Nonlinear Semigroups in Modular Function Spaces

Mohamed A. Khamsi
University of Texas -El Paso

Abstract

Let L_ρ be the modular function space determined by a function modular ρ. We study the existence and the behavior of nonlinear semigroups generated by an operator $A = I - T$, where T is a onexpansive mapping in the modular sense.

1980 Mathematics Subject classification: Primary 46E30, 47E10.
Key words and phrases: Semigroups, fixed points, nonexpansive mappings, modular function spaces.

Mailing address:

Mohamed A. Khamsi,
Department of Mathematical Sciences,
University of Texas at El Paso,
El Paso, Tx 79968-0514.

1 Introduction and Preliminaries.

In this paper we consider the classical Musielak-Orlicz spaces L^φ, in which we investigate the existence and the behavor of nonlinear semigroups. We obtain an existence result of semigroups generated by mappings $A = I - T$, where T is a nonexpansive mapping in the modular sense acting within L^φ. The advantages of this approch consist in : (1) an existence theorem even when ρ does not satisfy the Δ_2-condition (usually this implies that L^φ is a very bad space from the geometrical point of view); (2) our conditions on T can be much easier verified since it uses only the Musielak-Orlicz-modular, which is a simple integral functional.

Let us also add that the approch consists originally of solving an initial value problem. When ρ satisfies the Δ_2-condition, our existence result (Theorem 2.3) seems to be unknown. We start with a brief recollection of basic concepts and facts of the theory of Musielak-Orlicz spaces and modular spaces.

Definition 1.1. Let X be an arbitrary vector space.

(a) A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary x,y in X,

  (i) $\rho(x) = 0$ iff $x = 0$,

  (ii) $\rho(\alpha.x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

  (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$.

(b) If (iii) is replaced by

  (iii)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$,

  we say that ρ is a convex modular.

(c) A modular ρ defines a corresponding modular space, i.e the vector space X_ρ given by

$$X_\rho = \{x \in X; \rho(\lambda x) \to 0 \ \text{ as } \ \lambda \to 0\}.$$

Definition 1.2. The modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$||x||_\rho = inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}.$$

When ρ is convex we have

$$||x||_\rho = inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}.$$

In what follows we discuss a classical example of modular function spaces.

Example 1.3. Let (Ω, Σ, μ) be a measure space. A real function φ defined on $\Omega \times \Re_+$ will be said to belong to the class Φ if the following conditions are satisfied

(i) $\varphi(\omega, u)$ is a nondecreasing continuous function of u such that $\varphi(\omega, 0) = 0, \varphi(\omega, u) > 0$ for $u > 0$ and $\varphi(\omega, u) \to \infty$ as $u \to \infty$,

(ii) $\varphi(\omega, u)$ is a Σ-measurable function of ω for all $u \geq 0$,

(iii) $\varphi(\omega, u)$ is a convex function of u, for all $\omega \in \Omega$.

Moreover, consider X, the set off all real-valued Σ-measurable and finite μ-almost every where functions on Ω, with equality μ-almost every where. Since $\varphi(\omega, |x(\omega)|)$ is a Σ-measurable function of $\omega \in \Omega$ for every $x \in X$, set

$$\rho(x) = \int_\Omega \varphi(\omega, |x(\omega)|) d\mu(\omega). \quad (1)$$

It is easy to check that ρ is a convex modular on X. The associated modular function space X_ρ, is called Musielak-Orlicz space and will be denoted L^φ.

Throughout this work, we will only consider the Musielak-Orlicz spaces.

Definition 1.4.

(a) A subset C of L^φ is called ρ-bounded if

$$\delta_\rho(C) = sup\{\rho(f - g); f, g \in C\} < \infty,$$

(b) The sequence $\{f_n\} \subset L^\varphi$ is said to be ρ-convergent to $f \in L^\varphi$ if $\rho(f_n - f) \to 0$ as $n \to \infty$,

(c) A subset C of L^φ is called ρ-closed if the ρ-limit of a ρ-convergent sequence $\{f_n\} \subset C$ always belongs to C,

(d) The sequence $\{f_n\} \subset L^\varphi$ is said to be ρ-Cauchy if $\rho(f_n - f) \to 0$ as $n, m \to \infty$.

Notice that when $\{f_n\} \subset L^\varphi$ is norm-convergent to $f \in L^\varphi$, then $\rho[\alpha(f_n - f)] \to 0$ as $n \to \infty$, for any scalar α. The converse is also true. This clearly implies that norm-convergence is stronger than ρ-convergence.

Definition 1.5. The function modular ρ is said to satisfy the Δ_2-condition if $\rho(2f_n) \to 0$ as $n \to \infty$, whenever $\rho(f_n) \to 0$ as $n \to \infty$.

It is not hard to see that when ρ satisfies the Δ_2-condition, then ρ-convergence and norm-convergence are equivalent. On the Δ_2-condition and its properties one can consult [4],[5].

The proof of the following proposition can be found in [1].

Proposition 1.6. The following properties are satisfied by the Musielak-Orlicz modular,

(1) L^φ is ρ-complete, i.e any ρ-Cauchy sequence is ρ-convergent,

(2) (Fatou property) Let $\{f_n\}$ and $\{g_n\}$ be in L^φ and ρ-convergent respectively to f and g, then

$$\rho(f - g) \leq \liminf_{n\to\infty} \rho(f_n - g_n),$$

(3) ρ is left continuous, i.e $\rho(\lambda f) \to \rho(f)$ as $\lambda \uparrow 1$.

Remark that since ρ does not satisfy a priori the triangle inequality, we cannot expect that if $\{f_n\}$ and $\{g_n\}$ are ρ-convergent respectively to f and g then $\{f_n + g_n\}$ is ρ-convergent to $f + g$, neither that a ρ-convergent sequence is ρ-Cauchy.

Definition 1.7. Let C be a subset of L^φ and let $T : C \to C$ be an arbitrary mapping. T is said to be ρ-nonexpansive if $\rho(Tf - Tg) \leq \rho(f - g)$ for any f, g in C. The fixed point set of T will be denoted by $F(T)$, i.e $F(T) = \{f \in C; T(f) = f\}$.
Since in this work we are deeling with semigroups, the next definition is legitimate.

Definition 1.8. Let C be a subset of L^φ. A mapping $S : [0, \infty) \times C \to C$ is said to be a (ρ-nonexpansive)-semigroup if the following conditions are satisfied

(i) $S(0)f = f$ for all $f \in C$,

(ii) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \geq 0$,

((iii) the mapping $f \to S(t)f$ is ρ-nonexpansive for all $t \geq 0$).

Remark 1.9. One can ask what relation exists between ρ-nonexpansiveness and norm-nonexpansiveness. In [3] it is proved that a mapping T is norm-nonexpansive if and only if $\rho(\alpha(Tf - Tg)) \leq \rho(\alpha(f - g))$ for any $\alpha \geq 0$. Also an example is given of a mapping which is ρ-nonexpansive and not norm-nonexpansive. In order to be complete, we give the definition of this map. For more details one can consult [3].

Let $(\Omega, \Sigma, \mu) = ([0, \infty), \Sigma, dx)$ where Σ is the σ-algebra of all Lebsegue measurable subsets of $[0, \infty)$. Consider the Φ-function

$$\varphi(t, x) = \exp(-2)x^{t+1}.$$

The modular function ρ is defined by

$$\rho(f) = \exp(-2) \int_0^\infty |f(t)|^{t+1} dt.$$

Let $C = \{f \in L^\varphi; 0 \leq f \leq \frac{1}{2}\}$ and define the mapping $T : C \to C$ by

$$Tf(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ f(t-1) & \text{if } t \geq 1. \end{cases}$$

2 Semigroups in Musielak-Orlicz spaces.

In order to obtain an existence result concerning the semigroups in Musielak-Orlicz spaces, the following technical theorem is needed.

Theorem 2.1. Let C be a ρ-closed, ρ-bounded convex subset of L^{φ}. Let $T : C \to C$ be ρ-nonexpansive and norm-continuous. Let $f \in C$ be fixed and consider the recurrent sequence defined by

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = \exp(-t)f + \int_0^t \exp(s-t)T(u_n(s))ds \end{cases}$$

for $t \in [0, A]$, where A is a fixed positive number.
Then the sequence $\{u_n(t)\}$ is ρ-Cauchy for any $t \in [0, A]$. Therefore it converges with respect to ρ, to $u(t) \in C$ for any $t \in [0, A]$.

The proof of Theorem 2.1. is based on the following technical lemma.

Lemma 2.2 Let $x, y : [0, t] \to L^{\varphi}$ be norm-continuous mappings. Then, we have

$$\rho(\exp(-t)y(t)+\int_0^t \exp(s-t)x(s)ds) \leq \exp(-t)\rho(y(t))+K(t)sup\{\rho(x(s)); s \in [0,t]\}$$

where $K(t) = 1 - exp(-t) = \int_0^t \exp(s-t)ds$.
Proof of Lemma 2.2. Without any loss of generality, we can assume that $sup\{\rho(x(s)); s \in [0, t]\} < \infty$. Let $\tau = \{t_i; i = 0, 1, ..n\}$ be any subdivision of $[0, t]$. Set

$$S_\tau = \exp(-t)y(t) + \sum_{i=0}^{i=n-1} (t_{i+1} - t_i) \exp(t_i - t)x(t_i).$$

The family $\{S_\tau\}$ is norm-convergent to

$$\exp(-t)y(t) + \int_0^t \exp(s-t)x(s)ds$$

when $|\tau| = sup\{|t_{i+1} - t_i|; i = 0, 1, ..(n-1)\} \to 0$. The Fatou property implies that

$$\rho(\exp(-t)y(t) + \int_0^t \exp(s-t)x(s)ds) \leq \liminf_{|\tau|\to 0} \rho(S_\tau).$$

On the other hand, we have

$$\rho(S_\tau) \le \exp(-t)\rho(y(t) + (\sum_{i=0}^{n-1}(t_{i+1} - t_i)\exp(t_i - t)) \sup_{0\le s\le t}(\rho(x(s))),$$

since ρ is convex and

$$\exp(-t)+\sum_i(t_{i+1}-t_i)\exp(t_i-t) \le \exp(-t)+\int_0^t \exp(s-t)ds = \exp(-t)+K(t) = 1.$$

So $\rho(S_\tau) \le \exp(-t)\rho(y(t)) + K(t)sup\{\rho(x(s)); s \in [0,t]\}$.
This yields to the desired conclusion.

Let us go back to the proof of Theorem 2.1. First notice that by induction, we can prove that $u_n(t) \in C$ for any $n \in N$ and $t \in [0, A]$, since C is a ρ-closed (and therefore norm-closed) convex subset of L^φ. In order to prove that $(u_n(t))$ is ρ-convergent we establish the following inequality

$$\rho(u_{n+h}(t) - u_n(t)) \le K^{n+1}(A)\delta_\rho(C) \qquad (2)$$

for all $t \in [0, A]$ and $n, h \in N$.
For $n = 0$, we have $u_h(t) - u_0(t) = \int_0^t \exp(s - t)(Tu_{h-1}(s) - f)ds$. Since $Tu_{h-1}(s) \in C$ for all $s \in [0, t]$, the inequality (2) holds for $n = 0$ by using Lemma 2.2.
Assume that (2) holds for $n \in N$ and all $t \in [0, A]$, then

$$u_{n+1+h}(t) - u_{n+1}(t) = \int_0^t \exp(s - t)(Tu_{n+h}(s) - Tu_n(s))ds.$$

Using Lemma 2.2., we get

$$\rho(u_{n+1+h}(t) - u_{n+1}(t)) \le K(A) \sup_{0\le s\le t}\{\rho(Tu_{n+h}(s) - Tu_n(s))\}.$$

Since T is ρ-nonexpansive and $K(t) \le K(A)$, we obtain the inequality (2) for $n + 1$.
Therefore, by induction, the inequality (2) holds for every $n \in N$. Hence $\{u_n(t)\}$ is ρ-Cauchy for all $t \in [0, A]$. The proof of Theorem 2.1. is therefore complete.

It is not clear if the assumptions on C and T are enough to imply any good behavior of $u(t)$ on $[0, A]$ such as norm-continuity for example. But if ρ satisfies the Δ_2-condition then $u(t)$ is indeed continuous.

<u>Theorem 2.3.</u> under the assumptions of Theorem 2.1., if moreover ρ satisfies the Δ_2-condition, then $u(t)$ is solution of the following initial value problem,

$$\begin{cases} u'(t) + (I - T)u(t) = 0 \\ u(0) = f. \end{cases}$$

<u>Proof of Theorem 2.3.</u> for any function $v : [0, A] \to X_\rho$ and any subdivision $\tau = \{t_i; i = 0, 1, ..n\}$ of $[0, A]$, put

$$S_\tau(v)(t) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) \exp(t_i - t) v(t_i),$$

and $|\tau| = sup\{|t_{i+1} - t_i|; i = 0, 1, ..(n-1)\}$. Our assumptions on T and $\{u_n\}$ imply that

$$\lim_{|\tau| \to 0} ||S_\tau(Tu_n)(t) - \int_0^t \exp(s - t) T(u_n)(s)) ds||_\rho = 0, \tag{3}$$

for every $n \in N$. Using Lemma 2.2. and the inequality (2), we get

$$\rho(S_\tau(Tu)(t) - S_\tau(Tu_n)(t)) \leq K^{n+1}(A) \delta_\rho(C).$$

Since ρ satisfies the Δ_2-condition, this implies

$$\lim_{n \to \infty} ||S_\tau(Tu)(t) - S_\tau(Tu_n)(t)||_\rho = 0, \tag{4}$$

and also

$$\lim_{n \to \infty} ||u(t) - u_n(t)||_\rho = 0, \tag{5}$$

for all $t \in [0, A]$. But

$$||S_\tau(Tu)(t) - u(t) + \exp(-t)f||_\rho \leq ||S_\tau(Tu)(t) - S_\tau(Tu_n)(t)||_\rho +$$

$$||S_\tau(Tu_n)(t) - \textstyle\int_0^t \exp(s - t) Tu_n(s) ds||_\rho +$$

$$|| \int_0^t \exp(s-t)Tu_n(s)ds - u(t) + \exp(-t)f||_\rho.$$

Since $\int_0^t \exp(s-t)Tu_n(s)ds = u_{n+1}(t) - \exp(-t)f$, we obtain from (3), (4) and (5) that

$$\lim_{|\tau|\to 0} ||S_\tau(Tu)(t) - u(t) + \exp(-t)f||_\rho = 0.$$

So $\exp(s-t)Tu(s)$ is integrable on $[0,t]$ and

$$\int_0^t \exp(s-t)Tu(s)ds = u(t) - \exp(-t)f. \tag{6}$$

From (6) one can easily deduce that u is differentiable and is solution of the desired initial value problem.
The proof of Theorem 2.3. is therefore complete.

Remark 2.4 Notice that when ρ satisfies the Δ_2-condition there is no reason for T to be norm-nonexpansive. So the classical theorems related to the existence of solutions to the initial value problem won't apply (see [2],[6]).

Remark 2.5 Let $L > A$ and consider the following system

$$\begin{cases} U_0(t) = f \\ U_{n+1}(t) = \exp(-t)f + \int_0^t \exp(s-t)TU_n(s)ds. \end{cases}$$

Then $\{U_n(t)\}$ is ρ-convergent to $U(t)$ and $U(t) = u(t)$ for $t \in [0,A]$. This implies that there exists $u(t) \in C$ for all $t \in [0,\infty)$, such that the restriction of u to $[0,A]$ is the ρ-limit of the sequence $\{u_n(t)\}$ given in Theorem 2.2. From now on we will use the notation u_f to designate this function u associated to the initial condition $u(0) = f$.

In the next result we discuss the existence of ρ-nonexpansive semigroups in L^φ.

Theorem 2.6. Let C and T be as stated in Theorem 2.1. For any $f \in C$ consider $u_f(t) \in C$ for $t \in [0,\infty)$. Define $S : [0,\infty) \times C \to C$ by

$$S(t)f = u_f(t).$$

Then S defines a ρ-nonexpansive semigroup.
Proof. Clearly we have $S(0)f = f$ for all $f \in C$. Using Proposition 1.6, we get

$$\rho(S(t)f - S(t)g) \le \liminf_{n\to\infty} \rho(u_{f,n}(t) - u_{g,n}(t))$$

where $\{u_{f,n}\}$ is the sequence given by Theorem 2.1, with the initial value f. An easy induction, using Lemma 2.2 gives

$$\rho(u_{f,n} - u_{g,n}) \le \rho(f - g)$$

for all $t \ge 0$. Therefore,

$$\rho(S(t)f - S(t)g) \le \rho(f - g)$$

for all $t \ge 0$. So the mapping $S(t)$ is ρ-nonexpansive for all $t \ge 0$.
In order to complete the proof of Theorem 2.6, we need to show that $S(t+\mu) = S(t)S(\mu)$ for all $t \ge 0$ and $\mu \ge 0$. Let $f \in C$ and put $S(\mu)f = f_\mu$. Consider the following system

$$\begin{cases} U_0(0) = f_\mu \\ U_{n+1}(t) = \exp(-t)f_\mu + \int_0^t \exp(s-t)T(U_n(s))ds \end{cases}$$

for all $t \ge 0$.
We saw that $\{U_n(t)\}$ ρ-converges to $S(t)f_\mu$, for any $t \ge 0$. We denote by $\{u_n(t)\}$ the sequence given by the same system with f as initial value. Let us show that

$$\rho(U_n(t) - u_{n+m}(t+\mu)) \le \sum_{i=m+1}^{n+m+1} K^i(\mu)\delta_\rho(C) + K^{n+1}(t)\delta_\rho(C) \tag{7}$$

for any $n, m \in N$, and any $t, \mu \ge 0$.
We fix n and prove (7) by induction on n.
First notice that

$$u_n(t+\mu) = \exp(-t-\mu)f + \int_0^{t+\mu} \exp(s-t-\mu)Tu_{n-1}(s)ds$$

So

$$u_n(t+\mu) = \exp(-t)\{\exp(-\mu)f + \int_0^{\mu} \exp(s-\mu)Tu_{n-1}(s)ds\} + \exp(-t)\int_0^t \exp(s)Tu_{n-1}(s)ds.$$

Let us go back to the inequality (7) and let $n = 0$. We get

$$U_0(t) - u_m(t + \mu) = u(\mu) - u_m(t + \mu),$$

since $u(\mu) = f_\mu$, so

$$U_0(t)-u_m(t+\mu) = \exp(-t)(u(\mu)-u_m(\mu))+\int_0^t \exp(s-t)(u(\mu)-Tu_{m-1}(s+\mu))ds.$$

Then

$$\rho(U_0(t)-u_m(t+\mu)) \leq \exp(-t)\rho(u(\mu)-u_m(\mu))+K(t)\sup_{0\leq s\leq t}\{\rho(u(\mu)-Tu_{m-1}(s+\mu))\}.$$

Using the inequality (2) and the definition of $\delta_\rho(C)$ we get the inequality (7) for $n = 0$. Assume that this inequality holds for n and let us prove it for $n + 1$. Since

$$U_{n+1}(t)-u_{n+m+1}(t+\mu) = \exp(-t)(u(\mu)-u_{n+m+1}(\mu))+\int_0^t \exp(s-t)(TU_n(s)-Tu_{n+m}(s))ds,$$

we obtain

$$\rho(U_{n+1}(t)-u_{n+m+1}(t)) \leq \exp(-t)\rho(u(\mu)-u_{n+m+1}(\mu))+K(t)\sup_{0\leq s\leq t}\rho(TU_n(s)-Tu_{n+m}(s+\mu)).$$

But

$$\rho(TU_n(s)-Tu_{n+m}(s+\mu)) \leq \rho(U_n(s)-u_{n+m}(s+\mu)) \leq [\sum_{i=m+1}^{n+m+1} K^i(\mu)+K^{n+1}(s)]\delta_\rho(C).$$

Using the fact that $K(s) \leq K(t)$ for $s \leq t$ and inequality (2), we get

$$\rho(U_{n+1}(t)-u_{n+m+1}(t+\mu)) \leq [K^{n+m+1+1}(\mu)\exp(-t)+K(t)(\sum_{m+1}^{n+m+1} K^i(\mu)+K^{n+1}(t))]\delta_\rho(C)$$

Therefore

$$\rho(U_{n+1}(t) - u_{n+m+1}(t + \mu)) \leq \sum_{m+1}^{n+m+2} K(\mu)\delta_\rho(C) + K^{n+2}(t)\delta_\rho(C).$$

So the inequality (7) holds for $n+1$.
By induction the inequality (7) holds for any $n, m \in N$ and any $t, \mu \geq 0$.
Using now Fatou property and letting $m \to \infty$ in (7) we get

$$\rho(U_n(t) - u(t+\mu)) \leq \liminf_{m\to\infty} \rho(U_n(t) - u_{n+m}(t+\mu)) \leq K^{n+1}(t)\delta_\rho(C),$$

since the series $\sum_{i\geq 1} K^i(\mu)$ is convergent.
Therefore, we deduce that $\{U_n(t)\}$ converges with respect to ρ to $u(t+\mu)$.
The uniqueness of the ρ-limit yields to

$$S(t)(U_0(t)) = u(t+\mu) = S(t)(u(\mu)).$$

Hence $S(t)S(\mu) = S(t+\mu)$ for all $t, \mu \geq 0$.
The proof of Theorem 2.6 is therefore complete.

We conclude this work by a remark which links the set of fixed points of the semigroup S and the set of fixed points of T.

Remark 2.7. Define the set $F(S)$ to be the set of $f \in C$ such that $S(t)f = f$ for all $t \geq 0$. Let us prove that

$$F(S) = F(T).$$

Obviously we have $F(T) \subset F(S)$. Indeed, let $f \in F(T)$ then one can easily prove that the sequence $\{u_{n,f}\}$ is constant and $u_{n,f}(t) = f$ for all $t \geq 0$.
Conversely, let $f \in F(S)$. From the inequality (2) and Fatou property, one can deduce

$$\rho(f - u_n(t)) \leq K^{n+1}(A)\delta_\rho(C) \tag{8}$$

for any $n \geq 1$ and any $t \leq A$, with $A > 0$.
On the other hand, we have

$$\exp(-t)f + K(t)Tf - u_{n+1}(t) = \int_0^t \exp(s-t)[Tf - Tu_n(s)]ds.$$

So by using Lemma 2.2, we obtain

$$\rho(\exp(-t)f + K(t)Tf - u_{n+1}(t)) \leq K(t) \sup_{0\leq s\leq t} \rho(Tf - Tu_n(s)).$$

Since T is ρ-nonexpansive, we get from (8)

$$\rho(\exp(-t)f + K(t)Tf - u_{n+1}(t)) \leq K^{n+2}(t)\delta_\rho(C).$$

So $\{u_{n+1}(t)\}$ ρ-converges to $\exp(-t)f + K(t)Tf$ for any $t \geq 0$. Uniqueness of the ρ-limit implies that

$$S(t)f = \exp(-t)f + K(t)Tf,$$

which yields to $Tf = f$.
The proof of our statement is therefore complete.

The author would like to thank Professor S. Reich with whom he got fruitful discussions regarding this work.

References

[1] S. Chen, M.A. Khamsi, W.M. Kozlowski, "Some geometrical properties and fixed point theorems in Orlicz spaces", To appear in Jour. Math. Anal. Appl.

[2] M.C. Crandall, A. Pazy, "Semigroups of nonlinear contractions and dissipative sets", Jour. Funct. Anal., Vol.3(1963), 376-418.

[3] M.A. Khamsi, W.M. Kozlowski,S. Reich," Fixed point theory in Modular function spaces", Nonlinear Analysis 14(1990), 935-953.

[4] W.M. Kozlowski, " Modular function spaces", Dekker, New York, Basel 1988.

[5] J. Musielak, " Orlicz spaces and Modular spaces", Lecture Notes in Math. 1034, Springer-Verlag, Berlin, Heidelberg, New York 1983.

[6] S. Reich, "A note on the mean ergodic theorem for nonlinear semigroups", J. Math. Anal. Appl., Vol. 91(1983), 547-551.