Pergamon

Nonlinear Analysis, Theory, Methods & Applications, Vol. 26, No. 10, pp. 1733-1748, 1996

0362-546X/96 $15.00 + 0.00

0362-546X(94)00216-9

ON UNIFORM OPIAL CONDITION AND UNIFORM KADEC–KLEE PROPERTY IN BANACH AND METRIC SPACES

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(*Received* 19 *May* 1993; *received for publication* 18 *August* 1994)

Key words and phrases: Opial condition, Kadec–Klee property, hyperbolic metric spaces, normal structure property, fixed point property.

1. INTRODUCTION

Recently it has been shown [1–6] that many classical or nonclassical Banach spaces enjoy a uniform property with respect to a given topology. This enables the authors to prove a fixed point result via well known theorems [7–9]. In this work we show that there is a more general property that reduces to the main conclusion of these results. Therefore, these conclusions should not be seen as particular results in particular spaces. We will also define and study properties such as Opial condition, Kadec–Klee and ergodic properties in hyperbolic metric spaces.

2. BASIC DEFINITIONS

In this section, X will stand for a Banach space and τ for a topological vector space topology on X that is weaker than the norm topology.

Definition 2.1. X will be said to satisfy the property (L) for the topology τ if and only if there exists a continuous function $\delta(r, s)$ such that

$$\delta\left(\liminf_{n\to\infty}\|x_n - x\|, \|x - y\|\right) \le \liminf_{n\to\infty}\|x_n - y\|, \tag{1}$$

for every $\{x_n\}$ τ-convergent to x in X and for every $y \in X$. We will assume that δ is increasing with respect to every variable, that is

$$\delta(r, s) < \delta(r', s) \qquad \text{whenever } r < r',$$

and

$$\delta(r, s) < \delta(r, s') \qquad \text{whenever } s < s'.$$

Let us mention that this property originated from Lim's work [10]. Indeed Lim proved that for $\{x_n\} \in l^p$ $(p \in (1, \infty))$, weakly convergent to X, then

$$\liminf_{n\to\infty}\|x_n - y\| = \left(\liminf_{n\to\infty}\|x_n - x\|^p + \|x - y\|^p\right)^{1/p},$$

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for every $y \in l^p$. The same conclusion holds in l^1 for $\sigma(l^1, c_0)$-convergent sequences. Using this property Lim defined in any Banach space a property similar to the property (L). In his definition the inequality in the relation (1) is replaced by an equality. It is rather restricting to consider the equality since many examples do not satisfy this property.

Definition 2.2. Let δ be as in definition 2.1.

(i) We will say that δ is of type (I) if

$$\delta(r, 0) = r, \qquad \text{for every } r.$$

(ii) We will say that δ is of type (II) if

$$\delta(0, s) = s, \qquad \text{for every } s.$$

Using the continuity of δ, it is not hard to see that if δ is of type (I) (resp. (II)) then

$$\inf\{\delta(r, s_0) - r;\ r \in [0, A]\} > 0 \qquad (\text{resp. } \inf\{\delta(r_0, s) - s;\ s \in [0, A]\} > 0)$$

for every $r_0 > 0$, $s_0 > 0$ and $A > 0$.

Examples. (1) Let $\{x_n\} \in l^p$ ($p \geq 1$) be weakly convergent to 0. Then for every $x \in l^p$ we have

$$\liminf_{n\to\infty} \|x_n - x\|^p = \liminf_{n\to\infty} \|x_n\|^p + \|x\|^p.$$

This implies that l^p (for $p \geq 1$) satisfies the property (L) for the weak topology, with

$$\delta(r, s) = (r^p + s^p)^{1/p}.$$

Since the weak topology and the strong topology coincide in l^1, the property (L) for the weak topology reduces to a trivial relation in this case. Therefore, this example is interesting only when $p > 1$.

(2) Consider the space l^p ($1 < p < \infty$) with the new norm

$$|x| = \|x^+\| + \|x^-\|, \qquad x \in l^p$$

where x^+ and x^- are the positive and negative parts of x, respectively, and $\|\cdot\|$ stands for the l^p-norm. The new space is denoted by $l_{p,1}$. Let $u \in l_{p,1}$ and define the support of u to be the set $\operatorname{supp}(u) = \{n; u_n \neq 0\}$. Let u and v in $l_{p,1}$, we will write $\operatorname{supp}(u) < \operatorname{supp}(v)$ whenever for every $i \in \operatorname{supp}(u)$ and $j \in \operatorname{supp}(v)$ we have $i \leq j$. Let u and v be in $l_{p,1}$ such that $|u + v| \leq 1$, $u \neq 0$, $v \neq 0$ and $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$. Then we have the following inequality:

$$|u + v| \geq (|u|^p + |v|^p)^{1/p}.$$

Indeed, since

$$|u + v| = \|u^+ + v^+\| + \|u^- + v^-\| = (\|u^+\|^p + \|v^+\|^p)^{1/p} + (\|u^-\|^p + \|v^-\|^p)^{1/p},$$

then using the triangle inequality for the l^p-norm in $\mathbf{R}^2$, we get

$$|u + v| \geq (|u|^p + |v|^p)^{1/p}.$$

Therefore, $(l^p, |\cdot|)$ (for $p \geq 1$) satisfies the property (L) for the weak topology, with

$$\delta(r, s) = (r^p + s^p)^{1/p}.$$

(3) Let $\{x_n\} \in l^1$ be $\sigma(l^1, c_0)$-convergent to 0. Then for every $x \in l^1$ we have

$$\liminf_{n \to \infty} \|x_n - x\| = \liminf_{n \to \infty} \|x_n\| + \|x\|.$$

Hence l^1 satisfies the property (L) for the weak*-topology, with

$$\delta(r, s) = r + s.$$

A similar conclusion holds for $l_{1,p}$ $(1 < p < \infty)$, where $l_{1,p}$ stands for l^1 with the new norm

$$|x| = (\|x^+\|^p + \|x^-\|^p)^{1/p}, \qquad x \in l^1,$$

where x^+ and x^- are the positive and negative parts of x, respectively, and $\|\cdot\|$ stands for the l^1-norm. We use again the $\sigma(l^1, c_0)$-convergence and get

$$\delta(r, s) = (r^p + s^p)^{1/p}.$$

(4) Let X be a Banach space that is reflexive and has a weakly sequentially continuous duality map J_ϕ associated to a gauge function ϕ which is continuous, strictly increasing with $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. Set $\Phi(t) = \int_0^t \phi(x)\,\mathrm{d}x$. Then

$$\Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\phi(x + ty)\rangle\,\mathrm{d}t,$$

for all $x, y \in X$. Therefore, if $\{x_n\}$ converges weakly to 0 in X and $x \in X$, then

$$\Phi\left(\liminf_{n \to \infty} \|x_n + x\|\right) = \Phi\left(\liminf_{n \to \infty} \|x_n\|\right) + \Phi(\|x\|).$$

Therefore, X satisfies the property (L) for the weak topology and

$$\delta(r, s) = \Phi^{-1}(\Phi(r) + \Phi(s)).$$

For more on this example see [11].

(5) In [12] it is proved that if $\{f_n\}$ is a sequence of L^p-uniformly bounded functions on a measure space, and if $f_n \to f$ almost everywhere, then

$$\liminf_{n \to \infty} \|f_n\|^p = \liminf_{n \to \infty} \|f_n - f\|^p + \|f\|^p,$$

for all $p \in (0, \infty)$. A generalization of this relation to vector valued functions can be found in [1]. Therefore, L^p satisfies the property (L) for convergence almost everywhere with

$$\delta(r, s) = (r^p + s^p)^{1/p}.$$

It is not hard to see that this conclusion still holds for convergence in measure.

(6) Consider the Banach space $C_1(l^p, l^q)$ of nuclear operators, with $1/p + 1/q = 1$. It is known (see [13, 14]) that $C_1(l^p, l^q)$ is the dual of $K(l^q, l^p)$ the space of compact operators. Using Arazy's ideas [15] one can show that if $\{x_n\} \in C_1(l^p, l^q)$ converges weak* to 0, then for every $x \in C_1(l^p, l^q)$ we have

$$\liminf_{n\to\infty} \|x_n + x\|^q \geq \frac{1}{2^{q/p}} \liminf_{n\to\infty} \|x_n\|^q + \|x\|^q.$$

This clearly implies that $C_1(l^p, l^q)$ satisfies the property (L) for the weak* topology and

$$\delta(r, s) = \left(\frac{1}{2^{q/p}} r^q + s^q\right)^{1/q}.$$

(7) Consider the Hardy space $H^1(\Delta)$ where Δ is the open unit disc in the complex plane. Recall that

$$H^1(\Delta) = \{f: \Delta \to C; f \text{ holomorphic and } \|f\| < \infty\},$$

where

$$\|f\| = \sup_{0<r<1} \left\{\frac{1}{2\pi} \int_0^{2\pi} |f(r\, e^{i\theta})| \, d\theta\right\}.$$

It is well known that (see [16, 17])

$$H^1(\Delta) = (C(\Pi)/\bar{A}_0(\Delta))^*,$$

where Π is the unit sphere in the complex plane, $C(\Pi)$ is the space of all continuous functions defined on Π and

$$\bar{A}_0(\Delta) = \{f \text{ analytical functions on } \Delta \text{ such that } f(0) = 0\}.$$

using the ideas of Haagerup and Pisier [18] one can show that if $\{x_n\} \in H^1(\Delta)$ converges weak* to 0 and if $x \in H^1(\Delta)$ then

$$\frac{1}{2} \liminf_{n\to\infty} \|x_n\|^2 + \|x\|^2 \leq \liminf_{n\to\infty} \|x_n - x\|^2.$$

Hence, $H^1(\Delta)$ satisfies the property (L) for this weak* topology with

$$\delta(r, s) = (\tfrac{1}{2} r^2 + s^2)^{1/2}.$$

(8) Consider the spaces

$$J_i = \{\{x_n\} \in c_0; \|(x_n)\|_i < \infty\},$$

for $i = 1, 2, 3$ where

$$\|(x_n)\|_1 = \sup_{p_1<p_2<\cdots<p_m} \left(\sum_{j=1}^{m} |x_{p_j} - x_{p_{j-1}}|^2\right)^{1/2}$$

$$\|(x_n)\|_2 = \sup_{p_1<p_2<\cdots<p_{2k}} \left(\sum_{j=1}^{k} |x_{p_{2j}} - x_{p_{2j-1}}|^2\right)^{1/2}.$$

and

$$\|(x_n)\|_3 = \sup_{p_1 < p_2 < \cdots < p_m} \left(\sum_{j=2}^{m} |x_{p_j} - x_{p_{j-1}}|^2 + |x_{p_m} - x_{p_1}|^2 \right)^{1/2}.$$

The choice of norms depends on the property that one would like to get. These spaces play a central role in the geometry of Banach spaces (see [19–21]). It is not hard to see that if $u = \sum_{i<p} \alpha_i e_i$ and $v = \sum_{i>p+1} \beta_i e_i$ (where $\{e_n\}$ are the unit coordinate vectors) are in J_1 or J_2 then

$$\|u\|_i^2 + \|v\|_i^2 \leq \|u + v\|_i^2,$$

where $i = 1, 2$. From this one can deduce that if $\{x_n\}$ converges weakly to 0 in J_1 or J_2 then

$$\liminf_{n\to\infty} \|x_n\|_i^2 + \|x\|_i^2 \leq \liminf_{n\to\infty} \|x_n - x\|_i^2,$$

for every $x \in J_i$ and $i = 1, 2$. Therefore, the spaces J_1 and J_2 satisfy the property (L) for the weak topology and

$$\delta(r, s) = (r^2 + s^2)^{1/2},$$

for both spaces.

3. UNIFORM OPIAL CONDITION AND UNIFORM KADEC–KLEE PROPERTY

Throughout this section we will assume that the topology τ is lower semicontinuous with respect to the norm, that is if $\{x_n\}$ τ-converges to $x \in X$, then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

It is easy to see that this will happen if and only if the closed balls are sequentially τ-closed.

Definition 3.1. We will say that X satisfies τ-Opial condition if for every bounded $\{x_n\} \in X$ that τ-converges to $x \in X$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for every $y \neq x$.

We will say that X satisfies the uniform τ-Opial condition if for every $R > 0$, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $\{x_n\} \in X$ which τ-converges to $x \in X$ and for every $y \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| + \eta \leq \liminf_{n\to\infty} \|x_n - y\|,$$

provided $\liminf_{n\to\infty} \|x_n - x\| \leq R$ and $\|x - y\| \geq \varepsilon$.

Opial's property plays an important role in the study of τ-convergence of iterates nonexpansive mappings and of asymptotic behavior of nonlinear semigroups [22–26]. Clearly uniform τ-Opial condition implies τ-Opial condition. This property is related to the property (L) as the next theorem shows.

Theorem 3.1. Let X be a Banach space that satisfies the property (L) for τ-convergent sequences. Assume that the associated function δ is of type (I). Then X satisfies uniform τ-Opial condition.

Proof. Let $\varepsilon > 0$ and $\{x_n\}$ be τ-convergent to $x \in X$. Let $z \in X$ such that $\|x - z\| \geq \varepsilon$. Then since X satisfies the property (L) we have

$$\delta\left(\liminf_{n \to \infty} \|x_n - x\|, \|x - z\|\right) \leq \liminf_{n \to \infty} \|x_n - z\|.$$

Using the properties of δ we get

$$\delta\left(\liminf_{n \to \infty} \|x_n - x\|, \varepsilon\right) \leq \liminf_{n \to \infty} \|x_n - z\|.$$

Since δ is of type (I), then there exists η such that

$$\liminf_{n \to \infty} \|x_n - x\| + \eta \leq \delta\left(\liminf_{n \to \infty} \|x_n - x\|, \varepsilon\right).$$

This clearly implies the conclusion of theorem 3.1.

Using the examples discussed in the previous section we get the following corollary.

Corollary 3.1. (1) $l_{p,1}$, for $p \in [1, \infty)$, satisfies weak-uniform Opial condition.

(2) Let τ be the weak*-topology $\sigma(l^1, c_0)$. Then $l_{1,p}$, for $p \in [1, \infty)$, satisfies τ-uniform Opial condition.

(3) Let X be a reflexive Banach space which has a weakly sequentially continuous duality map. Then X satisfies weak-uniform Opial condition.

(4) Let τ be the topology of convergence in measure. The L^p, for $p > 0$, satisfies τ-uniform Opial condition.

(5) The James' spaces J_1 and J_2 satisfy weak-uniform Opial condition.

In the following we will discuss the case of spaces with δ that is of type (II). Before we proceed, let us introduce the Kadec–Klee property (in short K–K property).

Definition 3.2. We will say the X satisfies τ-K–K property if for some $\varepsilon > 0$ there exists $\eta > 0$ such that for every $\{x_n\}$ in the unit ball of X τ-convergent to x we have

$$\|x\| \leq 1 - \eta$$

provided that

$$\operatorname{sep}\{x_n\} = \inf\{\|x_n - x_m\|;\ n \neq m\} > \varepsilon.$$

We will say that X satisfies τ-uniform K–K property if the above still holds for every ε.

This property originated in [27, 28]. It was quickly related to the fixed point property through normal structure property [29, 30] (see the next section for the definitions). In [5, 31] the authors introduced a Kadec–Klee property for other than the classical weak or weak*-topologies.

THEOREM 3.2. Let X be a Banach space that satisfies the property (L) for τ convergent sequences. Assume that the associated function δ is of type (II). Then X satisfies τ-uniform Kadec–Klee property.

Proof. Let $\varepsilon > 0$ and $\{x_n\}$ in the unit ball be τ-convergent to x. Assume that $\operatorname{sep}\{x_n\} > \varepsilon$. Using the triangle inequality one can show that (after throwing out at most one element of $\{x_n\}$)

$$\frac{\varepsilon}{2} \leq \liminf_{n\to\infty} \|x_n - x\|.$$

Since

$$\delta\left(\liminf_{n\to\infty} \|x_n - x\|, \|x\|\right) \leq \liminf_{n\to\infty} \|x_n\|,$$

we get

$$\delta\left(\frac{\varepsilon}{2}, \|x\|\right) \leq 1.$$

using the fact that δ is of type (II), then there exists $\eta > 0$ such that

$$\|x\| \leq 1 - \eta.$$

This gives the desired conclusion.

All, the spaces discussed in the previous examples have an associated function δ that is of type (II). Then the next result will follow.

COROLLARY 3.2. All the spaces discussed in the examples of the previous section satisfy τ-uniform Kadec–Klee property.

It is worthy to mention that this corollary not only includes known results but it adds other unknown ones.

4. OPIAL CONDITION AND KADEC–KLEE PROPERTY IN HYPERBOLIC METRIC SPACES

Let (M, d) be a metric space. Suppose there exists a family $\mathfrak{F}$ of metric segments such that each two points x, y in M are endpoints of a unique metric segment $[x, y] \in \mathfrak{F}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote by $(1 - \beta)x \oplus \beta y$ the unique point Z of $[x, y]$ which satisfies

$$d(x, z) = \beta d(x, y), \qquad \text{and} \qquad d(z, y) = (1 - \beta)d(x, y).$$

Such metric spaces are usually called *convex metric spaces*. If, moreover, we have for all p, x, y in M

$$d(\tfrac{1}{2}p \oplus \tfrac{1}{2}x, \tfrac{1}{2}p \oplus \tfrac{1}{2}y) \leq \tfrac{1}{2}d(x, y)$$

then M is said to be a *hyperbolic metric space*. It is easy to check that in this case we have for all x, y, z, w in M and $\beta \in [0, 1]$

$$d((1-\beta)x \oplus \beta y, (1-\beta)z \oplus \beta w) \leq (1-\beta)d(x, z) + \beta d(y, w).$$

In general the point

$$\eta_1 x_1 \oplus \cdots \oplus \eta_n x_n$$

where $(x_1, \ldots, x_n)$ in M and (η_i) in $[0, 1]$ such that $\sum \eta_i = 1$, can be defined in a canonical fashion. In particular, for every $n_0 \geq 1$, for every $n = kn_0 + m$ and for every $x_1, x_2, \ldots, x_n \in M$, we have

$$\frac{1}{n}x_1 \oplus \frac{1}{n}x_2 \oplus \cdots \oplus \frac{1}{n}x_n = \frac{kn_0}{n}\left(\frac{1}{k}z_0 \oplus \cdots \oplus \frac{1}{k}z_{k-1}\right) \oplus \left(1 - \frac{kn_0}{n}\right)w$$

where

$$z_i = \frac{1}{n_0}x_{in_0+1} \oplus \cdots \oplus \frac{1}{n_0}x_{(i+1)n_0}, \qquad i = 1, \ldots, k-1$$

and

$$w = \frac{1}{n - kn_0}x_{kn_0+1} \oplus \cdots \oplus \frac{1}{n - kn_0}x_n.$$

This definition of convex combinations does not enjoy all the properties of linear convex combinations. Therefore, in this work, we will assume that the following property is satisfied by convex combinations

$$d\left(\frac{1}{n}x_1 \oplus \cdots \oplus \frac{1}{n}x_n, \frac{1}{n}x_2 \oplus \cdots \oplus \frac{1}{n}x_n \oplus \frac{1}{n}x_{n+1}\right) \leq \frac{1}{n}d(x_1, x_{n+1}),$$

for every $(x_1, \ldots, x_{n+1})$ in M and $n \geq 1$.

Clearly normed spaces are hyperbolic spaces. As nonlinear examples one can consider Hadamard manifolds [32] and the Hilbert open unit ball equipped with the hyperbolic metric [33]. We will say that a subset C of a hyperbolic metric space M is convex if $[x, y] \subset C$ whenever x, y are in C.

Let τ be another topology on M that is weaker than the strong topology. We will assume that τ is lower semicontinuous, that is

$$d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n)$$

for every $\{x_n\}$ and $\{y_n\}$ in M τ-convergent to x and y (respectively) in M. We will also assume that τ-compact subsets are τ-sequentially compact.

One can mimic the definitions 2.1 and 2.2 in this setting.

Definition 4.1. Let (M, d) be a hyperbolic metric space.

(1) We will say that M satisfies τ-Opial condition if for every $\{x_n\} \in M$ that τ-converges to $x \in M$ we have

$$\liminf_{n\to\infty} d(x_n, x) < \liminf_{n\to\infty} d(x_n, y),$$

for every $y \neq x$.

(2) We will say that M satisfies the uniform τ-Opial condition if for every $R > 0$, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $\{x_n\} \subset M$ which τ-converges to $x \in M$ and for every $y \in M$, we have

$$\liminf_{n\to\infty} d(x_n, x) + \eta \leq \liminf_{n\to\infty} d(x_n, y),$$

provided $\liminf_{n\to\infty} d(x_n, x) \leq R$ and $d(x, y) \geq \varepsilon$.

(3) We will say that M satisfies τ-Kadec–Klee property if for some $\varepsilon > 0$ and every $r > 0$ there exists $\eta > 0$ such that for every $\{x_n\}$ in M τ-convergent to x we have

$$d(x, a) \leq r(1 - \eta)$$

provided that $d(x_n, a) \leq r$ and

$$\mathrm{sep}\{x_n\} = \inf\{d(x_n, x_m);\ n \neq m\} > r\varepsilon.$$

We will say that M satisfies τ-uniform Kadec–Klee property if the above holds for every ε.

Next we will discuss the relation between these two properties and the fixed point property.

For $D \subset M$ we set

$$\mathrm{diam}(D) = \sup\{d(x, y);\ x, y \in D\},$$

$$r(x, D) = \sup\{d(x, y);\ y \in D\},$$

$$r(D) = \inf\{r(x, D);\ x \in D\},$$

$$\mathcal{C}(D) = \{x \in D;\ r(D) = r(x, D)\}.$$

Note that $r(x, D)$ minimizes the radius of balls centered at x which contain D and that $\mathcal{C}(D)$ may be empty. However, if D is τ-sequentially compact then $\mathcal{C}(D)$ is not empty. Indeed let $\{x_n\}$ be in D such that $\lim_{n\to\infty} r(x_n, D) = r(D)$. Since D is τ-sequentially compact then there exists a subsequence $\{x_{n'}\}$ that is τ-convergent to $x \in D$. Using the lower semicontinuity of τ then

$$r(x, D) \leq \liminf_{n\to\infty} r(x_{n'}, D) = r(D).$$

This clearly implies that $x \in \mathcal{C}(D)$.

We will say that M satisfies the τ-normal structure if and only if for every τ-compact bounded convex subset D of M not reduced to one point has a nondiametral point $x \in D$, that is

$$r(x, D) < \mathrm{diam}(D).$$

Definition 4.2. A selfmap T defined on a subset D of M, is said to be the nonexpansive if

$$d(T(x), T(y)) \leq d(x, y),$$

for every $x, y \in D$. The fixed point set of T is defined as

$$Fix(T) = \{x \in D;\ T(x) = x\}.$$

We will say that $D \subset M$ has the fixed point property if every nonexpansive map defined on D has a nonempty fixed point set.

Fixed point theory for nonexpansive mappings has its origins in the 1965 existence theorems [7–9]. Although such mappings are natural extensions of the contraction mappings, it was clear from the outset that the study of nonexpansive mappings required techniques which go far beyond the purely metric approach. For more on the fixed point property see [34, 35]. Our first result in this section relates the fixed point property to τ-Opial condition.

THEOREM 4.1. Let M be as described above and D be a τ-sequentially compact bounded convex subset of M. We will assume that M satisfies the τ-Opial condition and D is complete. Then D has the fixed point property.

Proof. Let $T: D \to D$ be a nonexpansive map. We will show first that there exists a sequence $\{x_n\} \in D$ such that

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0.$$

Such a sequence is called a quasi fixed sequence. Indeed let $\varepsilon \in (0, 1)$ and $x_0 \in D$. Set

$$T_\varepsilon(x) = (1 - \varepsilon)T(x) \oplus \varepsilon x_0.$$

Since D is convex, this map is well defined. Using the hyperbolicity of M, one can show that

$$d(T_\varepsilon(x), T_\varepsilon(y)) \leq (1 - \varepsilon)d(x, y),$$

for every $x, y \in D$. Therefore, T_ε has a unique fixed point in D, say x_ε. It is easy to see that

$$d(x_\varepsilon, T(x_\varepsilon)) \leq \varepsilon \operatorname{diam}(D).$$

If we denote by x_n the fixed point of $T_{1/n}$, we get

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0.$$

On the other hand since D is τ-sequentially compact there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ that is τ-convergent to $x \in D$. Using the nonexpansiveness of T we deduce that

$$\liminf_{n\to\infty} d(x_{n'}, T(x)) \leq \liminf_{n\to\infty} d(x_{n'}, x).$$

Since M satisfies τ-Opial condition we obtain $T(x) = x$. The conclusion of theorem 4.1 is, therefore, complete.

It is clear from the proof that one does not need the convexity of D. Indeed this assumption can be replaced by assuming that D is starshaped, that is there exists $x_0 \in D$ such that $[x_0, x]$ is in D whenever $x \in D$.

Let us add that under the assumptions of theorem 4.1 more can be said about the fixed point set when the map is affine. Recall that $T: D \to D$ (where D is convex) is said to be *affine* if for every $x_1, x_2, \ldots, x_n \in D$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ such that $\sum \alpha_i = 1$, then

$$T(\alpha_1 x_1 \oplus \cdots \oplus \alpha_n x_n) = \alpha_1 T(x_1) \oplus \cdots \oplus \alpha_n T(x_n).$$

THEOREM 4.2. Let (M, d) be a hyperbolic metric space and D be a τ-sequentially compact bounded convex subset of M. Assume that M satisfies the τ-Opial condition. Let $T: D \to D$ be a nonexpansive affine map. Then for every $x \in D$

$$P(x) = \tau - \lim_{n\to\infty}\left(\frac{1}{n}x \oplus \frac{1}{n}T(x) \oplus \cdots \oplus \frac{1}{n}T^{n-1}(x)\right),$$

exists. P defines a nonexpansive projection on $Fix(T)$.

Proof. First we know from theorem 4.1 that $Fix(T)$ is not empty. Let $x \in M$ and put

$$y_n = \frac{1}{n}x \oplus \frac{1}{n}T(x) \oplus \cdots \oplus \frac{1}{n}T^{n-1}(x)$$

for every $n \geq 1$. Let us show that $\{y_n\}$ is a quasi fixed sequence, that is $\lim d(y_n, T(y_n)) = 0$. Since T is affine then

$$T(y_n) = \frac{1}{n}T(x) \oplus \frac{1}{n}T^2(x) \oplus \cdots \oplus \frac{1}{n}T^n(x).$$

Therefore,

$$d(y_n, T(y_n)) \leq \frac{1}{n}d(x, T^n(x)) \leq \frac{1}{n}\operatorname{diam}(D).$$

Hence

$$\lim_{n\to\infty} d(y_n, T(y_n)) = 0.$$

Using the argument shown in the proof of theorem 4.1, we obtain that any τ-cluster point of $\{y_n\}$ is a fixed point of T. Let y be a τ-cluster point of $\{y_n\}$ and set

$$\eta_n = d(y_n, y)$$

for every $n \geq 1$. Let us prove that $\eta_{kn} \leq \eta_n$ for every $k \geq 1$. It is easy to see that

$$y_{kn} = \frac{1}{k}z_0 \oplus \frac{1}{k}z_1 \oplus \cdots \oplus \frac{1}{k}z_{k-1},$$

where

$$z_i = \frac{1}{n}T^{in}(x) \oplus \frac{1}{n}T^{in+1}(x) \oplus \cdots \oplus \frac{1}{n}T^{in+n-1}(x), \qquad i = 1, \ldots, k-1.$$

Since T is affine then

$$z_i = T^{in}\left(\frac{1}{n}x \oplus \frac{1}{n}T(x) \oplus \cdots \oplus \frac{1}{n}T^{n-1}(x)\right) = T^{in}(y_n)$$

for every $i \in [0, k-1]$. Then using the fact that $y \in Fix(T)$ and T is nonexpansive we get

$$\eta_{kn} \leq \frac{1}{k} \sum_{0 \leq i \leq k-1} d(z_i, y) \leq \frac{1}{k} \sum_{0 \leq i \leq k-1} d(y_n, y).$$

This clearly gives $\eta_{kn} \leq \eta_n$. The last step in our proof will be to show that

$$\lim_{n \to \infty} d(y_n, y) \text{ exists.}$$

Let $n_0 \geq 1$ and $n > n_0$. Then there exists $k \geq 1$ such that

$$kn_0 \leq n < kn_0 + n_0.$$

Let us rewrite y_n as

$$y_n = \frac{kn_0}{n} w_1 \oplus \left(1 - \frac{kn_0}{n}\right) w_2,$$

where $w_1 = y_{kn_0}$ and

$$w_2 = \frac{1}{n - kn_0} T^{kn_0}(x) \oplus \cdots \oplus \frac{1}{n - kn_0} T^{n-1}(x).$$

So

$$d(y_n, y_{kn_0}) \leq \left(1 - \frac{kn_0}{n}\right) d(w_2, y_{kn_0}) \leq \left(1 - \frac{kn_0}{n}\right) \operatorname{diam}(D).$$

Using the definition of k, we deduce that

$$\lim_{n \to \infty} \frac{kn_0}{n} = 1.$$

Therefore,

$$\lim_{n \to \infty} d(y_n, y_{kn_0}) = 0,$$

which implies

$$\limsup_{n \to \infty} d(y_n, y) = \limsup_{n \to \infty} d(y_{k(n)n_0}, y) \leq \eta_{n_0}.$$

Since n_0 was arbitrary we deduce

$$\limsup_{n \to \infty} d(y_n, y) \leq \liminf_{n \to \infty} d(y_n, y),$$

which gives the desired conclusion. Let us show that $\{y_n\}$ has one τ-cluster point. Assume not and let y and y' be two distinct τ-cluster points of $\{y_n\}$. Using the τ-Opial condition one can show easily that

$$\lim_{n \to \infty} d(y_n, y) < \lim_{n \to \infty} d(y_n, y').$$

This clearly implies a contradiction. Finally put

$$P(x) = \tau - \lim_{n \to \infty} y_n.$$

Then $P(x) \in Fix(T)$ and, therefore, we have $P(P(x)) = P(x)$. Since T is nonexpansive and d is lower semicontinuous, then P is nonexpansive. This completes the proof of theorem 4.2.

One would ask whether P is affine. This will happen if, moreover, we assume that convex combinations are τ-sequentially continuous, that is if $\{x_n\}$ and $\{y_n\}$ are τ-convergent to x and y, respectively, then $\{\alpha x_n \oplus (1 - \alpha)y_n\}$ τ-converges to $\alpha x \oplus (1 - \alpha)y$ for every $\alpha \in [0, 1]$. The next question will be to wonder whether a similar conclusion holds for nonaffine nonexpansive maps. Using Bruck's ideas [36] one can indeed prove that the fixed point set is a nonexpansive retract of M. Bruck's technique will not give an ergodic conclusion similar to the one of theorem 4.2. For more on this problem see [1, 37, 38].

Before getting to the result that relates τ-Kadec-Klee property to τ-normal structure property, we need the following technical lemma [39].

LEMMA 4.1. Assume that the hyperbolic metric space M has not the τ-normal structure property. Then there exists a bounded τ-convergent sequence $\{x_n\} \in M$ such that

$$\lim_{n\to\infty} d(x_n, x_i) = \operatorname{diam}(\{x_n\}),$$

for every $i \geq 1$.

Proof. Assume that M fails to have the τ-normal structure property. Then there exists D a τ-compact convex subset of M not reduced to one point such that

$$r(x, D) = \operatorname{diam}(D),$$

for every $x \in D$. Such a subset is called a diametral set. Let $x_1 \in D$. Since $r(x_1, D) = \operatorname{diam}(D)$, one can find $x_2 \in D$ such that

$$\operatorname{diam}(D)\left(1 - \frac{1}{2^2}\right) \leq d(x_1, x_2).$$

Assume that $x_1, x_2, \ldots, x_n$ are constructed. Since D is convex then

$$\frac{1}{n}x_1 \oplus \frac{1}{n}x_2 \oplus \cdots \oplus \frac{1}{n}x_n \in D.$$

Our assumption on D implies that there exists $x_{n+1} \in D$ such that

$$\operatorname{diam}(D)\left(1 - \frac{1}{(n+1)^2}\right) \leq d\left(x_{n+1}, \frac{1}{n}x_1 \oplus \cdots \oplus \frac{1}{n}x_n\right).$$

By induction the sequence $\{x_n\}$ is constructed. Let $i \geq 1$ and $n > i$. Then since

$$\begin{aligned} d\left(x_{n+1}, \frac{1}{n}x_1 \oplus \cdots \oplus \frac{1}{n}x_n\right) &\leq \frac{1}{n}d(x_{n+1}, x_i) + \sum_{j\neq i}\frac{1}{n}d(x_{n+1}, x_j) \\ &\leq \frac{1}{n}d(x_{n+1}, x_i) + \left(1 - \frac{1}{n}\right)\operatorname{diam}(D), \end{aligned}$$

we get

$$\operatorname{diam}(D)\left(1 - \frac{1}{(n+1)^2}\right) \le \frac{1}{n} d(x_{n+1}, x_i) + \left(1 - \frac{1}{n}\right) \operatorname{diam}(D).$$

This clearly implies

$$\operatorname{diam}(D)\left(1 - \frac{n}{(n+1)^2}\right) \le d(x_{n+1}, x_i) \le \operatorname{diam}(D).$$

Hence

$$\lim_{n \to \infty} d(x_n, x_i) = \operatorname{diam}(D).$$

This conclusion is still valid for any subsequence of $\{x_n\}$. Since D is τ-compact, it is also τ-sequentially compact. Then $\{x_n\}$ has a τ-convergent subsequence. This completes the proof of lemma 4.1.

THEOREM 4.3. Assume that M satisfies the τ-uniform Kadec–Klee property. Then M has τ-normal structure property.

Proof. Assume not. Then there exists D τ-compact bounded convex subset of M not reduced to one point such that $r(x, D) = \operatorname{diam}(D)$ for every $x \in D$. Using lemma 4.1 one finds a sequence $\{x_n\} \in D$ that is τ-convergent to $x \in D$ and such that

$$\lim_{n \to \infty} d(x_n, x_i) = \operatorname{diam}(D),$$

for every $i \ge 1$. One can assume that $\operatorname{sep}\{x_n\} \ge \operatorname{diam}(D)/3$. Since M satisfies τ-uniform Kadec–Klee property there exists $\eta > 0$ such that for every $y \in D$

$$d(x, y) \le \operatorname{diam}(D)(1 - \eta),$$

since $d(x_n, y) \le \operatorname{diam}(D)$ for every $n \ge 1$. Hence

$$r(x, D) \le \operatorname{diam}(D)(1 - \eta).$$

This clearly contradicts our assumption on D. Therefore, the conclusion of theorem 4.3 holds.

Finally it is worth mentioning that τ-normal structure property is related to the fixed point property via Kirk's theorem [40, 41].

THEOREM 4.4. Assume that M has τ-normal structure property. Let D be a τ-compact bounded convex subset of M. Then D has the fixed point property.

The proof of this theorem is standard and will not be given here.

It is worth mentioning that a constructive proof of theorem 4.4 exists (see [42]), which requires only τ-sequential compactness. Therefore, many definitions given above in terms of τ-compact subsets, can be weakened to τ-sequentially compact subsets.

Acknowledgement—My thanks to the referee for many constructive suggestions regarding this paper and especially for bringing to our attention reference [6] that led to the example (2).

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