Journal of Mathematical Analysis and Applications **265,** 249–263 (2002)
doi:10.1006/jmaa.2000.7275, available online at http://www.idealibrary.com on IDEAL®

Asymptotically Nonexpansive Mappings in Modular Function Spaces

T. Dominguez-Benavides[1]

Department of Mathematical Analysis, University of Seville, P.O. Box 1160, 41080 Seville, Spain
E-mail: tomasd@cica.es

M. A. Khamsi

Department of Mathematical Science, The University of Texas at El Paso, El Paso, Texas 79968
E-mail: mohamed@math.utep.edu

and

S. Samadi

Department of Mathematical Analysis, University of Seville, P.O. Box 1160, 41080 Seville, Spain
E-mail: sedki@cica.es

Submitted by W. A. Kirk

Received April 14, 2000

In this paper, we prove that if ρ is a convex, σ-finite modular function satisfying a Δ_2-type condition, C a convex, ρ-bounded, ρ-a.e. compact subset of L_ρ, and $T : C \to C$ a ρ-asymptotically nonexpansive mapping, then T has a fixed point. In particular, any asymptotically nonexpansive self-map defined on a convex subset of $L^1(\Omega, \mu)$ which is compact for the topology of local convergence in measure has a fixed point. © 2002 Elsevier Science

Key Words: asymptotically nonexpansive mappings; fixed point; modular functions.

[1] The first author is partially supported by PB-96-1338-C01-C02 and PAI-FMQ-0127.

0022-247X/02 $35.00

INTRODUCTION

Let (M, d) a metric space. A mapping $T : M \to M$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for any $x, y \in M$ and $n \in \mathbb{N}$. In 1970 Goebel and Kirk [5] proved that T has a fixed point whenever M is a convex bounded closed subset of a Banach space X. Further generalizations of this result were proved by Yu and Dai [14] when X is 2-uniformly rotund, by Martínez Yañez [10] and Xu [12] when X is k-uniformly rotund for some $k \geq 1$, by Xu [13] when X is nearly uniformly convex, and by Kim and Xu [9] when X has uniform normal stucture. Some special studies on the theory of the fixed point for asymptotically nonexpansive mappings were made by many other authors (see, for example, [2, 11]).

The first fixed point results in modular function spaces were given by Khamsi *et al.* [7]. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces. For instance, fixed point theorems are proved in [6, 7] for nonexpansive mappings, in [3] for asymptotically regular mappings, and in [4] for uniformly Lipschitzian mappings. In this paper we will prove the existence of fixed points for asymptotically nonexpansive mappings in modular function spaces when the modular ρ satisfies some convexity and Δ_2-type properties.

Our results can be, in particular, applied to $L^1(\Omega, \mu)$, showing that asymptotically nonexpansive mappings have a fixed point when they are defined on a convex subset of $L^1(\Omega, \mu)$ which is compact with respect to the topology of local convergence in measure.

1. PRELIMINARIES

We start by reviewing some basic facts about modular spaces as formulated by Kozłowski [8]. For more details the reader may consult [6, 7].

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$, and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

DEFINITION 1.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if:

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f, E) \leq \rho(g, E)$ whenever $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$, and $E \in \Sigma$,

(P_3) $\rho(f, .) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$,

(P_6) for any $\alpha > 0, \rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E);\ g \in \mathcal{E}, |g(\omega)| \leq |f(\omega)| \text{ for every } \omega \in \Omega\}.$$

For the sake of simplicity we write $\rho(f)$ instead of $\rho(f, \Omega)$.

DEFINITION 1.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(w)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{w \in \Omega; p(w) \text{ does not hold}\}$ is ρ-null.

For example, we will say frequently $f_n \to f$ ρ-a.e.

DEFINITION 1.3. A modular function ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(K_n) < \infty$ and $\Omega = \bigcup K_n$.

It is easy to see that the functional $\rho : \mathcal{M} \to [0, \infty]$ is modular and satisfies the following properties:

(i) $\rho(f) = 0$ iff $f = 0 \rho$-a.e.

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$.

In addition, if the following property is satisfied

(iii)' $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$; $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$,

we say that ρ is convex modular.

The modular ρ defines a corresponding modular space, i.e., the vector space L_ρ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

When ρ is convex, the formula

$$\|f\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \le 1\right\}$$

defines a norm in the modular space L_ρ which is frequently called the Luxemburg norm. We can also consider the space

$$E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \to 0 \text{ as } n \to \infty \text{ for every } A_n \in \Sigma \text{ that decreases to } \emptyset \text{ and } \alpha > 0\}.$$

DEFINITION 1.4. A function modular is said to satisfy the Δ_2-condition if

$$\sup_{n\ge 1} \rho(2f_n, D_k) \to 0 \quad \text{as } k \to \infty \text{ whenever } \{f_n\}_{n\ge 1} \subset \mathcal{M},\ D_k \in \Sigma \text{ decreases to } \emptyset \text{ and } \sup_{n\ge 1} \rho(f_n, D_k) \to 0 \text{ as } k \to \infty.$$

We know from [8] that $E_\rho = L_\rho$ when ρ satisfies the Δ_2-condition.

DEFINITION 1.5. A function modular is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \le K\rho(f)$.

In general, the Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that the Δ_2-type condition implies the Δ_2-condition on the modular space L_ρ.

DEFINITION 1.6. Let L_ρ be a modular space.

(1) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \not\to f(\omega)\}$ is ρ-null.

(3) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.

(4) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(5) A subset C of L_ρ is called ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.

(6) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(7) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f - g); f, g \in C\} < \infty.$$

We recall two basic results (see [7]) in the theory of modular spaces.

(i) If there exists a number $\alpha > 0$ such that $\rho(\alpha(f_n - f)) \to 0$, then there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ such that $g_n \to f$ ρ-a.e.

(ii) (Lebesgue's Theorem) If $f_n, f \in \mathcal{M}$, $f_n \to f$ ρ-a.e., and there exists a function $g \in E_\rho$ such that $|f_n| \leq |g|$ ρ-a.e. for all n, then $\|f_n - f\|_\rho \to 0$.

We know by [6, 7] that under the Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition.

DEFINITION 1.7. Let ρ be as above. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)}, f \in L_\rho \backslash \{0\}\right\} \qquad \text{for all } 0 \leq t < \infty.$$

We have the following:

LEMMA 1.1 [3]. *Let ρ be as above. Then the growth function ω has the following properties:*

(1) $\omega(t) < \infty$, $\forall t \in [0, \infty)$

(2) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. So, it is continuous.*

(3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$.

(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0, \infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

LEMMA 1.2 [3]. *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then*

$$\|f\|_\rho \leq \frac{1}{\omega^{-1}\big(1/\rho(f)\big)} \qquad \textit{whenever } f \in L_\rho.$$

The next lemma will be of major interest throughout this work.

LEMMA 1.3 [6]. *Let ρ be a function modular satisfying the Δ_2-condition and $\{f_n\}_n$ be a sequence in L_ρ such that $f_n \overset{\rho-a.e}{\longrightarrow} f \in L_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g) \qquad \textit{for all } g \in L_\rho.$$

Moreover, we have

$$\rho(f) \leq \liminf_{n\to\infty} \rho(f_n).$$

2. AN EQUIVALENT TOPOLOGY

The concept of ρ-a.e. closed, compact sets has been studied extensively in the sequential case. One of the problems that many authors have found hard to circumvent is whether these notions are related to a topology. In this section we will discuss this problem. In particular, we will construct a topology τ for which the ρ-a.e. compactness is equivalent to the usual compactness for τ. This is crucial when we try to use Zorn's lemma.

From now on, we assume that the modular function ρ is, in addition, σ-finite. Set

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho\left(\frac{|f-g|}{1+|f-g|} 1_{K_k} \right) \qquad \text{for any } f, g \in L_\rho.$$

Some basic properties satisfied by d are discussed in the following proposition.

PROPOSITION 2.1. *The functional d satisfies the following*:

(1) $d(f, g) = 0$ *if and only if* $f = g$ ρ-*a.e.*;

(2) $d(f, g) = d(g, f)$;

(3) $d(f, g) \leq \frac{\omega(2)}{2}\big(d(f, h) + d(h, g)\big)$;

for any f, g, and h in L_ρ.

Proof. Parts (1) and (2) are obvious. To prove (3) we only need to recall the inequality

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

for all positive numbers a, b and use the definition of the growth function ω. ∎

Remark 2.1. The functional d is not a distance because of (3). But there are many mathematical objects which fails the triangle inequality but are very useful tools. That is the case with d.

In the next proposition, we discuss the relationship between ρ-a.e. convergence and the convergence for the functional d.

PROPOSITION 2.2. *Let ρ be a convex, σ-finite modular satisfying the Δ_2-type condition and $\{f_n\}_n$ be a sequence of measurable functions. If $\{f_n\}_n$ is ρ-a.e. convergent to f, then*

$$\lim_{n\to\infty} d(f_n, f) = 0.$$

Moreover, if

$$\lim_{n\to\infty} d(f_n, f) = 0,$$

then there exists a subsequence $\{f_{n_k}\}_k$ which converges ρ-a.e. to f.

Proof. Assume that $\{f_n\}_n$ ρ-a.e. converges to f. We will show that $\lim_{n\to\infty} d(f_n, f) = 0$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \varepsilon$. We have

$$\begin{aligned}\lim_{n\to\infty} d(f_n, f) &\leq \lim_{n\to\infty} \sum_{k=1}^{N} \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \right) + \varepsilon \\ &= \sum_{k=1}^{N} \lim_{n\to\infty} \frac{1}{2^k} \frac{1}{\rho(1_{K_k})} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \right) + \varepsilon.\end{aligned}$$

Since

$$\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \overset{\rho-a.e}{\longrightarrow} 0 \qquad \text{as } n \to \infty$$

for any $k \in \mathbb{N}$ and $(|f_n - f|/(1 + |f_n - f|))1_{K_k} \leq 1_{K_k}$, from Lebesgue's Theorem we obtain $\lim_{n\to\infty} \rho((|f_n - f|/(1 + |f_n - f|))1_{K_k}) = 0$ for every nonnull integer k. Thus $\lim_{n\to\infty} d(f_n, f) \leq \varepsilon$ for each $\varepsilon > 0$ which means that $\lim_{n\to\infty} d(f_n, f) = 0$.

Assume now that $\lim_{n\to\infty} d(f_n, f) = 0$. For every nonnull integer k we have

$$\lim_{n\to\infty} \rho\left(\frac{|f_n - f|}{1 + |f_n - f|} 1_{K_k} \right) = 0.$$

Thus, there exists a subsequence $\{f_n^1\}_n$ of $\{f_n\}_n$ such that $(|f_n^1 - f|/(1 + |f_n^1 - f|))1_{K_1} \overset{\rho-a.e}{\longrightarrow} 0$ and so $f_n^1 \overset{\rho-a.e}{\longrightarrow} f$ in K_1, i.e., $\lim_{n\to\infty} f_n^1(x) = f(x)$ whenever $x \in K_1 \backslash A_1$ where $A_1 \subset K_1$ and $\rho(1_{A_1}) = 0$.

By induction and using a diagonal argument we obtain a subsequence of $\{f_n\}_n$ which converges ρ-a.e. to f. ∎

DEFINITION 2.1. Let C be a subset of L_ρ.

(a) C is said to be d-closed iff for any sequence $\{f_n\}_n$ in C which d-converges to f, then we have $f \in C$.

(b) C is d-open iff $L_\rho \backslash C$ is d-closed.

(c) C is said to be d-sequentially compact if for each sequence $\{f_n\}_n$ there exists a subsequence $\{f_{n_k}\}_k$ which d- converges to a point in C.

It is easily seen that the family of all d-open subsets of L_ρ form a topology on L_ρ. Furthermore, from Proposition 2.2, d-sequentially compact sets and ρ-a.e. compact sets are identical. On the other hand, even though d satisfies (3) instead of the triangular inequality, the usual arguments which prove that sequential compactness and compactness are identical in metric spaces hold in this setting. We also have that d-sequential compactness and d-compactness are identical.

3. TECHNICAL LEMMAS

In the sequel we assume that ρ is a convex, σ-finite modular function satisfying the Δ_2-type condition, C is a convex, ρ-bounded, and ρ-a.e. compact subset of the modular function space L_ρ, and $T : C \to C$ is a ρ-asymptotically nonexpansive mapping; i.e., there exists a sequence of positive integers $\{k_n\}_n$ which converge to 1 such that for every $n \in \mathbb{N}$ and $f, g \in C$ we have $\rho(T^n f - T^n g) \leq k_n \rho(f - g)$.

LEMMA 3.1. *Under the above assumptions, let* $\{f_n\}_n$ *be a sequence of elements of* C. *Consider the functional* $\Phi : C \to R$ *defined by* $\Phi(g) = \limsup_{n\to\infty} \rho(f_n - g)$. *Then for any sequence* $\{g_m\}_m$ *in* C *which* ρ*-a.e. converges to* $g \in C$ *we have*

$$\Phi(g) \leq \liminf_{m\to\infty} \Phi(g_m).$$

Proof. Since C is ρ-a.e. compact, there exists a subsequence $\{f_{\phi(n)}\}_n$ of $\{f_n\}_n$ such that $f_{\phi(n)} \overset{\rho-a.e}{\longrightarrow} f \in C$ and $\lim_{n\to\infty} \rho(f_{\phi(n)} - g) = \limsup_{n\to\infty} \rho(f_n - g)$. Hence

$$\begin{aligned}\Phi(g_m) &= \limsup_{n\to\infty} \rho(f_n - g_m)\\ &\geq \limsup_{n\to\infty} \rho(f_{\phi(n)} - g_m)\\ &\geq \liminf_{n\to\infty} \rho(f_{\phi(n)} - g_m).\end{aligned}$$

Lemma 1.3 implies

$$\liminf_{n\to\infty} \rho(f_{\phi(n)} - g_m) = \liminf_{n\to\infty} \rho(f_{\phi(n)} - f) + \rho(f - g_m).$$

Thus, $\Phi(g_m) \geq \liminf_{n\to\infty} \rho(f_{\phi(n)} - f) + \rho(f - g_m)$, for any $m \leq 1$. Hence

$$\liminf_{m\to\infty} \Phi(g_m) \geq \liminf_{n\to\infty} \rho(f_{\phi(n)} - f) + \liminf_{m\to\infty} \rho(f - g_m).$$

Again using Lemma 1.3, we have

$$\liminf_{m\to\infty} \rho(f - g_m) = \liminf_{m\to\infty} \rho(g_m - g) + \rho(g - f),$$

which implies

$$\liminf_{m\to\infty} \Phi(g_m) \geq \liminf_{n\to\infty} \rho(f_{\phi(n)} - f) + \liminf_{m\to\infty} \rho(g_m - g) + \rho(g - f). \quad \text{(I)}$$

On the other hand,

$$\Phi(g) = \limsup_{n\to\infty} \rho(f_n - g) = \lim_{n\to\infty} \rho(f_{\phi(n)} - g) = \liminf_{n\to\infty} \rho(f_{\phi(n)} - g)$$

which implies

$$\Phi(g) = \liminf_{n\to\infty} \rho(f_{\phi(n)} - f) + \rho(f - g). \tag{II}$$

From (I) and (II), it is clear that

$$\Phi(g) \le \liminf_{m\to\infty} \Phi(g_m),$$

which completes the proof. ∎

Denote by $\mathfrak{F}$ the family of all subsets K of C satisfying the following property: K is a nonempty, convex, and ρ-a.e. closed subset of C such that

$$f \in K \text{ implies } \Omega_{\rho-a.e}(f) \subset K, \tag{3.1}$$

where $\Omega_{\rho-a.e}(f) = \{g \in L_\rho : g = \lim_{i\to\infty} T^{n_i}(f)\rho\text{-a.e for some } n_i \uparrow \infty\}$. Ordering $\mathfrak{F}$ by inclusion, there exists a nonempty minimal element H in $\mathfrak{F}$ which satisfies (3.1) by using Zorn's lemma because C is compact for the topology generated by d.

The following lemma is the counterpart in modular function spaces of Lemma 2.1 in [13] for Banach spaces.

LEMMA 3.2. *Under the above assumptions, for each $f \in H$ define the functional*

$$r_f(g) = \limsup_{n\to\infty} \rho(T^n f - g)$$

for any $g \in L_\rho$. Then the functional $r_f(.)$ is constant on H and this constant is independent of f in H.

Proof. Let $t > 0$ and $f \in H$. Set

$$H_t(f) = \{g \in H, r_f(g) \le t\}.$$

It is easily seen that $H_t(f)$ is convex. We claim that $H_t(f)$ is ρ-a.e. closed. Indeed, assume that $\{g_m\}_m \in H_t(f)$ ρ-a.e. converges to $g \in H$. Using Lemma 3.1, we get

$$\limsup_{n\to\infty} \rho(T^n f - g) \le \liminf_{m\to\infty} \limsup_{n\to\infty} \rho(T^n f - g_m) \le t.$$

Hence $g \in H_t(f)$, which clearly implies that $H_t(f)$ is ρ-a.e. closed. Since H is is ρ-a.e. compact we have that $H_t(f)$ is ρ-a.e. compact. Next, we claim that $H_t(f)$ satisfies property (3.1). Indeed, let $g \in H_t(f)$ and $h \in \Omega_{\rho-a.e}(g)$. We need to check that $h \in H_t(f)$. By definition of $\Omega_{\rho-a.e}(g)$, there exists an increasing sequence of integers $\{n_i\}_i$ such that $T^{n_i}(g) \overset{\rho-a.e}{\longrightarrow} h$. Lemma 3.1 implies

$$\begin{aligned} r_f(h) &= \limsup_{n\to\infty} \rho(T^n f - h) \le \liminf_{i\to\infty} \limsup_{n\to\infty} \rho(T^n f - T^{n_i} g) \\ &\le \liminf_{i\to\infty} r_f\big(T^{n_i}(g)\big) \le \limsup_{i\to\infty} r_f\big(T^{n_i}(g)\big) \le \limsup_{m\to\infty} r_f\big(T^m(g)\big) \\ &\le \limsup_{m\to\infty}\Big(\limsup_{n\to\infty} \rho(T^n f - T^m g)\Big) \\ &\le \limsup_{m\to\infty}\Big(k_m \limsup_{n\to\infty} \rho(T^{n-m} f - g)\Big) \\ &\le \limsup_{m\to\infty} \limsup_{n\to\infty} \rho(T^n f - g) \le t. \end{aligned}$$

Hence $h \in H_t(f)$ as claimed. The minimality of H implies that $H_t(f)$ is $\emptyset$ or equal to H. From this, it is clear that $r_t(\cdot)$ is constant on H. In order to complete the proof of this lemma, we need to prove that r_f is independent of f. Let $f, g \in H$. Since C is ρ-a.e. compact, there exists a subsequence $\{T^{n_i}(g)\}_i$ of $\{T^n(g)\}_n$ which ρ-a.e. converges to $h \in C$. Since H satisfies property (3.1), we have $h \in H$. Lemma 1.3 implies

$$\rho(T^n f - h) \le \liminf_{i\to\infty} \rho(T^n f - T^{n_i} g).$$

Hence

$$\begin{aligned} r_f = r_f(h) &= \limsup_{n\to\infty} \rho(T^n f - h) \\ &\le \limsup_{n\to\infty} \liminf_{i\to\infty} \rho(T^n f - T^{n_i} g) \\ &\le \limsup_{n\to\infty} \limsup_{m\to\infty} \rho(T^n f - T^m g) \\ &\le \limsup_{m\to\infty} \rho(f - T^m g) = r_g(f) = r_g, \end{aligned}$$

which obviously implies $r_g = r_f$. ∎

Recall that if ρ satisfies the Δ_2-type condition, then ρ-convergence and norm (i.e., Luxemburg norm) convergence coincide. We have the following result:

LEMMA 3.3. *Let ρ be a convex modular function satisfying the Δ_2-type condition. Let S be a nonempty, norm-compact subset of L_ρ with* $\operatorname{diam}_\rho(S) > 0$. *Then there exists $f \in \overline{\operatorname{conv}}(S)$ such that*

$$\sup\{\rho(g - f) : g \in S\} < \operatorname{diam}_\rho(S).$$

Proof. The proof is similar to the classical one known in Banach spaces. Indeed, since S is compact and ρ is norm continuous, there exist $f_0, f_1 \in S$ such that $\rho(f_0 - f_1) = \operatorname{diam}_\rho(S)$. Let S_0 be a maximal subset of S such that $f_0, f_1 \in S_0$ and for any $f, g \in S_0$, $f \neq g$, we have $\rho(f - g) = \operatorname{diam}_\rho(S)$. Since S is compact, S_0 must be finite. Write $S_0 = \{f_0, f_1, f_2, \dots, f_n\}$ and define

$$h = \frac{f_0 + f_1 + \cdots + f_n}{n+1}.$$

Since S is compact, there exists $g_0 \in S$ such that

$$\rho(g_0 - h) = \sup\{\rho(g - h) : g \in S\}.$$

On the other hand, using the convexity of ρ, we get

$$\begin{aligned}\rho(g_0 - h) &= \rho\left(\sum_{k=0}^{k=n}\left(\frac{1}{n+1}\right)g_0 - \sum_{k=0}^{k=n}\left(\frac{1}{n+1}\right)f_k\right)\\ &\le \sum_{k=0}^{k=n}\left(\frac{1}{n+1}\right)\rho(g_0 - f_k) \le \operatorname{diam}_\rho(S).\end{aligned}$$

If $\rho(g_0 - h) = \operatorname{diam}_\rho(S)$, then we must have $\rho(g_0 - f_k) = \operatorname{diam}_\rho(S)$, for $k = 0, 1, \dots, n$. This will contradict the maximality of S_0. Hence

$$\sup\{\rho(g - h) : g \in S\} = \rho(g_0 - h) < \operatorname{diam}_\rho(S). \qquad \blacksquare$$

4. MAIN RESULTS

THEOREM 4.1. *Let ρ be a convex, ρ is a convex, σ-finite function modular satisfying the Δ_2-type condition and C be a ρ-bounded, ρ-a.e. compact subset of L_ρ. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Let H be a convex subset of C such that*:

(i) *if $f \in H$ then $\Omega_{\rho-a.e}(f) \subset H$;*

(ii) *for each $f \in H$, any subsequence $\{T^{n_i}(f)\}_i$ of $\{T^n(f)\}_n$, has a ρ-convergent subsequence.*

Then T has a fixed point.

Proof. Consider the family $\mathcal{F}$ of nonempty ρ-a.e. compact subset of H which satisfies property (3.1). $\mathcal{F}$ is not empty since $H \in \mathcal{F}$. By the previous results, $\mathcal{F}$ has a minimal element. Let K be a minimal element of $\mathcal{F}$. Assume that K has more than one point, i.e., $\operatorname{diam}_\rho(K) > 0$. Let $f \in K$. Set

$$S = \Omega_{\|\cdot\|}(f) = \{g \in H; T^{n_i}(f)\ \|\cdot\|\text{-converges to } g \text{ for some } n_i \uparrow \infty\}.$$

It is easy to see that $S \subset K$. We claim that $S = T(S)$. Indeed, let $g \in S$. Then there exists a sequence $\{T^{n_i}(f)\}_i$ which $\|\cdot\|$-converges to g. Since T is continuous, we have $T^{n_i+1}(f) \overset{\|\cdot\|}{\longrightarrow} T(g)$. By definition of S, we get $T(g) \in S$, i.e., $T(S) \subset S$. Let us show the other inclusion, i.e., $S \subset T(S)$. Let $g \in S$. Again by definition of S, there exists a sequence $\{T^{n_i}(f)\}_i$ which $\|\cdot\|$-converges to g. The sequence $\{T^{n_i-1}(f)\}_i$ has a norm convergent subsequence, say $\{T^{n_{\phi(i)}-1}(f)\}_i$. Let h be its $\|\cdot\|$-limit. Since T is continuous, we get

$$T(h) = T\Big(\lim_{i\to\infty} T^{n_{\phi(i)}-1}(f)\Big) = \lim_{i\to\infty} T^{n_{\phi(i)}}(f) = g.$$

Hence $g \in T(S)$, i.e., $S \subset T(S)$. So our claim is proved, i.e., $T(S) = S$. Next, notice that the assumption (ii) implies that S is norm compact. Lemma 3.3 implies the existence of $f_0 \in \overline{\mathrm{conv}}(S) \subset K$ such that

$$\sup\{\rho(g - f_0) : g \in S\} < \mathrm{diam}_\rho(S). \tag{A}$$

Let $r = \sup\{\rho(g - f_0) : g \in S\}$. Set

$$D = \Big\{h \in K; \sup_{g\in S} \rho(g - h) \le r\Big\}.$$

Since $f_0 \in D$ and ρ is convex, D is a nonempty convex subset of K. We claim that $D = K$. Indeed, let us first show that D is ρ-a.e. compact. By the assumption (ii), it is enough to show that D is ρ-a.e. closed. Let $\{h_n\}_n$ be a sequence in D such that $h_n \overset{\rho-a.e.}{\longrightarrow} h \in L_\rho$. Fix $g \in S$. Since $g - h_n \overset{\rho-a.e}{\longrightarrow} g - h$, Lemma 1.3 implies

$$\rho(g - h) \le \liminf_{n\to\infty} \rho(g - h_n)$$

which yields

$$\rho(g - h) \le \liminf_{n\to\infty} \Big(\sup\{\rho(f - h_n) : f \in S\}\Big) \le r.$$

Hence $\sup\{\rho(h - g) : g \in S\} \le r$, i.e., $h \in D$. Next we check that D satisfies property (3.1). Indeed, let $f \in D$ and $g \in \Omega_{\rho-a.e}(f)$. Then there exists a sequence $\{T^{n_i}(f)\} \overset{\rho-a.e.}{\longrightarrow} g$. Using Lemma 1.3 we obtain

$$\rho(g - h) \le \liminf_{n\to\infty} \rho(T^{n_i}(f) - h) \le \limsup_{n\to\infty} \rho(T^n f - h)$$

for any $h \in S$. Since $T(S) = S$, there exists a sequence $\{u_n\}_n$ in S such that $h = T^n(u_n)$, for any $n \ge 1$. Hence

$$\begin{aligned}\rho(g - h) &\le \limsup_{n\to\infty} \rho(T^n f - T^n u_n) \le \limsup_{n\to\infty} k_n \rho(f - u_n)\\ &\le \limsup_{n\to\infty} \rho(f - u_n) \le \sup\{\rho(f - u) : u \in S\} \le r.\end{aligned}$$

So $\sup\{\rho(g-h) : h \in S\} \leq r$ which gives $g \in D$. Thus D satisfies property (3.1) and by minimality of K, we obtain $D = K$. But

$$\operatorname{diam}_\rho(D) \leq r < \operatorname{diam}_\rho(S) \leq \operatorname{diam}_\rho(K),$$

which is a contradiction. Therefore, K is reduced to one point. Property (3.1) will force this point to be a fixed point of T. ∎

Now we are ready to state and prove the main result of this work.

THEOREM 4.2. *Let ρ be a convex, σ-finite function modular satisfying the Δ_2-type condition and C be a convex ρ-bounded and ρ-a.e. compact subset of L_ρ. Let $T : C \to C$ be ρ-asymptotically nonexpansive. Then T has a fixed point.*

Proof. Let $\mathcal{F}$ be the family of nonempty convex subsets of C which satisfy the property (3.1). $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ has a minimal element. Let H be a minimal element of $\mathcal{F}$. Let us show that H satisfies the hypothesis of Theorem (4.1). It suffices to check that H satisfies property (ii). Let r be defined on H as in Lemma (3.2). If $r = 0$ we have

$$\lim_{n\to\infty} T^n f = g$$

for any $f, g \in H$, which implies (ii). Otherwise, assume that $r > 0$. Let $f \in H$ such that there exists a sequence $\{T^{n_i} f\}_i$ which has no norm-convergent subsequence. Thus, there exists $\varepsilon > 0$ and a subsequence $\{T^{n(k)} f\}_k$ such that

$$\operatorname{Sep}\big(\{T^{n(k)} f\}_k\big) = \inf\big\{\rho\big(T^{n(k)} f - T^{n(k')} f\big) k \neq k'\big\} \geq \varepsilon.$$

Since H is ρ-a.e. compact, there exists $f_\infty \in H$ such that $T^{n(k)} f \overset{\rho-a.e}{\longrightarrow} f_\infty \in H$ as $k \to \infty$. Without loss of generality, we may assume the existence of

$$\lim_{k\to\infty} \rho(T^{n(k)} f - f_\infty) = l.$$

Since $\limsup_{n\to\infty} \rho(T^n f - f) = r$, we choose $\eta > 0$ such that $\eta < \varepsilon/2$, and an integer $n_0 \geq 1$, such that for all $n \geq n_0$ we have

$$\rho(T^n f - f) < r + \eta.$$

Fix $n \geq n_0$. There exists $k_0 \geq 1$ such that for all $k \geq k_0$, we have $n(k) \geq n + n_0$ and

$$\begin{aligned}\rho(T^n f - T^{n(k)} f) &= \rho(T^n f - T^{n+(n(k)-n)} f) = \rho(T^n f - T^n(T^{n(k)-n} f)) \\ &\leq k_n \rho(f - T^{n(k)-n} f) < k_n(r + \eta).\end{aligned}$$

Note that if $f_n \stackrel{\rho-a.e}{\longrightarrow} f$ and $\mathrm{Sep}\{f_n\}_n \geq \varepsilon$, then by Lemma 1.3, we have

$$\varepsilon \leq \liminf_{m\to\infty} \liminf_{n\to\infty} \rho(f_n - f_m) \leq 2 \liminf_{n\to\infty} \rho(f_n - f).$$

Combined with Lemma 1.3, we get

$$\liminf_{n\to\infty} \rho(f_n) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f) \geq \frac{\epsilon}{2} + \rho(f).$$

In particular, since $\{T^{n(k)}f - T^n f\}_k$ is ρ-a.e. convergent to $f_\infty - T^n f$ as $k \to \infty$ and satisfies $\mathrm{Sep}(\{T^{n(k)}f - T^n f\}_k) \geq \epsilon$, we get

$$\rho(T^n f - f_\infty) \leq \liminf_{k\to\infty} \rho(T^{n(k)}f - T^n f) - \frac{\epsilon}{2}.$$

Hence

$$\rho(f_\infty - T^n f) \leq r + \eta - \frac{\epsilon}{2}$$

which implies

$$r = \limsup_{n\to\infty} \rho(f_\infty - T^n f) \leq r + \eta - \frac{\epsilon}{2} < r.$$

This contradiction completes the proof of Theorem 4.2. ∎

Assume that $L_\rho = L_p(\Omega, \mu)$ for a σ-finite measure μ. If C is a convex, bounded, and closed subset of L_p for $1 < p < \infty$ and $T : C \to C$ is asymptotically nonexpansive, it is known that C has a fixed point because L_p is uniformly convex. However, the result does not holds for $p = 1$ (even for nonexpansive mappings, see [1]). Since L_1 is a modular space, Theorem 4.1 implies the existence of a fixed point if $p = 1$ when C is ρ-a.e. compact. Thus we can state.

COROLLARY 4.1. *Let (Ω, μ) be as above, $C \subset L_1(\Omega, \mu)$ a convex bounded set which is compact for the topology of the convergence local in measure, and $T : C \to C$ asymptotically nonexpansive. Then, T has a fixed point.*

Proof. Under the above hypothesis ρ-a.e. compact sets and compact sets in the topology of convergence local in measure are identical. ∎

REFERENCES

1. D. E. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* **82** (1981), 423–424.
2. S. C. Bose, Weak convergence to the fixed point of an asymptotically nonexpansive map, *Proc. Amer. Math. Soc.* **68** (1978), 305–308.
3. T. Dominguez Benavides, M. A. Khamsi, and S. Samadi, Asymptotically regular mappings in modular function spaces, preprint.

4. T. Dominguez Benavides, M. A. Khamsi, and S. Samadi, Uniformly Lipschitzian mappings in modular function spaces, preprint.
5. K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35** (1972), 171–174.
6. M. A. Khamsi, Fixed point theory in modular function spaces, *in* "Recent Advances on Metric Fixed Point Theory," pp. 31–58, Universidad de Sevilla, Sevilla, 1996.
7. M. A. Khamsi, W. M. Kozlowski, and S. Reich, Fixed point theory in modular function spaces, *Nonlinear Anal.* **14** (1990), 935–953.
8. W. M. Kozlowski, "Modular Function Spaces," Dekker, New York/Basel, 1988.
9. T.-H. Kim and H.-K. Xu, Remarks on asymptotically nonexpansive mappings, *Nonlinear Anal.*, in press.
10. C. Martínez Yañez, A fixed point theorem on k-uniformly rotund spaces, *Nonlinear Anal.* **13** (1988), 857–861.
11. G. Passty, Construction of fixed points for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **84** (1982), 213–216.
12. H.-K. Xu, k-Uniform rotundity and fixed points of mappings of asymptotically nonexpansive type, to appear in Chinese.
13. H.-K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, *Nonlinear Anal.* **16** (1991), 1139–1146.
14. X. T. Yu and X. Dai, A fixed point theorem of asymptotically nonexpansive mappings, *J. Math. (PRC)* **6** (1986), 255–262.