Note on a fixed point theorem in Banach lattices.

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Abstract

In this note we give the complete and detailed proof of Lemma 1 of [1]. We also discuss a counterexample that was suggested for this lemma.

1 Definitions and Main results

Let E be a real Banach space endowed with a lattice structure satisfying

$$(x^+ \leq y^+ \ \ and \ \ x^- \leq y^-) \Longrightarrow ||x|| \leq ||y||$$

for all $x, y \in E$.
Let τ be the coarsest topology on E for which the map $x \to ||\ |x| \wedge u\ ||$ from E to $R+$ is continuous at 0 for every $u \in E_+$.

Example. Let E be a real Banach space endowed with an unconditional Schauder basis (e_n). Then E is a vector lattice for the "coordinatewise order" defined by

$$\sum x_n e_n \leq \sum y_n e_n \ \ when \ \ x_n \leq y_n$$

for $n = 0, 1, 2..$ The topology τ is the topology of coordinatewise convergence.

It is easy to see that, in general, τ is a Hausdorff linear topology coarser than the topology defined by the norm. Furthermore we have the following result.

Proposition. Every convex τ-compact subset C of E is norm-bounded.
Proof. We will prove a stronger result. Indeed, let θ be a Hausdorff linear topology on E coarser than the topology defined by the norm and C be a nonempty θ-compact convex subset of E. Let us show that C is norm-bounded.
Let E_C be the linear subspace spanned by C, i.e $E_C = \bigcup t(C - C); t > 0$. Define

$$||x||_C = \inf\{t > 0; x \in t(C - C)\}$$

for $x \in E_C$. Clearly $||.||_C$ is a norm on E_C since $C - C$ is θ-compact. Let us show that $(E_C, ||.||_C)$ is a Banach space. So consider (x_n) to be a cauchy sequence in $(E_C, ||.||_C)$. Since $\sup ||x_n||_C < \infty$ then the θ-closure of $\{x_n; n \in N\}$ is θ-compact. Therefore, there exists $x \in E$ a θ-cluster point of (x_n). Let $\epsilon > 0$ because (x_n) is cauchy there exists $n_0 \in N$ such that

$$x_n - x_m \in \epsilon(C - C) \ \ for \ \ n, m \geq n_0.$$

Since $x - x_m$ is a θ-cluster point of $(x_n - x_m)_n$ we deduce that

$$x - x_m \in \epsilon(C - C) \ \ for \ m \geq n_0$$

because $\epsilon(C-C)$ is θ-closed. Therefore $x \in E_C$ and $||x - x_m||_C \leq \epsilon$ for all $m \geq n_0$,i.e. (x_n) converges to x in $(E_C, ||.||_C)$.
Following the same idea one can easily show that the graph of the canonical injection $i_C : E_C \to E$ is closed in $(E_C, ||.||_C) \times (E, \theta)$. A fortiori it is also closed in $(E_C, ||.||_C) \times (E, \theta)$. Therefore $i_C : (E_C, ||.||_C) \to (E, ||.||)$ is continuous. This clearly implies that $C - C$,and therefore C, is bounded. The proof of our claim is therefore complete.

Remark. It was suggested as counterexample to Lemma 1, the following:
Let E be a Banach space with (e_n) as Schauder basis. Consider the closed convex hull C of $\{0, e_1, 2e_2, ..., ne_n, ..\}$. Then C is compact for the pointwise topology but it is not bounded!.

First, we give an example when this situation does not occur. Indeed, let E and C be as described above and $\sum \lambda_n$ be a convergent series of positive numbers. Put

$$x_n = \sum_{i=1}^{i=n} \lambda_i i e_i \ \ for \ all \ \ n \in N.$$

Then $x_n \in C$ for all $n \in N$. Assume that (x_n) has a subsequence which is converging pointwise. Then the limit point should be $x = \sum_{n=1}^{\infty} \lambda_n n e_n$. So one should have $\lim_n n\lambda_n = 0$ because x would be in E. To see that this can fail to happen one can take

$$\lambda_n = k^{-3} \ \ for \ \ n = k^4 \ \ and \, 0 \ \ otherwise.$$

Then $\sum_n \lambda_n < \infty$ and $\limsup n\lambda_n = \infty$.
the mystake in the counterexample described above is that the convex hull C is compact in R^N the space of all real sequences, which is strictly bigger than E.

References

[1] M. A. Khamsi and Ph. Turpin,"Fixed points of nonexpansive mappings in Banach lattices", Proc. Amer. Math. Soc., Volume 105(1989), 102-110.