $(B,\rho)$ enjoys some geometric properties similar to uniformly convex Banach spaces. For example, $(B,\rho)$ has a kind of weak compactness, that is the family of closed bounded convex subsets of $B$ has the finite infinite intersection property. Moreover, we have

$$\left\{ \begin{array}{l} \rho(a,x) \leq r \\ \rho(a,y) \leq r \\ \rho(x,y) \geq r\varepsilon \end{array} \right. \implies \rho\Big(a, \frac{1}{2}x \oplus \frac{1}{2}y\Big) \leq r(1-\delta(r,\varepsilon)) \ ,$$

for any $a, x, y$ in $B$ and any positive $r$ and $\varepsilon$, where

$$\delta(r,\varepsilon) = 1 - \frac{1}{r}\text{Argth}\left(\frac{\sinh(r(1+\varepsilon/2))\sinh(r(1-\varepsilon/2))}{\cosh(r)}\right)^{1/2} .$$

It is easy to check that for $r > 0$ and $\varepsilon > 0$, we have $\delta(r,\varepsilon) > 0$. This is the analogue of the uniform convexity in the nonlinear case. It is not hard to check that the Chebyshev center of any nonempty convex closed bounded set is reduced to one point. Note that holomorphic mappings are nonexpansive mappings for the Poincare distance $\rho$.

**Another Wonderful Example [?].** A metric space $(X,d)$ is said to be a *length space* if each two points of $X$ are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of $X$ is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, $d$ is said to be a *length metric* (otherwise known an *inner metric* or *intrinsic metric*). In case no rectifiable path joins two points of the space the distance between them is said to be $\infty$.

A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, $c$ is an isometry and $d(x,y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) *segment* joining $x$ and $y$. $(X,d)$ is said to be a *geodesic space* if every two points of $X$ are joined by a geodesic. $X$ is said to be *uniquely geodesic* if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$.

Denote by $M_\kappa^n$ the following classical metric spaces:

(1) if $\kappa = 0$ then $M_0^n$ is the Euclidean space $\mathbb{R}^n$;

(2) if $\kappa > 0$ then $M_\kappa^n$ is obtained from the sphere $\mathbb{S}^n$ by multiplying the spherical distance by $1/\sqrt{\kappa}$;

(3) if $\kappa < 0$ then $M_\kappa^n$ is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the hyperbolic distance by $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(X,d)$ consists of three points in $X$ (the *vertices* of $\Delta$) and a geodesic segment between each pair of vertices (the *edges* of $\Delta$). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $M_\kappa^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. If $\kappa > 0$ it is further assumed that the perimeter of $\Delta(x_1, x_2, x_3)$ is less than $2D_\kappa$, where $D_\kappa$ denotes the diameter of $M_\kappa^2$. Such a triangle always exists.

A geodesic metric space is said to be a CAT($\kappa$) space if all geodesic triangles of appropriate size satisfy the following CAT($\kappa$) comparison axiom.

CAT($\kappa$): Let $\Delta$ be a geodesic triangle in $X$ and let $\overline{\Delta} \subset M_\kappa^2$ be a comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the CAT($\kappa$) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces.*

Observe that if $x, y_1, y_2$ are points of a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$ then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2$$

because equality holds in the Euclidean metric. In fact *a geodesic metric space is a CAT*(0) *space if and only if it satisfies inequality* (which is known as the CN inequality of Bruhat and Tits). In particular, $d(x, y_1) \leq R$, $d(x, y_2) \leq R$, and $d(y_1, y_2) \geq r$ imply

$$d(x, y_0) \leq \left(1 - \delta\left(\frac{r}{R}\right)\right) R$$

where $\delta(\varepsilon) = \sqrt{1 - \frac{\varepsilon^2}{4}}$ and so one has the usual Euclidean modulus of convexity in CAT(0) spaces.

## 2.2 The Convexity Structures

*Let $M$ be an abstract set. A family $\Sigma$ of subsets of $M$ is called a convexity structure if*

*(i) the empty set $\emptyset \in \Sigma$;*

*(ii) $M \in \Sigma$;*

*(iii) $\Sigma$ is closed under arbitrary intersections.*

The convex subsets of $M$ are the elements of $\Sigma$. If $M$ is a metric space, we will always assume that closed balls are convex. The smallest convexity structure which contains the closed balls is $\mathcal{A}(M)$ the family of admissible subsets of $M$.

Recall that $A$ is an admissible subset of $M$ if it is an intersection of closed balls. In his attempt to generalize the concept of weak-compactness, Kijima and Takahashi [**?**] considered compact metric spaces, which was very restrictive. Penot [**?**] on the other hand, defined compactness for convexity structures which leads to weak-compactness in the linear case. Indeed, a convexity structure $\Sigma$ is said to be compact if and only if every family of subsets of $\Sigma$ which has the finite intersection property has a nonempty intersection, i.e. if $(A_i)_{i\in I}$, with $A_i \in \Sigma$, then

$$\bigcap_{i\in I} A_i \neq \emptyset$$

provided $\displaystyle\bigcap_{i\in I_f} A_i \neq \emptyset$ for any finite subset $I_f$ of $I$.

In the linear case (Banach spaces), the natural family which defines the convexity is the family of convex sets. Clearly this family is bigger than the family of admissible sets. In 1979, it was discovered that in the case of $l^\infty$, the family of admissible sets enjoy a geometric property (uniform normal structure) which the family of closed convex sets fail (see [**?**] and [**?**]). It was an amazing result which opened the door to major discoveries.

Recall the following definitions

**Definition.** Let $\Sigma$ be a convexity structure. We will say that $\Sigma$ is normal (resp. uniformly normal) if for any $A \in \Sigma$ not reduced to one point, there exists $a \in A$ such that

$$\sup\{d(a,x);\ x \in A\} < \sup\{d(x,y);\ x,y \in A\} = \mathrm{diam}(A)$$

resp.

$$\sup\{d(a,x);\ x \in A\} \leq \alpha\ \mathrm{diam}(A)$$

where $\alpha < 1$ and is independent of $A$ (depends only on $\Sigma$).

**A Wonderful Example.** Consider the Banach space $l_\infty$. Then the family of admissible subsets is compact and normal. In fact we have for any nonempty $A \in \mathcal{A}(l_\infty)$, there exists $a \in A$ such that

$$\sup\{d(a,x);\ x \in A\} = \frac{1}{2}\,\mathrm{diam}(A)\ .$$

This is not the case for the family of all closed convex subsets of $l_\infty$. Investigating this example closely, one can generalize this conclusion to a large class of metric spaces called hyperconvex metric spaces. The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [**?**] who discovered it when investigating an extension of Hahn-Banach theorem in metric spaces. The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin (see for instance [**?**, **?**, **?**, **?**]). The nonlinear theory is still developing. The recent interest into these spaces goes back to the results of

Sine [?] and Soardi [?] who proved independently that fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces.

In general weak compactness is used to prove the existence of a minimal invariant set via Zorn's lemma. In 1979, Gillespie and Williams [?] showed that a constructive proof (of Kirk's fixed point theorem [?]) may be found which uses only countable compactness. In other words, the convexity structure $\Sigma$ is assumed to satisfy a countable intersection property, i.e. for any $(A_n)_{1\leq n}$, with $A_n \in \Sigma$, then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

provided $\bigcap_{n=1}^{m} A_n \neq \emptyset$ for any $m \geq 1$. This weakening is very important since in many practical cases, we do not have a compactness generated by a topology but a compactness defined sequentially. The latest usually generates some kind of countable compactness. Note also that if the convexity structure $\Sigma$ is uniformly normal then it is countably compact [?]. This is an amazing metric translation of a well known similar result in Banach spaces due to Maluta [?]. It is natural to ask whether a convexity structure which is countably compact is basically compact. The answer is yes if we are dealing with uniform normal structure. In fact, if $\mathcal{A}(M)$ is countably compact and normal, then $\mathcal{A}(M)$ is compact.

## 3 Generalized Metric Spaces

In 1986 Jawhari, Misane and Pouzet [?] were able to show that Sine and Soardi's fixed point theorems are equivalent to the classical Tarski's fixed point theorem [?] in complete ordered sets (via generalized metric spaces).
Traditionally discrete techniques are often used in metric spaces (via Zorn's lemma for example). The other way is not that common. But let us generalize the common distance into the discrete case. For that let M be an arbitrary set and $\mathcal{V}$ a set with a binary operation which will be denoted $\oplus$. We assume that $\oplus$ enjoys most of the properties that the classical addition does. In particular, we have a zero element $0 \in \mathcal{V}$ which satisfies $u \oplus 0 = 0 \oplus u = u$, for any $u \in \mathcal{V}$. We also assume that $\mathcal{V}$ is ordered by $\leq$ such that $0 \leq u$ and $u \oplus v \leq u' \oplus v'$ whenever $u \leq u'$ and $v \leq v'$. Note that $\mathcal{V}$ is not totally behaving like the set of positive numbers since we do not have a multiplication operation on $\mathcal{V}$.

**Definition.** Let $\mathcal{V}$ be a set as described above and $M$ be an arbitrary set. The mapping $d : M \times M \to \mathcal{V}$ is called a generalized distance if

(i) $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = \tau\Big(d(y, x)\Big)$, where $\tau$ is an involution, that is $\tau(\tau(x)) = x$;

(iii) $d(x,y) \leq d(x,z) \oplus d(z,y)$

To appreciate the power behind the above new concept, let us show how any ordered set is a generalized metric space. Let $(\mathcal{M}, \prec)$ be an ordered set. Set $\mathcal{V} = \{0, \alpha, \beta, 1\}$ ordered as: $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $\alpha$ and $\beta$ are not comparable. Define $\oplus$ as the max operation, i.e. $u \oplus v = \max(u,v)$. Define the generalized distance $d$ by

(i) $d(x,x) = 0$;

(ii) $d(x,y) = \alpha$, if $x \prec y$;

(iii) $d(x,y) = \beta$, if $y \prec x$;

(iv) $d(x,y) = 1$, if $x$ and $y$ are not comparable;

(v) $\tau(0) = 0$, $\tau(\alpha) = \beta$, $\tau(\beta) = \alpha$, and $\tau(1) = 1$.

With $d$, $\mathcal{M}$ becomes a generalized metric space. Armed with this distance, we can talk about balls (left and right to be precise) and admissible sets. It is quite amazing to show that if $(\mathcal{M}, \prec)$ is a complete lattice, then $\mathcal{A}(\mathcal{M})$ is compact and normal. Also it is an amazing fact to see that nonexpansive maps are exactly the monotone increasing maps. The famous Tarski theorem [**?**] becomes Kirk theorem [**?**]. More can be said and little is known....

## 3.1 More on Hyperconvex Metric Spaces

Recall That a metric space $M$ is said to be hyperconvex if for any collection of points $\{x_\alpha\}_{\alpha \in \Gamma}$ in $M$ and positive numbers $\{r_\alpha\}_{\alpha \in \Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any $\alpha$ and $\beta$ in $\Gamma$, we must have

$$\bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha) \neq \emptyset \; .$$

Also a metric space $M$ is said to be *injective* if it has the following extension property: Whenever $Y$ is a subspace of $X$ and $f : Y \to M$ is nonexpansive, then $f$ has a nonexpansive extension $\tilde{f} : X \to M$. This fact has several nice consequences.

**Theorem 3.1.** *Let $H$ be a metric space. The following statements are equivalent:*

*(i) $H$ is hyperconvex;*

*(ii) $H$ is injective.*

A similar result to Aronszajn and Panitchpakdi's main theorem may be stated in terms of retractions as follows.

**Theorem 3.2.** *Let $H$ be a metric space. The following statements are equivalent:*

*(i) $H$ is hyperconvex;*

*(ii) for every metric space $M$ which contains $H$ metrically, there exists a nonexpansive retraction $R : M \to H$;*

Note that statement (ii) is also known as an absolute retract property. This is why hyperconvex metric spaces are also called absolute nonexpansive retract (or in short ANR).

# 4 Some Open Problems

Sine and Soardi results are at the origin of the recent interest to hyperconvex metric spaces. Both Sine and Soardi showed that nonexpansive mappings defined on a bounded hyperconvex metric space have fixed points. Their results were stated in different context but the underlying spaces are simply hyperconvex spaces.

**Theorem 4.1.** *Let $H$ be a bounded hyperconvex metric space. Any nonexpansive map $T : H \to H$ has a fixed point. Moreover, the fixed point set of $T$, $Fix(T)$, is hyperconvex.*

Note that since $\mathrm{Fix}(T)$ is hyperconvex, then any commuting nonexpansive maps $T_i$, $i = 1, 2, ..., n$, defined on a bounded hyperconvex set $H$, have a common fixed point. Moreover their common fixed point set $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \cap \cdots \cap \mathrm{Fix}(T_n)$ is hyperconvex.

Combining these results with Baillon's theorem, we get the following:

**Theorem 4.2.** *Let $H$ be a bounded hyperconvex metric space. Any commuting family of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : H \to H$, has a common fixed point. Moreover, the common fixed point set $\bigcap_{i\in I} Fix(T_i)$ is hyperconvex.*

**Remark 4.3.** *Baillon asked whether boundedness may be relaxed. He precisely asked whether the conclusion holds if the nonexpansive map has a bounded orbit. In the classical Kirk's fixed point theorem [**?**], having a bounded orbit implies the existence of a fixed point. Prus answered this question in the negative [**?**]. Indeed, consider the hyperconvex Banach space $H = l_\infty$ and the map $T : H \to H$ defined by*

$$T\Big((x_n)\Big) = (1 + \lim_{\mathcal{U}} x_n, x_1, x_2, ...)$$

*where $\mathcal{U}$ is a nontrivial ultrafilter on the set of positive integers. We may also take a Banach limit instead of a limit over an ultrafilter. The map $T$ is an isometry and has no fixed point. On the other hand, we have*

$$T^n(0) = (1, 1, ..., 1, 0, 0, ...)$$

*where the first block of length $n$ has all its entries equal to 1 and 0 after that. So $T$ has bounded orbits.*

Recently, we wondered whether Sine and Soardi's theorem holds for asymptotically nonexpansive mappings. Recall that a map $T$ is said to be **asymptotically nonexpansive** if

$$d(T^n(x), T^n(y)) \leq \lambda_n d(x, y)$$

and $\lim\limits_{n} \lambda_n = 1$. This question is till unknown. But a partial positive answer is known for approximate fixed points [**?**]. Before we state this result, recall that if $T : H \to H$ is a map, then $x \in H$ is an **$\varepsilon$-fixed point** if $d(x, T(x)) \leq \varepsilon$ where $\varepsilon \geq 0$. The set of $\varepsilon$-fixed points of $T$ is denoted by $\mathrm{Fix}_\varepsilon(T)$. Sine obtained the following wonderful result:

**Theorem 4.4.** *Let $H$ be a bounded hyperconvex metric space and $T : H \to H$ a nonexpansive map. For any $\varepsilon > 0$, $Fix_\varepsilon(T)$ is not empty and is hyperconvex.*

Now we are ready to state the following result [**?**].

**Theorem 4.5.** *Let $H$ be a bounded hyperconvex metric space and $T : H \to H$ be asymptotically nonexpansive. For any $\varepsilon > 0$, $Fix_\varepsilon(T)$ is not empty, in other words we have*

$$\inf_{x \in H} d(x, T(x)) = 0.$$

Next we discuss the case of uniformly Lipschitzian mappings defined in hyperconvex metric spaces.

**Definition 4.6.** *Let $M$ be a metric space. A mapping $T : C \to C$ of a subset $C$ of $M$ is said to be* Lipschitzian *if there exists a non-negative number $k$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x$ and $y$ in $C$. The smallest such $k$ is called* Lipschitz constant *and will be denoted by $Lip(T)$. Same mapping is called* uniformly Lipschitzian *if $\sup\limits_{n \geq 1} Lip(T^n) < \infty$.*

It is well-known fact that if a map is uniformly Lipschitzian, then one may find an equivalent distance for which the map is nonexpanive (see [**?**] and [**?**]). Indeed, let $T : C \to C$ be uniformly Lipschitzian. Setting

$$\rho(x, y) = sup\{d(T^n x, T^n y) : n = 0, 1, 2...\}$$

for $x, y \in C$, one can obtain a metric $\rho$ on $C$ which is equivalent to the metric $d$ and relative to which $T$ is nonexpansive. In this context, it is natural to ask the question: if a set $C$ has the fixed point property (fpp) for nonexpansive mappings with respect to the metric $d$, then does $C$ also have (fpp) for mappings which are nonexpansive relative to an equivalent metric? This is known as the stability of (fpp). The first result in this direction is due to Goebel and Kirk [**?**]. Motivated by such questions, the following fixed point theorems of uniformly Lipschitzian mappings in hyperconvex metric spaces is still open:

**Problem.** Let $M$ be a bounded hyperconvex metric space. Let $T : M \to M$ be a uniformly Lipschitzian map such that

$$\sigma(T) = \sup_{n \geq 1} Lip(T^n) \leq k \ .$$

For what value of $k$ does $T$ have a fixed point?

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