Math Competition: Review 1
Khamsi

Problem 1. [2005] Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Problem 2. [2005] Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Problem 3. [2005] Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx.$$

Problem 4. [2009] Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{cases} f' = 2f^2gh + \dfrac{1}{gh}, & f(0) = 1, \\[2ex] g' = fg^2h + \dfrac{4}{fh}, & g(0) = 1, \\[2ex] h' = 3fgh^2 + \dfrac{1}{fg}, & h(0) = 1. \end{cases}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.