The numerical index of the L_p space

by

Elmouloudi ED-DARI and Mohamed Amine KHAMSI
(France) (USA)

Abstract. We give a partial answer to the problem of the numerical index of L_p-space $1 < p < \infty$. Also we give a geometric characterization of Banach spaces with the Radon-Nikodým property to have numerical index 1. Finally, we show that the numerical index of the trace class operator S_1 cannot be equal to one.

1 Introduction.

Given a Banach space X over $\mathbb{R}$ or $\mathbb{C}$, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and $B(X)$ is the Banach algebra of all bounded linear operators on X. The *numerical range* of an operator $T \in B(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx);\ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}\cdot$$

The *numerical radius* is then given by

$$v(T) = \sup\{|\lambda|,\ \lambda \in V(T)\}\cdot$$

Clearly, v is a semi norm on $B(X)$, and $v(T) \leq \|T\|$ for every $T \in B(X)$. It was shown by Glickfeld [**13**] (and essentially by Bohnenblust and Karlin [**4**]) that if X is a complex space, then $e^{-1}\|T\| \leq v(T)$ for every $T \in B(X)$ where $e = \exp 1$, so that for complex spaces v is always a norm and it is equivalent to the operator norm $\|\ \|\cdot$ Thus it is natural to consider the so-called *numerical index* of the Banach space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T),\ T \in S_{B(X)}\}\cdot$$

Obviously, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in B(X)\cdot$ Note that for any complex Banach space X, $e^{-1} \leq n(X) \leq 1$.
The concept of the numerical index was first suggested by G. Lumer [**18**] in a lecture to the North British Functional Analysis Seminar in 1968. At that time, it was known that if X is a complex Hilbert space (with $dimX > 1$) then $n(X) = \frac{1}{2}$

and if it is real then $n(X) = 0$ so that for real spaces, $0 \leq n(X) \leq 1$.
Later, Duncan, McGregor, Pryce and White [**8**] determined the range of values of the numerical index. More precisely they proved that

$$\{n(X),\ X \ real\ Banach\ space\} = [0,1]$$

$$\{n(X),\ X \ complex\ Banach\ space\} = [e^{-1},1].$$

Recently, Finet, Martín and Payá [**12**] also studied the values of the numerical index from the isomorphic point of view and they proved that

$$\mathcal{N}\ (c_0) = \mathcal{N}\ (l_1) = \mathcal{N}\ (l_\infty) = \begin{cases} [0,1] & \text{in the real case} \\ [e^{-1},1] & \text{in the complex case} \end{cases}$$

where $\mathcal{N}\ (X) := \{n(X,p) : p \in \mathcal{E}\ (X)\}$, $\mathcal{E}\ (X)$ denotes the set of all equivalent norms on the Banach space X and $n(X,p)$ is the numerical index of X equipped with the norm p.

The authors in [**8**] proved that the extreme case $n(X) = 1$ occurs for a large class of interesting spaces including all real or complex L-spaces and M-spaces. Lately, López, Martín and Payá [**19**] studied some real Banach spaces with numerical index 1. In fact, they proved that an infinite dimensional real Banach space with numerical index 1 and satisfying the Radon-Nikodym property contains l_1. This result is a partial answer to the conjecture [**21**] : Every infinite dimensional Banach space with numerical index 1 contains either l_1 or c_0. Very recently, Martín and Payá [**23**] studied the numerical index of vector-valued function spaces and they proved that if K is a compact Hausdorff space and μ is a positive measure then the Banach spaces $C(K,X)$ and $L_1(\mu,X)$ have the same numerical index as the Banach space X. For general information and background on numerical ranges we refer to the books by Bonsall and Duncan [**2**], [**3**]. Further developments in the Hilbert space may be found in [**14**].
The computation of $n(\text{Ł}_p)$ for $1 < p < \infty$, $p \neq 2$ is much more complicated, in fact it is an open problem since 1968.

In this paper we give a partial answer to this problem (Theorem 2.1). Actually, we prove that for $1 \leq p < \infty$, the numerical index of the Banach space $L_p([0,1],\mu)$ where μ is the Lebesgue measure on the unit interval is equal to the numerical index of the l_p space. And it is well known that the numerical index of the Banach space l_p is the limit of the sequence of numerical index of finite-dimensional subspaces l_p^m $m = 1,2,....$ [**11**]. The computation of the numerical index of the l_p^m-space gives then a complete answer to the problem of the numerical index of the L_p-space. On the other hand we give a geometric characterization of Banach spaces with the Radon-Nikodým property to have numerical index 1 (Theorem 2.4) . And as an application we show that the numerical index of the Banach space S_1 of the trace-class operator is less than 1.

2 Main results.

Theorem 2.1. *For $1 < p < \infty$, the numerical index of the Banach space $L_p[0,1]$ is equal to the numerical index of l_p, i.e.*

$$n(L_p[0,1]) = n(l_p).$$

Before, we prove this theorem, we need to introduce some notations.
Let μ be the Lebesgue measure on $\Omega = [0,1]$. For each integer $n \geq 1$ we denote by π_{2^n} the partition of Ω into 2^n dyadic intervals: $[0, \frac{1}{2^n}[, [\frac{1}{2^n}, \frac{2}{2^n}[, \cdots, [\frac{2^n-1}{2^n}, 1]$.
By V_n we denote the subspace of $L_p(\Omega,\mu)$ defined by

$$V_n = \left\{ \sum_{k=1}^{2^n} a_k 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}[}; a_k \in \mathbb{C} \right\}.$$

P_n denotes the projection of $L_p(\Omega,\mu)$ onto V_n defined by

$$P_n(f) = \sum_{k=1}^{2^n} \left[\frac{1}{2^{-n}} \int_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} f(t)\, d\mu(t) \right] 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}]}.$$

And V denotes the union of all subspaces V_n of $L_p(\Omega,\mu)$. We recall that V is a dense subspace of $L_p(\Omega,\mu)$ ([**6**], p 140), that is, for each $f \in L_p(\Omega,\mu)$ the sequence $(P_n f)_n$ converges to f in $L_p(\Omega,\mu)$.

Lemma 2.2. *For each integer $n \geq 1$ and $T \in B(V_n)$ there exists $\tilde{T} \in B(V_{n+1})$ satisfying the following conditions*
(i) $\tilde{T}_{|V_n} = T$
(ii) $\|\tilde{T}\| = \|T\|$
(ii) $v(\tilde{T}) = v(T)$.

Proof. Let $E_k^{2^n}$ denotes the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}[$ for $k = 1, 2, ..., 2^n$.
STEP **1.** *We shall first prove that the conditions in Lemma 2.2 are satisfied for n=1.*
Let $T \in B(V_1)$ represented by its matrix $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ on the unit normal basis $\left\{ \frac{1_{E_1^{2^1}}}{\mu(E_1^{2^1})^{\frac{1}{p}}}, \frac{1_{E_2^{2^1}}}{\mu(E_2^{2^1})^{\frac{1}{p}}} \right\}$ of V_1. Since $\mu(E_1^{2^1})^{\frac{1}{p}} = \mu(E_2^{2^1})^{\frac{1}{p}} = 2^{-\frac{1}{p}}$, the same matrix represents T on the basis $\left\{ 1_{E_1^{2^1}}, 1_{E_2^{2^1}} \right\}$. Consider the operator $\tilde{T} \in B(V_2)$

represented by its matrix

$$\begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix}$$

on the normal basis $\left\{ \dfrac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}} \right\}_{k=1,\ldots,4}$ of V_2. Also, since $\mu(E_k^{2^2}) = 2^{\frac{-2^2}{p}}$, $k = 1, ..., 4$ the same matrix represents $\tilde{T}$ on the basis $\{1_{E_k^{2^2}}\}_{k=1,\ldots,4}$.

For (i), we have

$$\begin{aligned} \tilde{T}(1_{E_1^{2^1}}) &= \tilde{T}(1_{E_1^{2^2}}) + \tilde{T}(1_{E_2^{2^2}}) \\ &= t_{11}1_{E_1^{2^2}} + t_{21}1_{E_3^{2^2}} + t_{11}1_{E_2^{2^2}} + t_{21}1_{E_4^{2^2}} = T(1_{E_1^{2^1}}) \cdot \end{aligned}$$

Also, $\tilde{T}(1_{E_2^{2^1}}) = T(1_{E_2^{2^1}})$. This means that $\tilde{T}_{|V_1} = T$.

For (ii), let $x = \sum_{k=1}^{4} x_k \dfrac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}} \in V_2$. Clearly, $\|x\|^p = \sum_{k=1}^{4} |x_k|^p$ and we have

$$\tilde{T}(x) = \begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix}.$$

From this we obtain

$$\begin{aligned} \|\tilde{T}(x)\|^p &= |t_{11}x_1 + t_{12}x_3|^p + |t_{11}x_2 + t_{12}x_4|^p + |t_{21}x_1 + t_{22}x_3|^p + |t_{21}x_2 + t_{22}x_4|^p \\ &= \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \right\|^p + \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \right\|^p \\ &\leq \|T\|^p(|x_1|^p + |x_2|^p + |x_3|^p + |x_4|^p). \end{aligned}$$

This means that $\|\tilde{T}\| \leq \|T\|$ and since $\tilde{T}_{|V_1} = T$ we have $\|T\| \leq \|\tilde{T}\|$. Therefore $\|\tilde{T}\| = \|T\|$.

For (iii), we have

$$v(\tilde{T}) = \sup\{|x_x^*(\tilde{T}x)|; x \in S_{V_2}\} = |x_x^*(\tilde{T}x)|$$

for some $x = \sum_{k=1}^{4} x_k \dfrac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}}$; $\|x\|^p = \sum_{k=1}^{4} |x_k|^p = 1$, and $x_x^* = \sum_{k=1}^{4} \varepsilon_k |x_k|^{p-1} \dfrac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{q}}}$ where ε_k is a scalar number such that $\varepsilon_k x_k = |x_k|$. From the previous expression of $\tilde{T}(x)$ we have

$$\begin{aligned} x_x^*(\tilde{T}x) &= \left(\varepsilon_1|x_1|^{p-1}, ..., \varepsilon_4|x_4|^{p-1}\right) \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix} \\ &= \left(\varepsilon_1|x_1|^{p-1}, \varepsilon_3|x_3|^{p-1}\right) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \\ &\qquad \left(\varepsilon_2|x_2|^{p-1}, \varepsilon_4|x_4|^{p-1}\right) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \left|x_x^*(\tilde{T}x)\right| &\leq v(T)\left\|\left(\varepsilon_1|x_1|^{p-1}, \varepsilon_3|x_3|^{p-1}\right)\right\|_q \left\|(x_1, x_3)\right\|_p + \\ &\quad v(T)\left\|\left(\varepsilon_2|x_2|^{p-1}, \varepsilon_4|x_4|^{p-1}\right)\right\|_q \left\|(x_2, x_4)\right\|_p \leq v(T)\cdot \end{aligned}$$

Since $\tilde{T}_{|V_1} = T$ we have also $v(T) \leq v(\tilde{T})$ and consequently $v(\tilde{T}) = v(T)$.
STEP **2.** *For every integer $n \geq 1$ and $T \in B(V_n)$ there exists $\tilde{T} \in B(V_n)$ satisfying the conditions (i), (ii) and (iii).*
Indeed, let $n \geq 1$ and $T \in B(V_n)$ represented by its matrix

$$\begin{bmatrix} t_{11} & \cdots & t_{12^n} \\ \vdots & & \vdots \\ t_{2^n1} & \cdots & t_{2^n2^n} \end{bmatrix}$$

on the normal basis $\left\{\dfrac{1_{E_k^{2^n}}}{\mu(E_k^{2^n})^{\frac{1}{p}}}\right\}_{k=1,\ldots,2^n}$ of V_n. As in step 1, we check that the operator $\tilde{T} \in B(V_{n+1})$ defined by its matrix

$$\begin{bmatrix} \begin{pmatrix} t_{11} & 0 \\ 0 & t_{11} \end{pmatrix} & \cdots & \begin{pmatrix} t_{12^n} & 0 \\ 0 & t_{12^n} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} t_{2^n1} & 0 \\ 0 & t_{2^n1} \end{pmatrix} & \cdots & \begin{pmatrix} t_{2^n2^n} & 0 \\ 0 & t_{2^n2^n} \end{pmatrix} \end{bmatrix}$$

as in the normal basis $\left\{\dfrac{1_{E_k^{2(n+1)}}}{\mu(E_k^{2(n+1)})^{\frac{1}{p}}}\right\}_k$ as in $\left\{1_{E_k^{2(n+1)}}\right\}_k$ of V_{n+1} satisfy the three conditions (i),(ii) and (iii). □

Proof of Theorem 2.1. We shall prove that the numerical index of $L_p[0,1]$ is less than the numerical index of l_p.

Let $n \geq 1$ and $U := T_n \in B(V_n)$. Following the previous lemma we can find a sequence of operators $\{T_m\}_{m\geq n}$, $T_m : V_m \to V_m$, satisfying the following conditions:

(i) $T_{m+1|_{V_m}} = T_m$

(ii) $\|T_{m+1}\| = \|T_m\|$

(iii) $v(T_{m+1}) = v(T_m)$.

Let $x \in L_p[0,1]$ and $x_m \in V_m$ for $m \geq n$ such that x_m converges to x in $L_p[0,1]$. The sequence $\{T_m(x_m)\}_{m\geq n}$ converges in $L_p[0,1]$. Indeed, for $m, m^{'} \geq n$ with $m^{'} \geq m$, we have

$$T_{m'}(x_{m'}) - T_m(x_m) = T_{m'}(x_{m'} - x_m) + (T_{m'} - T_m)(x_m)\cdot$$

Since $T_{m'|_{V_m}} = T_m$ and $\|T_{m'}\| = \|T_n\|$ we obtain

$$\|T_{m'}(x_{m'}) - T_m(x_m)\| = \|T_{m'}(x_{m'} - x_m)\| \leq \|T_n\|\|x_{m'} - x_m\|\cdot$$

Let $T(x) = \lim_m T_m(x_m)$ (note that $T(x)$ is independent of the choice of $\{x_m\}_m$). Clearly, T is a bounded linear operator on $L_p[0,1]$. Moreover,

(i) $T_{|_{V_n}} = T_n$

(ii) $\|T\| = \|T_n\|$

(iii) $v(T) = v(T_n)$.

The first conditions (i) and (ii) are clear. For (iii), let $\varepsilon > 0$. There exists $x \in L_p[0,1]$ such that

$$v(T) - \varepsilon \leq |x_x^*(Tx)|\cdot$$

Here $x_x^* = \varepsilon_x|x|$ where ε_x is the sign function of x, that is, $\varepsilon_x(t)x(t) = |x(t)|$ for all $t \in [0,1]$. But $T(x) = \lim_m T_m(x_m)$ for x_m converges to x in $L_p[0,1]$ and $x_m \in S_{V_m}$ for all m. Since the norm of $L_p[0,1]$ is Fréchet differentiable [**6**], the sequence $x_{x_m}^*$ converges to x_x^* in $L_q[0,1]$. Consequently

$$v(T) - \varepsilon \leq |x_x^*(Tx)| = \lim_m |x_{x_m}^*(T_m x_m)| \leq \lim_m v(T_m) = v(T_n)$$

which means that $v(T) \leq v(T_n)$. Since $T_{|_{V_n}} = T_n$ we have also $v(T_n) \leq v(T)$ and therefore $v(T) = v(T_n)$. It follows from this that

$$\{v(T_n); T_n \in S_{V_n}\} \subset \{v(T); T \in S_{L_p[0,1]}\},$$

and then

$$n(L_p[0,1]) \leq n(V_n).$$

Since V_n is isometric to $l_p^{2^n}$, we obtain

$$n(L_p[0,1]) \leq n(l_p^{2^n}).$$

According to Theorem 2.5 [**11**] we get

$$n(L_p[0,1]) \leq \lim_n n(l_p^{2^n}) = n(l_p)\cdot$$

Also, it was proved (Theorem 3.1 [**11**]) that

$$n(L_p(\mu)) \geq n(l_p)$$

for any Banach space $L_p(\mu), 1 \leq p < \infty$. This completes the proof of Theorem 2.1. □

D. Li and C. Finet asked if the numerical index of the real Banach space l_p is equal to 0 for some $p \neq 2$.
In the following we prove that the numerical index of all Banach spaces $l_p^m, p \neq 2, m = 1, 2, ...$ cannot be equal to 0.

Theorem 2.3. *For every real number $p, 1 \leq p < 2$ and $m = 2, 3, \cdots$, the numerical index of the Banach space l_p^m is positive.*

Proof. Fix a real number p, $1 < p < 2$ and let m be an integer with $m \geq 2$. Consider

$$y = \frac{e_i + \varepsilon t e_j}{(1+t^p)^{\frac{1}{p}}} \text{ and } y^* = \frac{e_i^* + \varepsilon t^{p-1} e_j^*}{(1+t^p)^{\frac{p-1}{p}}}, \;\; 0 \leq t \leq 1, \;\; \varepsilon \in \{-1, 1\}.$$

Here $\{e_i\}$ is the canonical basis of l_p^m. Clearly, $\|y\|_{l_p^m} = \|y^*\|_{l_q^m} = y^*(y) = 1$. For $T = (t_{lk})_{lk} \in B(l_p^m)$ and by a simple calculus we have

$$y^*(Ty) = \frac{1}{1+t^p}(t_{ii} + \varepsilon t\ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}).$$

From this we obtain

$$|t_{ii} + \varepsilon t\ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| \leq (1+t^p) v(T)\cdot$$

Then

$$\max_{\varepsilon \in \{-1,1\}} |t_{ii} + \varepsilon t\ t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| = |t_{ii} + t^p t_{jj}| + |t\ t_{ij} + t^{p-1} t_{ji}| \leq (1+t^p) v(T)\cdot$$

Since

$$|t_{ii} + t^p t_{jj}| \geq |t_{ii}| - t^p|t_{jj}| \text{ and } |t\ t_{ij} + t^{p-1}t_{ji}| \geq t^{p-1}|t_{ji}| - t|t_{ij}|,$$

we have then

$$|t_{ii}| - t^p|t_{jj}| + t^{p-1}|t_{ji}| - t|t_{ij}| \leq (1 + t^p)v(T)\cdot$$

Similarly

$$|t_{jj}| - t^p|t_{ii}| + t^{p-1}|t_{ij}| - t|t_{ji}| \leq (1 + t^p)v(T)\cdot$$

From last two inequalities we obtain

$$(1 - t^p)(|t_{ii}| + |t_{jj}|) + (t^{p-1} - t)(|t_{ij}| + |t_{ji}|) \leq 2(1 + t^p)v(T)\cdot$$

Since $(1 - t^p) \geq t^p - t$, we have

$$(t^{p-1} - t)(|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq 2(1 + t^p)v(T)\cdot$$

Hence

$$\frac{M_p}{2}(|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq v(T) \qquad (*)$$

where $M_p = \sup_{t\in[0,1]} \dfrac{t^{p-1} - t}{1 + t^p}$. Thus if $v(T) = 0$ for $T \in B(l_p^m)$ we have necessarily $\|T\| = 0$ and consequently $n(l_p^m) = v(T) > 0$ for some $T \in S_{l_p^m}$. □

Remark. Following (*) in the previous proof, v is a norm on $B(l_p)$ for $p \neq 2$. And still unknown if it is an equivalent norm to the operator norm.

On the McGregor's characterization of finite dimensional spaces with numerical index 1.

Before we proceed, we need more notations and definitions. Indeed, let K be a convex subset of X, the set $ext(K)$ denotes the subset of extreme points of K and the set $exp(K)$ the subset of exposed points of K. Recall that $x \in exp(K)$ if and only there exists $x^* \in X^*$ such that $\Re x^*(y) < \Re x^*(x)$ for every $y \in K\backslash\{x\}\cdot$
By $sexp(K)$ the set of strongly exposed points of K, and $x \in sexp(K)$ if there exists $x^* \in X^*$such that
(i) $\Re x^*(y) < \Re x^*(x)$ for every $y \in K\backslash\{x\}$
and
(ii) $\lim_{\delta\to 0} diam\ S(x^*, K, \delta) = 0$
where $S(x^*, K, \delta) = \{y \in K,\ \Re x^*(y) > \Re x^*(x) - \delta\}\cdot$ Finally, for $x \in S_X$, $D(x)$ denoted the subset $\{x^* \in X^*,\ x^*(x) = 1 = \|x^*\|\}$.

Let us recall McGregor's characterization of finite dimensional spaces with numerical index 1.

Theorem. ([**20**], Theorem 3.1) *Let X be a finite dimensional normed space over $\mathbb{R}$ or $\mathbb{C}\cdot$ Then the following are equivalent,*
(i) $n(X) = 1$
(ii) for all $x \in ext(B_X)$ and all $x^* \in ext(B_{X^*})$, $|x^*(x)| = 1$
(iii) for all $x \in ext(B_X)$ and all $y \in S_X$, there exists a scalar λ with $|\lambda| = 1$ such that $D(\lambda x) \cap D(y) \neq \emptyset$.

We should make two observations about finite dimensional normed spaces. First if X has finite dimension, then the topology of X is independent of the norm, so that, for example, as sets, X^* and $exp(K)$ are independent of the norm on X. Second if X has finite dimension, then B_X is norm compact and it is equal to $conv(extB_X)$ (see, for example Klee [**16**], [**17**]). However in any infinite dimensional Banach space X, B_X is never norm-compact, and we may have $ext(B_X) = \emptyset$ (for example, c_0, $L_1[0,1]$).

Theorem 2.4. *Let X be an infinite dimensional Banach Space over $\mathbb{R}$ or $\mathbb{C}$ with the Radon-Nikodým property (RNP). Then, the following are equivalent,*
(i) $n(X) = 1$
(ii) for all $x \in sexp(B_X)$ and all $x^* \in ext(B_{X^*})$, $|x^*(x)| = 1$.

It is well known that strongly exposed points are denting points, then following [**19**] (lemma 1) (ii) yields (i). For the sake of completeness we give here another proof and under the assumption : X has the RNP, the converse is true.

Proof. Assume that (i) is satisfied. We recall that if $x^* \in ext(B_{X^*})$ then according to [**7**, p.107] the set $\{S^*(y, B_{X^*}, \delta)\}_{\substack{y\in S_X\\ \delta\in]0,1[}}$ is a base of x^* for the weak* topology in B_{X^*}. So if $\{x^*_{(y,\delta)}\}$ is a net in B_{X^*} with $x^*_{(y,\delta)} \in S^*(y, B_{X^*}, \delta)$ then $x^*_{(y,\delta)} \xrightarrow{W^*} x^*$.
Let $x^* \in ext(B_{X^*})$ and $x \in sexp(B_X)$. There is $y^* \in S_{X^*}$ which exposes strongly x, that is,
(i) $1 = y^*(x) > \Re y^*(B_X \backslash \{x\})$
(ii) $\lim\limits_{\delta\to 0} diam\ S(y^*, B_X, \delta) = 0\cdot$
This is equivalent to say that for every sequence (x_n) in B_X such that $y^*(x_n)$ converges to $y^*(x)$ then x_n converges to x in norm. Let $\varepsilon > 0$ and $(y,\delta) \in S_X\times]0,1[$. We consider the operator $T : X \to X \quad z \mapsto y^*(z)y\cdot$ Since $n(X) = 1$, then according to [**11**] (Lemma 2.2) we have

$$1 = \|T\| = v(T) = \sup\{|x^*_x(Tx)|, x \in S_X\} \quad (x^*_x \in D(x))\cdot$$

Let $(x_n) \subset S_X$ such that $|x^*_{x_n}(Tx_n)| = |x^*_{x_n}(y)||y^*(x_n)| \to 1$. We may assume that $|x^*_{x_n}(y)|y^*(x_n) \to 1$, then $y^*(x_n) \to 1$ and $|x^*_{x_n}(y)| = e^{i\theta_n} x^*_{x_n}(y) \to 1$. Take $x_{(y,\delta)} \in S_X$ and $\theta_{(y,\delta)} \in \mathbb{R}$ such that $\|x_{(y,\delta)} - x\| < \varepsilon$ and $e^{i\theta_{(y,\delta)}} x^*_{x_{(y,\delta)}} \in S^*(y, B_{X^*}, \delta)$. Now we have

$$x^*(x) = (x^* - e^{i\theta_{(y,\delta)}} x^*_{x_{(y,\delta)}})x + e^{i\theta_{(y,\delta)}} x^*_{x_{(y,\delta)}}(x - x_{(y,\delta)}) + e^{i\theta_{(y,\delta)}}.$$

So

$$1 - |x^*(x)| \leq |x^*(x) - e^{i\theta_{(y,\delta)}}| \leq |(x^* - e^{i\theta_{(y,\delta)}} x^*_{x_{(y,\delta)}})x| + \varepsilon,$$

which implies the desired result.
Assume now that *(ii)* is satisfied. Note that $n(X) = 1$ if and only if $v(T) \geq 1$ for every $T \in S_{B(X)}$. Let $T \in S_{B(X)}$ and $\varepsilon > 0$. By assumption X has the RNP, thus $B_X = \overline{conv}(sexpB_X)$ ([**10**], Theorem 1, p 259) and we can choose $x_0 \in sexp(B_X) \subset ext(B_X)$ such that $1 - \varepsilon \leq \|T(x_0)\|$. There exists $x_0^* \in ext(B_{X^*})$ such that $\|T(x_0)\| = |x_0^*(Tx_0)|$. By *(ii)* we obtain $1 - \varepsilon < |x_0^*(Tx_0)| \leq v(T)$ which yields $n(X) = 1$. □

In [**22**] Martín gave other characterizations of Banach spaces having the Radon-Nikodým property and numerical index 1.

Remark. We find Mc Gregor's result from the previous theorem. Indeed, If X is a finite dimensional normed space, it is reflexive and then it possesses the Radon-Nikodým property [**5**] (Corollary 5.12). Furthermore, a simple application of Milman's theorem shows that strongly exposed points in B_X are norm-dense in extreme points of B_X.

Corollary 2.5. *Let X^* be a separable dual space. Then the following assertions are equivalent:*
(i) $n(Y) = 1$
(ii) $|y^*(y)| = 1$ *for all* $y \in sexp(B_Y)$ *and all* $y^* \in ext(B_{Y^*})$.

Proof. Follows from the previous Theorem and the the fact that every separable dual space possesses the Radon-Nikodým property [**9**].

Application : We shall prove that the numerical index of the Banach space S_1 of the trace-class operator is less than 1.

First let us recall the following two theorems.

Theorem 2.6. *([**1**], Arazy) Let S_1 denote the Banach space of the trace-class operator acting on an infinite-dimensional complex Hilbert space, and assume that*

$T \in B_{S_1}$. *The following statements are equivalent:*
(a) T *has rank 1 and* $tr(T^*T) = 1$*;*
(b) T *is an extreme point of* B_{S_1}*;*
(c) T *is a strongly exposed point of* B_{S_1}.

Theorem 2.7. *([**15**], p. 263) Let* H *be a Hilbert space. The following statement are equivalent:*
(a) The operator $U \in B(H)$ *is an extreme point of* $B_{B(H)}$*;*
(b) U *or its adjoint-operator* U^* *is an isometry.*

Now let $\{e_n\}$ be the canonical basis of the Hilbert space l_2 endowed with the inner product $<,>$ and consider the rank 1 operator $T \in B(l_2)$ defined by
$T(x) =< x, e_1 > e_1, \ \ x \in l_2$. A simple calculus shows that $T^* = T$ and $tr(T^*T) = \sum_{k=1}^{\infty} < T^*T(e_k), e_k >=< e_1, e_1 >= 1$. This shows that $T \in sexp(B_{S_1})$ (Theorem 2.6). Let $U \in B(l_2) = S_1^*$ be the shift operator defined by $U(e_k) = e_{k+1}$, $k = 1, 2, ...$ It is clear that U is an isometry thus $U \in ext(B_{B(l_2)})$ (Theorem 2.7), and we have $tr(UT) = 0$. Theorem 2.4 asserts that $n(S_1) < 1$.

Corollary 2.8. *Let* H *be an infinite dimensional complex Hilbert space. Then the numerical index of the Banach space* $B(H)$ *is less than 1.*

References

[1] J. Arazy, *Basic sequences, embeddings, and the uniqueness of the symmetric structure in unitary matrix spaces*, J. Funct. Anal. 40 (1981), no. 3, 302-340.

[2] F. F. Bonsall and J. Duncan, *Numerical ranges of Operators on Normed Spaces and of elements of Normed Algebras,* London Math. Soc. Lecture Note Ser. 2, (Cambridge Univ. Press, 1971).

[3] F. F. Bonsall and J. Duncan, *Numerical ranges II,* London Math. Soc. Lecture Note Ser. 10 (Cambridge Univ. Press, 1971).

[4] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere in Banach algebra*, Ann. of Math. 62 (1955), 217-229.

[5] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear, Functional Analysis, Vol. 1.* Colloquium publications (American Mathematical Society), v. 48, 2000.

[6] B. Beauzamy, Introduction to Banach Spaces and their Geometry, Second revised edition, Nort-Holland-Amsterdam. New York. Oxford (1985).

[7] G. Choquet. *Lectures on Analysis*, Vol. II. W. A. Benjamin, New York, 1969.

[8] J. Duncan, C. M. McGregor, J. D. Pryce and A. J. White, *The numerical index of a normed space,* J. London Math. Soc. (2) 2 (1970), 481-488.

[9] N. Dunford and B. J. Pettis, *Linear operations on summable functions,* Trans. AMS., 47 (1940), 323-392.

[10] J. Diestel, *Geometry of Banach Spaces-Selected Topics,* (Lecture Notes in Mathematics, 485), Springer-Verlag, 1975.

[11] E. Ed-dari, *On the numerical index of Banach spaces*, to appear.

[12] C. Finet, M. Martín and R. Payá, *Numerical index and renorming*, Proc. Amer. Math. Soc. 131 (2003), no. 3, 871-877.

[13] B. W. Glickfeld, *On an inequality of Banach algebra geometry and semi-inner product space theory*, Illinois. J. Math. 14(1970), 76-81.

[14] K. E. Gustafsan and D. K. M. Rao, *Numerical Range. The Field of Values of Linear Operators and Matrices,* Springer, New York, 1997.

[15] P. R. Halmos, A Hilbert Space Book, Second Edition, Springer-Verlag, New York, Hiedelberg, Berlin, 1982.

[16] V. L. Klee, Jr, *Extremal structure of convex sets*, Arch. Math, 8(1957), 234-240.

[17] V. L. Klee, Jr, *Extremal structure of convex sets II*, Math. Z, 69(1958), 90-104.

[18] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29-43.

[19] G. López, M. Martín and R. Payá, *Real Banach spaces with numerical index 1*, Bull. London Math. Soc. 31 (1999) 207-212.

[20] C. M. McGregor, *Finite dimensional normed linear spaces with numerical index 1*, J. London Math. Soc. (2) 3 (1971), 717-721.

[21] M. Martín, *A survey on the numerical index of Banach space*, Extracta Math 15 (2000), 265-276.

[22] M. Martín, *Banach spaces having the Radon-Nikodým property and numerical index 1*, Proc. Amer. Math. Soc 131 (2003), no. 11, 3407-3410.

[23] M. Martín and R. Payá, *Numerical index of vector-valued function spaces*, Studia Mathematica 142 (3) (2000) 269-280.

E. ED-DARI
Université D'Artois
Faculté des sciences Jean Perrin
SP 18
62307-Lens Cedex
FRANCE
Elmouloudi.Eddari@euler.univ-artois.fr

M. A. KHAMSI
University of Texas at El Paso
Dept. of Mathematical Sciences
500 West University
Ave. El Paso, Texas 79968-0514
USA
mohamed@math.utep.edu