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Uniformly Lipschitzian mappings in modular function spaces

T. Dominguez Benavides[a,1], M.A. Khamsi[b,*], S. Samadi[a] ,

[a] Department of Mathematical Analysis, University of Seville, P.O. Box 1160, 41080 Seville, Spain
[b] Department of Mathematical Science, The University of Texas at El Paso, El Paso, TX 79968, USA

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0. Introduction

The theory of modular spaces was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see, for instance, [15]). Another direction is based on considering an abstractly given functional which controls the growth of the functions. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces (see, for instance, [8] and references therein). In this paper, we study the existence of fixed points for a more general class of mappings: uniformly Lipschitzian mappings. Fixed point theorems for this class of mappings in Banach spaces have been studied in [2,3] and in metric spaces in [11,12] (for further information about this subject, see [1, Chapter VIII] and references therein). The main tool in our approach is the coefficient of normal structure $\tilde{\mathrm{N}}(L_\rho)$. We prove that under suitable conditions a k-uniformly Lipschitzian mapping has a fixed point if $k < (\tilde{\mathrm{N}}(L_\rho))^{-1/2}$. In the last section we show a class of modular spaces where $\tilde{\mathrm{N}}(L_\rho) < 1$ and so, the above theorem can be successfully applied.

* Corresponding author. Tel.: +1-915-747-6763; fax: +1-915-747-6502.
E-mail address: tomas@mathiutep.edu (T.D. Benavides), mohamed@math.utep.edu (M.A. Khamsi), sedesi@cica.es (S. Samadi).

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1. Preliminaries

We start by recording a brief collection of basic concepts and facts of modular spaces as formulated by Kozlowski. For more details the reader is refered to [7,8,10,13].

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathscr{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $\Omega = \bigcup K_n$. In other words, the family $\mathscr{P}$ plays the role of the δ-ring of subsets of finite measure. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from $\mathscr{P}$. By $\mathscr{M}$ we will denote the space of all measurable functions, i.e. all functions $f : \Omega \to \mathfrak{R}$ such that there exists a sequence $\{g_n\} \in \mathscr{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.1. A functional $\rho : \mathscr{E} \times \Sigma \to [0,\infty]$ is called a function modular if
(P$_1$) $\rho(0,E) = 0$ for any $E \in \Sigma$,
(P$_2$) $\rho(f,E) \le \rho(g,E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega$, $f,g \in \mathscr{E}$ and $E \in \Sigma$,
(P$_3$) $\rho(f,.) : \Sigma \to [0,\infty]$ is a σ-subadditive measure for every $f \in \mathscr{E}$,
(P$_4$) $\rho(\alpha,A) \to 0$ as α decreases to 0 for every $A \in \mathscr{P}$, where $\rho(\alpha,A) = \rho(\alpha 1_A, A)$,
(P$_5$) if there exists $\alpha > 0$ such that $\rho(\alpha,A) = 0$, then $\rho(\beta,A) = 0$ for every $\beta > 0$,
(P$_6$) for any $\alpha > 0$ $\rho(\alpha,.)$ is order continuous on $\mathscr{P}$, that is $\rho(\alpha,A_n) \to 0$ if $\{A_n\} \in \mathscr{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathscr{M}$ by

$$\rho(f,E) = \sup\{\rho(g,E); g \in \mathscr{E},\ |g(\omega)| \le |f(\omega)|\, \omega \in \Omega\}.$$

A set E is said to be ρ-null if $\rho(\alpha,E) = 0$ for every $\alpha > 0$. For the sake of simplicity we write $\rho(f)$ instead of $\rho(f,\Omega)$.

It is easy to see that the functional $\rho : \mathscr{M} \to [0,\infty]$ is a modular because it satisfies the following properties:
(i) $\rho(f) = 0$ iff $f = 0$ ρ-a.e.
(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathscr{M}$.
(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \ge 0, \beta \ge 0$ and $f,g \in \mathscr{M}$.
In addition, if the following property is satisfied
(iii)$'$ $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$; $\alpha \ge 0, \beta \ge 0$ and $f,g \in \mathscr{M}$, we say that ρ is a convex modular.
The modular ρ defines a corresponding modular space, i.e the vector space L_ρ given by

$$L_\rho = \{f \in \mathscr{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

We can also consider the space $E_\rho = \{f \in \mathscr{M}; \rho(\alpha f, A_n) \to 0 \text{ as } n \to \infty$ for every $A_n \in \Sigma$ that decreases to $\emptyset$ and $\alpha > 0\}$.

A function modular is said to satisfy the Δ_2-condition if $\sup_{n\ge 1} \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{f_n\}_{n\ge 1} \subset \mathscr{M}$, $\mathscr{D}_k \in \Sigma$ decreases to $\emptyset$ and $\sup_{n\ge 1} \rho(f_n, D_k) \to 0$ as $k \to \infty$. We know (see [10]) that $E_\rho = L_\rho$ when ρ satisfies the Δ_2-condition. When ρ

is convex, the formula

$$\|f\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\right\}$$

defines a norm in the modular space L_ρ which is frequently called the Luxemburg norm.

Definition 1.2. (1) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$,

(2) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \nrightarrow f(\omega)\}$ is ρ-null.

(3) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞,

(4) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(5) A subset C of L_ρ is called ρ-a.e. sequentially closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.

(6) A subset C of L_ρ is called ρ-a.e. sequentially compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(7) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f - g); f, g \in C\} < \infty.$$

Let B be a bounded subset of L_ρ. We define the ρ-ball of center $f \in L_\rho$ and radius $r > 0$ by $B(f,r) = \{g \in L_\rho,\ \rho(g - f) \leq r\}$. We will denote $r(f,B) = \sup\{\rho(f - g),\ g \in B\}$, $\delta(B) = \sup\{r(f,B),\ f \in B\}$, $R(B) = \inf\{r(f,B),\ f \in B\}$. We define the admissible hull of B as the intersection of all ρ-ball containing B, i.e.

$$ad(B) = \bigcap\{A\colon B \subset A \subset L_\rho, \text{ where } A \text{ is a } \rho\text{-ball}\}.$$

B is said admissible if $ad(B) = B$. We define the normal structure coefficient $\tilde{\mathrm{N}}(L_\rho)$ of L_ρ by

$$\tilde{\mathrm{N}}(L_\rho) = \sup\{R(B)/\delta(B),\ B \text{ is admissible, } \rho\text{-bounded and } \rho\text{-a.e. sequentially compact}\}.$$

The useful following proposition is easily seen:

Proposition 1.1. *Let B be a ρ-bounded subset of L_ρ and $f \in L_\rho$. Then*
(1) $r(f, ad(B)) = r(f,B)$.
(2) $\delta(ad(B)) = \delta(B)$.

We say that ρ satisfies the Δ_2-type condition if there exists $K > 0$ such that $\rho(2f) \leq K\rho(f)$ for all $f \in L_\rho$. In general, Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that Δ_2-type condition implies Δ_2-condition. Assume that ρ is convex and satisfies the Δ_2-type condition. We define a growth function

ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)},\ 0 < \rho(f) < \infty\right\} \quad \text{for all } 0 \le t < \infty.$$

The following properties of the growth function can be easily seen.

Lemma 1.1. *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then the growth function ω has the following properties*:

(1) $\omega(t) < \infty,\ \forall t \in [0,\infty)$.
(2) $\omega : [0,\infty) \to [0,\infty)$ *is a convex, strictly increasing function. So, it is continuous.*
(3) $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0,\infty)$.
(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \le \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0,\infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 1.2 (Dominguez Benavides et al. [4]). *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then*

$$\|f\|_\rho \le \frac{1}{\omega^{-1}(1/\rho(f))} \quad \textit{whenever } f \in L_\rho.$$

The following lemma can be found in [7].

Lemma 1.3. *Let ρ be a function modular satisfying the Δ_2-condition and $\{f_n\}_n$ be a sequence in L_ρ such that $f_n \overset{\rho\text{-a.e.}}{\to} f \in L_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g) \quad \textit{for all } g \in L_\rho.$$

Lemma 1.4. *Let ρ be a modular function satisfying the Δ_2-type condition. Let B be a ρ-a.e. sequentially closed and ρ-bounded subset of L_ρ. Let $\{g_n\}_n$ be a sequence in B such that $g_n \overset{\rho\text{-a.e.}}{\to} g$. Then,*

(1) $\rho(g) \le \liminf_{n\to\infty} \rho(g_n)$.
(2) $B(0,r) \cap B$ is *ρ-a.e. sequentially closed.*
(3) $ad(A) \cap B$ *is ρ-a.e. sequentially closed, for all $A \subset L_\rho$.*

Proof. Condition (1) is a straighforward consequence of Lemma 1.3 applied to the sequence $g_n \overset{\rho\text{-a.e.}}{\to} g$ and the null function. Conditions (2) and (3) can be easily deduced from (1). □

2. Fixed point for uniformly Lipschitzian mappings

The following lemma is the key of our fixed point result.

Lemma 2.1. *Let ρ be a modular function satisfying the Δ_2-type condition and B a ρ-bounded and ρ-a.e. sequentially compact subset of L_ρ. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in B. Then, there exists $g \in \bigcap_{n=1}^{\infty} ad(g_j,\ j \ge n) \cap B$ such that*

$$\limsup_{n\to\infty} \rho(g - f_n) \le \limsup_{j\to\infty} \limsup_{n\to\infty} \rho(g_j - f_n)$$

Proof. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in B. We define $\theta(h) = \limsup_{n\to\infty} \rho(h - f_n)$ for all $h \in B$. Since B is ρ-sequentially compact and ρ-bounded, there exist a subsequence $\{g_{\phi(n)}\}_n \subset \{g_n\}_n$ such that $g_{\phi(n)} \overset{\rho\text{-a.e.}}{\to} g$ and a subsequence $\{f_{\psi(n)}\}_n \subset \{f_n\}_n$ such that $\lim_{n\to\infty} \rho(f_{\psi(n)} - g) = \limsup_{n\to\infty} \rho(f_n - g)$ and $f_{\psi(n)} \overset{\rho\text{-a.e.}}{\to} f \in B$. Since $g_{\phi(n)} \in ad(g_j,\ j \ge n) \cap B$ which is ρ-a.e. sequentially closed (by property (3) of Lemma 1.4) and $g_{\phi(n)} \overset{\rho\text{-a.e.}}{\to} g$, we obtain $g \in ad(g_j,\ j \ge n) \cap B$ for all $n \ge 1$. We will see that $\theta(g) \le \limsup_{j\to\infty} \theta(g_j)$. Indeed, from Lemma 2.3 we have $\theta(g_j) = \limsup_{n\to\infty} \rho(f_n - g_j) \ge \liminf_{n\to\infty} \rho(f_{\psi(n)} - g_j) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \rho(f - g_j)$. Thus, again using Lemma 2.3, we obtain

$$\begin{aligned}
\limsup_{j\to\infty} \theta(g_j) &\ge \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \limsup_{j\to\infty} \rho(f - g_j) \\
&\ge \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \liminf_{j\to\infty} \rho(f - g_{\phi(j)}) \\
&= \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \liminf_{j\to\infty} \rho(g_{\phi(j)} - g) + \rho(f - g).
\end{aligned}$$

On the other hand, $\theta(g) = \limsup_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - g) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \rho(f - g)$. Therefore, $\theta(g) \le \limsup_{j\to\infty} \theta(g_j)$. □

The following lemma is inspired on [2] where a similar lemma is proved in reflexive Banach spaces (see also [12, Lemma 6] for a version in metric spaces with additional properties).

Lemma 2.2. *Let ρ be a function modular satisfying the Δ_2-type condition and B an be admissible, ρ-a.e. sequentially compact and ρ-bounded subset of L_ρ. Let $\{f_n\}$ be a sequence in B and c a constant such that $c > \tilde{N}(L_\rho)$. Then there exists $f \in B$ such that*
(1) $\limsup_{n\to\infty} \rho(f - f_n) \le c\delta(\{f_n\}_n)$.
(2) $\rho(f - g) \le \limsup_{n\to\infty} \rho(f_n - g)$ *for all* $g \in B$.

Proof. Let $\{f_n\}_n$ be a sequence of B. Denote $A_m = ad(f_j\colon j \ge m) \subset B$ and $A = \bigcap_{m=1}^{\infty} A_m$. Since B is ρ-a.e. sequentially compact, there exists a subsequence of $\{f_n\}_n$ ρ-a.e. convergent, say to h. It is clear that $h \in A$ and so $A \ne \emptyset$. Furthermore, from Proposition 1.1(2), we have $\delta(A_n) \le \delta(\{f_n\}_n)$. On the other hand, for any $f \in A$ and $g \in B$ we have $\rho(g - f) \le r(g, A) \le r(g, A_n) = r(g, \{f_j\colon j \ge n\}) = \sup_{j\ge n} \rho(g - f_j)$. Therefore, $\rho(g - f) \le \limsup_{n\to\infty} \rho(g - f_n)$ and (2) holds for any $f \in A$. We will prove that there exists $f \in A$ satisfying (1). Without loss of generality we may assume that $\delta(\{f_n\}_n) > 0$. Choose $\varepsilon > 0$ such that $\tilde{N}(L_\rho)\delta(\{f_n\}_n) + \varepsilon \le c\delta(\{f_n\}_n)$. By definition of $R(A_n)$, there exists $g_n \in A_n$ such that $r(g_n, A_n) < R(A_n) + \varepsilon \le \tilde{N}(L_\rho)\delta(A_n) + \varepsilon \le$

$\tilde{\mathrm{N}}(L_\rho)\delta(\{f_n\}_n)+\varepsilon \le c\delta(\{f_n\}_n)$. Since $r(g_n,A_n)=r(g_n,\{f_j\}_{j\ge n})=\sup_{j\ge n}\rho(g_n-f_j)$, we have

$$\limsup_{j\to\infty}\rho(g_n-f_j)\le c\delta(\{f_n\}_n). \tag{A}$$

Using Lemma 2.5, there exists $f\in\bigcap_{n=1}^{\infty} ad(g_i,\ i\ge n)$ such that

$$\limsup_{j\to\infty}\rho(f-f_j)\le \limsup_{n\to\infty}\limsup_{j\to\infty}\rho(g_n-f_j). \tag{B}$$

We will check that $f\in A$. Indeed, for all i,n integers such that $i\ge n$ we have $g_i\in A_i\subset A_n$. Thus, $\{g_i\}_{i\ge n}\subset A_n$ which implies $ad(g_i,\ i\ge n)\subset A_n$ and $f\in A$. Using (B) and (A) it is clear that $\limsup_{j\to\infty}\rho(f-f_j)\le c\delta(\{f_n\}_n)$. □

Theorem 2.1. *Let ρ be a convex function modular satisfying the Δ_2-condition and B an admissible, ρ-a.e. sequentially compact and ρ-bounded subset of L_ρ. Suppose that $\tilde{\mathrm{N}}(L_\rho)<1$ and let $T:B\to B$ be a k-uniformly Lipschitzian mapping satisfying $k<(\tilde{\mathrm{N}}(L_\rho))^{-1/2}$. Then, T has a fixed point.*

Proof. We can assume that $k>1$; otherwise T will be nonexpansive and the existence of a fixed point is a consequence of [8, Theorem 3.5]. Choose a constant c, $\tilde{\mathrm{N}}(L_\rho)<c<1$ such that $1<k<c^{-1/2}$. Fix $f_0\in B$. By Lemma 2.6, we can inductively construct a sequence $\{f_j\}_{j\ge 0}\subset B$ such that for each $j\ge 0$

(1) $\limsup_{n\to\infty}\rho(T^n(f_j)-f_{j+1})\le c\delta(\{T^n(f_j)\}_n)$.

(2) $\rho(f_{j+1}-g)\le \limsup_{n\to\infty}\rho(T^n(f_j)-g)$ for all $g\in B$.

Denote $D_j=\limsup_{n\to\infty}\rho(T^n(f_j)-f_{j+1})$ and $h=ck^2<1$. For $n\ge m\ge 0$, we have

$$\begin{aligned}\rho(T^m f_j-T^n f_j)&\le k\rho(f_j-T^{n-m}f_j)\\ &\le k\limsup_{i\to\infty}\rho(T^i f_{j-1}-T^{n-m}f_j)\\ &\le k^2\limsup_{i\to\infty}\rho(T^{i-(n-m)}f_{j-1}-f_j)\\ &\le k^2 D_{j-1}.\end{aligned}$$

Since $D_j=\limsup_{n\to\infty}\rho(T^n(f_j)-f_{j+1})\le c\delta(\{T^n(f_j)\}_n)$, we obtain $D_j\le ck^2D_{j-1}=hD_{j-1}$. Thus, $D_j\le h^jD_0$ and we have

$$\begin{aligned}\rho(f_{j+1}-f_j)&\le \omega(2)(\rho(f_{j+1}-T^nf_j)+\rho(f_j-T^nf_j))\\ &\le \omega(2)(\rho(f_{j+1}-T^nf_j)+\limsup_{m\to\infty}\rho(T^mf_{j-1}-T^nf_j))\\ &\le \omega(2)(\rho(f_{j+1}-T^nf_j)+k\limsup_{m\to\infty}\rho(T^{m-n}f_{j-1}-f_j))\\ &\le \omega(2)(\rho(f_{j+1}-T^nf_j)+kD_{j-1}).\end{aligned}$$

Taking lim sup as $n\to\infty$, we obtain

$$\rho(f_{j+1}-f_j)\le \omega(2)(D_j+kD_{j-1})$$

$$\leq \omega(2)(h^j + kh^{j-1})D_0$$
$$\leq \omega(2)(h+k)h^{j-1}D_0$$
$$\leq Ah^j, \text{ where } A = \omega(2)\frac{h+k}{h}D_0.$$

Hence, there exists an integer N and some $\beta < 1$ such that for $j > N$ we have $\rho(f_{j+1} - f_j) \leq \beta^j$, which implies $1/\beta^j \leq 1/\rho(f_{j+1} - f_j)$. Using properties (2) and (3) of Lemma 1.1 we obtain

$$\omega^{-1}\left(\frac{1}{\beta^j}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{j+1} - f_j)}\right)$$

and

$$\left(\omega^{-1}\left(\frac{1}{\beta}\right)\right)^j \leq \omega^{-1}\left(\frac{1}{\rho(f_{j+1} - f_j)}\right).$$

Therefore, by Lemma 2.2 we have

$$\|f_{j+1} - f_j\|_\rho \leq \frac{1}{\omega^{-1}\left(\frac{1}{\rho(f_{j+1} - f_j)}\right)} \leq \frac{1}{\left(\omega^{-1}\left(\frac{1}{\beta}\right)\right)^j}.$$

Hence $\{f_j\}$ is a Cauchy sequence in $(L_\rho, \|\,.\,\|_\rho)$, there exists $f \in L_\rho$ such that $\|f_j - f\|_\rho \to 0$, because $(L_\rho, \|\,.\,\|_\rho)$ is complete. Since under Δ_2-condition norm-convergence and modular-convergence are identical, $\{f_j\}$ is modular convergent to f. Thus, there exists a subsequence of $\{f_j\}_j$ ρ-a.e. convergent to f [1, Theorem 1] and f belongs to B because B is ρ-a.e. sequentially closed. We will prove that f is a fixed point of T. Indeed,

$$\begin{aligned}\rho(f - Tf) &\leq \omega(3)(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j) + \rho(T^n f_j - Tf))\\ &\leq \omega(3)(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j) + k\rho(T^{n-1} f_j - f))\\ &\leq \omega(3)g(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j)\\ &\quad + k\omega(2)(\rho(T^{n-1} f_j - f_{j+1}) + \rho(f_{j+1} - f))g).\end{aligned}$$

Taking limsup as $n \to \infty$, we have

$$\rho(f - Tf) \leq \omega(3)(\rho(f - f_{j+1}) + D_j + k\omega(2)(D_j + \rho(f_{j+1} - f))).$$

Now, taking lim as $j \to \infty$, we obtain $\rho(f - Tf) = 0$, i.e. $T(f) = f$. $\square$

3. Uniformly convex modular spaces

Our goal in this section is to give some classes of modular funtions spaces such that $\tilde{\mathrm{N}}(L_\rho) < 1$. We begin by recalling the definitions of ρ-modulus of uniform convexity [9].

For any ε and any $r>0$, the ρ-modulus of uniform convexity is defined by

$$\begin{aligned}\delta_\rho(r,\varepsilon) &= \inf\left\{1-\frac{1}{r}\rho\left(\frac{f+g}{2}\right); \rho(f)\le r, \rho(g)\le r, \rho\left(\frac{f-g}{2}\right)\ge r\varepsilon\right\}\\ &= \inf\left\{1-\frac{1}{r}\rho\left(f+\frac{h}{2}\right); \rho(f)\le r, \rho(f+h)\le r; \rho\left(\frac{h}{2}\right)\ge r\varepsilon\right\}.\end{aligned}$$

This following lemma gives a relationship between $\tilde{\mathrm{N}}(L_\rho)$ and the ρ-modulus of uniform convexity.

Lemma 3.1. *Let ρ be a convex function modular satisfying Δ_2-condition. Then,*

$$\tilde{N}(L_\rho)\le 1-\inf_{d>0}\delta_\rho(d,\gamma) \text{ for all } \gamma\in\left(0,\frac{1}{\omega(2)}\right).$$

Proof. Let B be an admissible, ρ-bounded and ρ-a.e. sequentially compact subset of L_ρ. We know that B is a convex set because it is an intersection of ρ-balls which are convex, as a consequence of the convexity of ρ. Denote $d=\delta(B)$ and $r=R(B)$. Let $\varepsilon\in(0,1)$. There exist $f,g\in B$ such that $\rho(f-g)\ge\varepsilon\delta(B)$. Hence $\rho((f-g)/2)\ge \rho(f-g)/\omega(2)\ge(d-\varepsilon)/\omega(2)$. Let $h\in B$. We know that $\rho(h-f)\le d$, $\rho(h-g)\le d$ and $\rho((h-f)-(h-g)/2)\ge d\varepsilon/\omega(2)$. By definition of $\delta_\rho(d,\varepsilon/\omega(2))$, we have

$$\begin{aligned}\rho\left(h-\frac{f+g}{2}\right) &= \rho\left(\frac{(h-f)+(h-g)}{2}\right)\\ &\le d\left(1-\delta_\rho\left(d,\frac{\varepsilon}{\omega(2)}\right)\right)\end{aligned}$$

for all $h\in B$. Thus,

$$\frac{r}{d}\le 1-\delta_\rho\left(d,\frac{\varepsilon}{\omega(2)}\right).$$

Therefore,

$$\begin{aligned}\tilde{N}(L_\rho) &\le \sup_{d>0}\left(1-\delta_\rho\left(d,\frac{\varepsilon}{\omega(2)}\right)\right)\\ &\le 1-\inf_{d>0}\delta_\rho\left(d,\frac{\varepsilon}{\omega(2)}\right). \qquad \square\end{aligned}$$

Let $\Phi : R\to R^+$ is said to be an N-function if Φ is a convex symmetric function which satisfies
1. $\Phi(0)=0$
2. Φ is strictly increasing on $[0,\infty)$
3. $\lim_{u\to 0}\Phi(u)/u=0$ and $\lim_{u\to\infty}\Phi(u)/u=\infty$.

Let (G,Σ,μ) be a measure space, μ being finite and atomless. Consider the space $L^0(G)$ consisting of all measurable real-valued functions on G, and define the Orlicz function modular $\rho(f,B)=\int_{t\in B}\Phi(f(t))\,\mathrm{d}\mu(t)$ for every $f\in L^0(G)$ and $B\in\Sigma$. The modular function space L_ρ is the Orlicz space defined by

$$L_\rho=\{f\in L^0(G),\rho(\lambda f)<\infty \text{ for some } \lambda>0\}.$$

If Φ satisfies the Δ_2-condition at zero and at infinity i.e. $\limsup_{u\to 0}\Phi(2u)/\Phi(u)<\infty$ and $\limsup_{u\to\infty}\Phi(2u)/\Phi(u)<\infty$, the convex Orlicz modular associated to Φ satisfies the Δ_2-type condition. We recall that the function Φ is said to be uniformly convex [8] if for all $\varepsilon>0$ there exists $\delta(\varepsilon)\in(0,1)$ such that

$$0\le u \text{ and } v\le\varepsilon u \text{ implies } \Phi\left(\frac{u+v}{2}\right)\le(1-\delta(\varepsilon))\frac{\Phi(u)+\Phi(v)}{2}.$$

(Some equivalent definitions can be found in [6].)

The following lemma connects the uniform convexity of Φ and the ρ-modulus of uniform convexity of the modular.

Lemma 3.2. *Let Φ be a uniformly convex, N-function satisfying the Δ_2-condition at zero and at infinity and ρ the Orlicz function modular associated to Φ. Then there exists $\varepsilon_0\in(0,1)$, such that for every $\varepsilon\in(\varepsilon_0,1)$ there exists $\gamma(\varepsilon)\in(0,1/\omega(2))$ with* $\inf_{r>0}\delta_\rho(r,\gamma(\varepsilon))>0$.

Proof. We can find $\varepsilon_0\in(0,1)$ such that $(1-\varepsilon)/2\varepsilon<1/\omega(2)$ for all $\varepsilon\in(\varepsilon_0,1)$, Choose $\varepsilon\in(\varepsilon_0,1)$. By definition of uniform convexity, there exists $\delta(\varepsilon)\in(0,1)$ such that

$$0\le u \text{ and } v\le\varepsilon u \text{ implies } \Phi\left(\frac{u+v}{2}\right)\le(1-\delta(\varepsilon))\frac{\Phi(u)+\Phi(v)}{2}.$$

Choose $\gamma(\varepsilon)>0$ such that $(1-\varepsilon)/2\varepsilon<\gamma(\varepsilon)<1/\omega(2)$. Let r be a positive number and consider functions $f,g\in L_\rho$ such that $\rho(f)\le r,\rho(f+g)\le r$ and $\rho(h/2)\ge r\gamma(\varepsilon)$.

We consider the following sets:

$$G_1=\{t\in G/0\le f(t),\ f(t)<\varepsilon(f(t)+h(t))\},$$

$$G_2=\{t\in G/0\le f(t),\ f(t)+h(t)<\varepsilon f(t)\},$$

$$G_3=\{t\in G/f(t)<0,\ \varepsilon(f(t)+h(t))\le f(t)\},$$

$$G_4=\{t\in G/f(t)<0,\ \varepsilon f(t)\le f(t)+h(t)\}.$$

We have

$$\rho\left(f+\frac{h}{2}\right)=\int_{G\setminus\bigcup_{i=1}^{i=4}G_i}\Phi\left(f(t)+\frac{h(t)}{2}\right)\mathrm{d}t+\int_{\bigcup_{i=1}^{i=4}G_i}\Phi\left(f(t)+\frac{h(t)}{2}\right)\mathrm{d}t.$$

Using the definition of the uniform convexity for the function Φ on G_1,G_2,G_3 and G_4 we obtain

$$\Phi\left(\frac{f(t)+(f(t)+h(t))}{2}\right)\le(1-\delta(\varepsilon))\frac{\Phi(f(t))+\Phi(f(t)+h(t))}{2}$$

for every $t \in \bigcup_{i=1}^{i=4} G_i$. Hence, using the convexity of Φ in $G \setminus \bigcup_{i=1}^{i=4} G_i$ we have

$$\begin{aligned}
\int_G \Phi\left(f(t) + \frac{h(t)}{2}\right) \mathrm{d}t &\le \int_{G\setminus \bigcup_{i=1}^{i=4} G_i} \Phi\left(f(t) + \frac{h(t)}{2}\right) \mathrm{d}t \\
&\quad + (1-\delta(\varepsilon)) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} \mathrm{d}t \\
&\le \int_{G\setminus \bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} \mathrm{d}t \\
&\quad + (1-\delta(\varepsilon)) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} \mathrm{d}t \\
&= \int_G \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} \mathrm{d}t \\
&\quad - \delta(\varepsilon) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} \mathrm{d}t \\
&\le r - \delta(\varepsilon) \int_{\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) \mathrm{d}t, \qquad \text{(I)}
\end{aligned}$$

where the last inequality is again a consequence of the convexity and symmetry of Φ, because

$$\begin{aligned}
\Phi\left(\frac{h(t)}{2}\right) &= \Phi\left(\frac{(f(t)+h(t)) - f(t)}{2}\right) \\
&\le \frac{\Phi(f(t)+h(t)) + \Phi(-f(t))}{2} \\
&= \frac{\Phi(f(t)+h(t)) + \Phi(f(t))}{2}.
\end{aligned}$$

We claim that

$$\Phi\left(\frac{h(t)}{2}\right) \le \frac{1-\varepsilon}{2\varepsilon}\Phi(f(t)) \qquad \text{(II)}$$

for every $t \in G \setminus \bigcup_{i=1}^{i=4} G_i$. To prove this inequality we will consider two cases:

Case 1: Assume $f(t) \ge 0$. Since $t \in G \setminus \bigcup_{i=1}^{i=4} G_i$ we have

$$-\frac{1-\varepsilon}{2\varepsilon} f(t) \le \frac{h(t)}{2} \le \frac{1-\varepsilon}{2\varepsilon} f(t).$$

Therefore, by the symmetry and convexity of Φ, we obtain

$$\Phi\left(\frac{h(t)}{2}\right) = \Phi\left(\left|\frac{h(t)}{2}\right|\right) \le \Phi\left(\frac{1-\varepsilon}{2\varepsilon} f(t)\right) \le \frac{1-\varepsilon}{2\varepsilon}\Phi(f(t)).$$

Case 2: Assume $f(t) < 0$. Since $t \in G \setminus \bigcup_{i=1}^{i=4} G_i$ we have

$$-\frac{\varepsilon - 1}{2\varepsilon} f(t) < \frac{h(t)}{2} < \frac{\varepsilon - 1}{2\varepsilon} f(t)$$

and we obtain

$$\Phi\left(\frac{h(t)}{2}\right) = \Phi\left(\left|\frac{h(t)}{2}\right|\right) < \frac{1-\varepsilon}{2\varepsilon}\Phi(f)$$

as above.

Thus we proved the inequality (II). Hence,

$$\int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) \mathrm{d}t \le \frac{1-\varepsilon}{2\varepsilon} \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi(f(t))\, \mathrm{d}t$$

$$\le \frac{1-\varepsilon}{2\varepsilon} r$$

and we obtain

$$\int_{\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) \mathrm{d}t = \int_G \Phi\left(\frac{h(t)}{2}\right) \mathrm{d}t - \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) \mathrm{d}t$$

$$\ge r\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon} r. \tag{III}$$

From inequalities (I) and (III) we obtain

$$\rho\left(f + \frac{h}{2}\right) \le r\left(1 - \delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right)\right)$$

and

$$\delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right) \le 1 - \frac{\rho\left(f + \frac{h}{2}\right)}{r}.$$

Thus

$$\delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right) \le \delta_\rho(r, \gamma(\varepsilon)) \quad \text{for every } r > 0$$

and therefore

$$\inf_{r>0} \delta_\rho(r, \gamma(\varepsilon)) \ge \delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right) > 0. \qquad \square$$

Using Lemmas 3.1 and 3.2 we obtain the following corollary:

Corollary 3.1. *Let Φ be a uniformly convex N-function satisfying the Δ_2-condition at zero and at infinity. Then the modular function space L_ρ associated to Φ satisfies $\tilde{N}(L_\rho) < 1$.*

Remark 3.1. It is not difficult to find examples of functions satisfying the conditions in the above corollary. Besides $\Phi(t) = |t|^p$ for $p > 1$ we can obtain some other examples using the following result [5]: Φ is uniformly convex if $\limsup_{t\to 0} \Phi'(at)/\Phi'(t) < 1$ and $\limsup_{t\to\infty} \Phi'(at)/\Phi'(t) < 1$ for every $a \in (0,1)$. It is easy to check that $\Phi(t) = t^2 - \log(1+t^2)$ satisfies these conditions.

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References

[1] J.M. Ayerbe, T. Dominguez Benavides, G. Lopez Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Basel, Bikhauser, 1997.
[2] E. Casini, E. Maluta, Fixed points of uniformly Lipchitzian mappings in spaces with uniformly normal structure, Nonlinear Anal. 9 (1985) 103–108.
[3] T. Dominguez Benavides, Fixed point theorems for uniformly Lipschitzian mappings and asymptotically regular mappings, Nonlinear Anal. 32 (1) (1998) 15–27.
[4] T. Dominguez Benavides, M.A. Khamsi, S. Samadi, Asymptotically regular mappings in modular function spaces, preprint.
[5] H. Hudzik, A. Kaminska, J. Musielak, On the convexity coefficient of Orlicz spaces, Math. Z. 197 (1988) 291–295.
[6] A. Kaminska, On uniform convexity of Orlicz spaces, Mathematics, Proceedings. Konink. Nederl, Ak. Wet. Amsterdam A 85 (1) (1982) 27–36.
[7] M.A. Khamsi, Fixed point theory in modular function spaces, in: Recent Advances on Metric Fixed Point Theory, Universidad de Sevilla, Sevilla (1996), 31–58.
[8] M.A. Khamsi, W.M. Kozlowski, S. Reich, Fixed point theory in modular function spaces, Nonlinear Anal. 14 (1990) 935–953.
[9] M.A. Khamsi, W.M. Kozlowski, C. Shutao, Some geometrical properties and fixed point theorems in Orlicz Spaces, J. Math. Anal. Appl. 155 (2) (1991) 393–412.
[10] W.M. Koslowski, Modular Function Spaces, New York, Dekker, Basel, 1988.
[11] E.A. Lifschitz, Fixed point theorems for operators in strongly convex spaces, Voronez. Gos. Univ. Trudy Mat. Fak. 16 (1975) 23–28 (in Russian).
[12] T.C. Lim, H.K. Xu, Uniformly Lipschitzian mappings in metric spaces with uniform normal structure, Nonlinear Anal. 25 (11) (1995) 1231–1235.
[13] J. Musielak, W. Orlicz, On modular spaces, Studia Math. 18 (1959) 591–597.
[14] H. Nakano, Modulared semi-ordered spaces, Tokyo, 1950.
[15] D. Van Dultst, V. de Valk, (KK)-properties, normal structure and fixed points of nonexpansive mappings in Orlicz spaces, Canad. J. Math. 38 (1986) 728–750.