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KKM mappings in metric type spaces

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ABSTRACT

In this work we discuss some recent results about KKM mappings in cone metric spaces. We also discuss the fixed point existence results of multivalued mappings defined on such metric spaces. In particular we show that most of the new results are merely copies of the classical ones and do not necessitate the underlying Banach space nor the associated cone.

1. Introduction

Cone metric spaces were introduced in [1]. A similar notion was also considered by Rzepecki in [2]. After carefully defining convergence and completeness in cone metric spaces, the authors in [1] proved some fixed point theorems of contractive mappings. Recently, more fixed point results in cone metric spaces appeared in [3,4]. Topological questions in cone metric spaces were studied in [3] where it was proved that every cone metric space is a first countable topological space. Hence, continuity is equivalent to sequential continuity and compactness is equivalent to sequential compactness. It is worth mentioning the pioneering work of Quilliot [5] who introduced the concept of generalized metric spaces. His approach was very successful and is used by many (see the references in [6]). It is our belief that cone metric spaces are a special case of generalized metric spaces. In this work, we introduce a metric type structure in cone metric spaces and show that classical proofs related to KKM mappings proved in [7] do apply almost identically in these metric type spaces. This approach suggests that any extension of known fixed point results to cone metric spaces is redundant. Moreover the underlying Banach space and the associated cone subset are not necessary.

For more on metric fixed point theory, the reader may consult book [8].

2. Basic definitions and results

First let us start by defining some basic definitions.

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Definition 1. Let E be a real Banach space with norm $\|.\|$ and P a subset of E. Then P is called a cone if and only if

1. P is closed, nonempty and $P \neq \{0\}$;
2. if $a, b \geq 0$, and $x, y \in P$, then $ax + by \in P$;
3. if $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone P in a Banach space E, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{Int}(P)$ (where $\mathrm{Int}(P)$ denotes the interior of P). The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in E$, we have

$$0 \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying this inequality is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone P is called regular if every decreasing sequence which is bounded from below is convergent. Regular cones are normal and there exist normal cones which are not regular. Moreover the cone P allows for the definition of a new natural norm $\|.\|_+$ in E, known as the order norm, defined by

$$\|x\|_+ = \inf\{\lambda \geq 0; -\lambda u \preceq x \preceq \lambda v \text{ for some } u, v \in B_1\},$$

where B_1 is the closed unit ball of E. It is easy to check that the normal constant of P with respect to $\|.\|_+$ is 1. But in general $\|.\|_+$ is not equivalent to $\|.\|$. For more on this norm, please see [9].

Throughout, the Banach space E and the cone P will be omitted.

Definition 2. A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:

1. $d(x, y) \in P$, i.e. $0 \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Convergence is defined as follows

Definition 3. Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in P$ with $c \gg 0$, there is $N \geq 1$ such that for all $n \geq N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent. We will say that $\{x_n\}$ converges to x and write $\lim_{n\to\infty} x_n = x$.

It is easy to show that if $\{x_n\}$ is convergent, then its limit is unique. Cauchy sequences and completeness are defined by

Definition 4. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in P$ with $c \gg 0$, there is $N \geq 1$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

The basic properties of convergent and Cauchy sequences may be found in [1]. In fact the properties and their proofs are identical to the classical metric ones. Since this work concerns the fixed point property of mappings, we will need the following property.

Definition 5. Let (X, d) be a cone metric space. A mapping $T : X \to X$ is called Lipschitzian if there exists $k \in \mathbb{R}$ such that

$$d(Tx, Ty) \preceq kd(x, y),$$

for all $x, y \in X$. The smallest constant k which satisfies the above inequality is called the Lipschitz constant of T, denoted $Lip(T)$.

As we mentioned earlier cone metric spaces have a metric type structure. Indeed we have the following result:

Theorem 2.1 (*[10]*)**.** *Let (X, d) be a metric cone over the Banach space E with the cone P which is normal with the normal constant K. The mapping $D : X \times X \to [0, \infty)$ defined by $D(x, y) = \|d(x, y)\|$ satisfies the following properties*

(1) $D(x, y) = 0$ *if and only if* $x = y$;
(2) $D(x, y) = D(y, x)$, *for any* $x, y \in X$;
(3) $D(x, y) \leq K\Big(D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y)\Big)$, *for any points* $x, y, z_i \in X$, $i = 1, 2, \ldots, n$.

Note that property (3) is discouraging since it does not give the classical triangle inequality satisfied by a distance. But there are many examples where the triangle inequality fails (see [11] for example).

The above result suggests the following definition.

Definition 6. Let X be a set. Let $D : X \times X \to [0, \infty)$ be a function which satisfies

(1) $D(x, y) = 0$ if and only if $x = y$;
(2) $D(x, y) = D(y, x)$, for any $x, y \in X$;
(3) $D(x, y) \leq K\Big(D(x, z) + D(z, y)\Big)$, for any points $x, y, z \in X$, for some constant $K > 0$.

The pair (X, D) is called a metric type space.

Similarly we define convergence and completeness in metric type spaces.

Definition 7. Let (X, D) be a metric type space.

1. The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} D(x_n, x) = 0$.
2. The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

(X, D) is complete if and only if any Cauchy sequence in X is convergent.

An example of such a D is given below.

Example 1. Let X be the set of Lebesgue measurable functions on [0, 1] such that

$$\int_0^1 |f(x)|^2 \mathrm{d}x < \infty.$$

Define $D : X \times X \to [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 \mathrm{d}x.$$

Then D satisfies the following properties

(1) $D(f, g) = 0$ if and only if $f = g$;
(2) $D(f, g) = D(g, f)$, for any $f, g \in X$;
(3′) $D(f, g) \leq 2\Big(D(f, h) + D(h, g)\Big)$, for any points $f, g, h \in X$.

Most of the examples in [7] will easily translate into metric type spaces; in particular, Example 28 describes a cone metric space with examples of admissible and non-admissible subsets.

3. Topological metric type spaces

In this section we discuss a natural topology defined in any metric type space. This topology enjoys most of the metric like properties.

Definition 8. Let (X, D) be a metric type space. A subset $A \subset X$ is said to be open if and only if for any $a \in A$, there exists $\varepsilon > 0$ such that the open ball $B_o(a, \varepsilon) \subset A$. The family of all open subsets of X will be denoted by τ.

The following result is easy to show.

Proposition 1. *τ defines a topology on (X, D).*

Next we discuss some properties of closed sets.

Proposition 2. *Let (X, D) be a metric type space and τ be the topology defined above. Then for any nonempty subset $A \subset X$ we have*

(1) *A is closed if and only if for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$;*
(2) *if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\varepsilon > 0$, we have*

$$B_o(x, \varepsilon) \cap A \neq \emptyset.$$

Proof. Let us prove (1) first. Assume that A is closed and let $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} x_n = x$. Let us prove that $x \in A$. Assume not, i.e. $x \notin A$. Since A is closed, then there exists $\varepsilon > 0$ such that $B_o(x, \varepsilon) \cap A = \emptyset$. Since $\{x_n\}$ converges to x, then there exists $N \geq 1$ such that for any $n \geq N$ we have $x_n \in B_o(x, \varepsilon)$. Hence $x_n \in B_o(x, \varepsilon) \cap A$, which leads to a contradiction. Conversely assume that for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$. Let us prove that A is closed. Let $x \notin A$. We need to prove that there exists $\varepsilon > 0$ such that $B_o(x, \varepsilon) \cap A = \emptyset$. Assume not, i.e. for any $\varepsilon > 0$, we have $B_o(x, \varepsilon) \cap A \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B_o(x, 1/n) \cap A$. Clearly we have $\{x_n\}$ converges to x. Our assumption on A implies $x \in A$, a contradiction.

Let us prove (2). Clearly $\overline{A}$ is the smallest closed subset which contains A. Set

$$A^* = \{x \in X;\ \text{for any } \varepsilon > 0,\ \text{there exists } a \in A \text{ such that: } D(x, a) < \varepsilon\}.$$

We have $A \subset A^*$. Next we prove that A^* is closed. For this we use property (1). Let $\{x_n\}$ be a sequence in A^* such that $\{x_n\}$ converges to x. Let us prove that $x \in A^*$. Let $\varepsilon > 0$. Since $\{x_n\}$ converges to x, there exists $N \geq 1$ such that for any $n \geq N$ we have

$$D(x, x_n) < \frac{\varepsilon}{2(K+1)},$$

where K is the constant associated to the triangle inequality satisfied by D. Since $x_n \in A^*$, there exists $a \in A$ such that

$$D(x_n, a) < \frac{\varepsilon}{2(K+1)}.$$

Hence

$$D(x, a) \leq K\Big(D(x, x_n) + D(x_n, a)\Big) < K\Big(\frac{\varepsilon}{2(K+1)} + \frac{\varepsilon}{2(K+1)}\Big) < \varepsilon,$$

which implies $x \in A^*$. Therefore A^* is closed and contains A. The definition of $\overline{A}$ will force $\overline{A} \subset A^*$, which implies the conclusion of (2). □

Next we discuss the compactness in metric type spaces.

Proposition 3. *Let (X, D) be a metric type space and τ the topology defined above. Let A be a nonempty subset of X. The following properties are equivalent*

(1) *A is compact.*
(2) *For any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k \to \infty} x_{n_k} \in A$.*

Proof. Assume that A is a nonempty compact subset of X. It is easy to see that any decreasing sequence of nonempty closed subsets of A have a nonempty intersection. Let $\{x_n\}$ be a sequence in A. Set $C_n = \{x_m;\, m \geq n\}$. Then we have $\bigcap_{n \geq 1} \overline{C_n} \neq \emptyset$. Let $x \in \bigcap_{n \geq 1} \overline{C_n}$. Then for any $\varepsilon > 0$ and for any $n \geq 1$, there exists $m_n \geq n$ such that $D(x, x_{m_n}) < \varepsilon$. This clearly implies the existence of a subsequence of $\{x_n\}$ which converges to x. Since A is closed, then we must have $x \in A$.

Conversely let A be a nonempty subset of X such that the conclusion of (2) is true. Let us prove that A is compact. First note that for any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subset B_o(x_1, \varepsilon) \cup \cdots \cup B_o(x_n, \varepsilon).$$

Assume not, then there exists $\varepsilon_0 > 0$, such that for any finite number of points $x_1, x_2, \ldots, x_n \in A$, we have

$$A \not\subset B_o(x_1, \varepsilon_0) \cup \cdots \cup B_o(x_n, \varepsilon_0).$$

Fix $x_1 \in A$. Since $A \not\subset B_o(x_1, \varepsilon_0)$, there exists $x_2 \in A \setminus B_o(x_1, \varepsilon_0)$. By induction we build a sequence $\{x_n\}$ such that

$$x_{n+1} \in A \setminus \Big(B_o(x_1, \varepsilon_0) \cup \cdots \cup B_o(x_n, \varepsilon_0)\Big)$$

for all $n \geq 1$. Clearly we have $D(x_n, x_m) \geq \varepsilon_0$, for all $n, m \geq 1$, with $n \neq m$. This condition implies that no subsequence of $\{x_n\}$ will be Cauchy or convergent. This contradicts our assumption on A. Next let $\{\mathcal{O}_\alpha\}_{\alpha \in \Gamma}$ be an open cover of A. Let us prove that only finitely many $\mathcal{O}_\alpha$ cover A. First note that there exists $\varepsilon_0 > 0$ such that for any $x \in A$, there exists $\alpha \in \Gamma$ such that $B_o(x, \varepsilon_0) \subset \mathcal{O}_\alpha$. Assume not, then for any $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that for any $\alpha \in \Gamma$, we have $B_o(x_\varepsilon, \varepsilon) \not\subset \mathcal{O}_\alpha$. In particular, for any $n \geq 1$, there exists $x_n \in A$ such that for any $\alpha \in \Gamma$, we have $B_o(x_n, 1/n) \not\subset \mathcal{O}_\alpha$. By our assumption on A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point $x \in A$. Since the family $\{\mathcal{O}_\alpha\}_{\alpha \in \Gamma}$ covers A, there exists $\alpha_0 \in \Gamma$ such that $x \in \mathcal{O}_{\alpha_0}$. Since $\mathcal{O}_{\alpha_0}$ is open, there exists $\varepsilon_0 > 0$ such that $B_o(x, \varepsilon_0) \subset \mathcal{O}_{\alpha_0}$. For any $n_k \geq 1$ and $a \in B_o(x_{n_k}, 1/n_k)$, we have

$$D(x, a) \leq K\Big(D(x, x_{n_k}) + D(x_{n_k}, a)\Big) < K\Big(D(x, x_{n_k}) + \frac{1}{n_k}\Big).$$

For n_k large enough, we will get $D(x, a) < \varepsilon_0$, for any $a \in B_o(x_{n_k}, 1/n_k)$. In other words, we have $B_o(x_{n_k}, 1/n_k) \subset B_o(x, \varepsilon_0)$, which implies $B_o(x_{n_k}, 1/n_k) \subset \mathcal{O}_{\alpha_0}$. This is in clear contradiction with the way the sequence $\{x_n\}$ was constructed. Therefore there exists $\varepsilon_0 > 0$ such that for any $x \in A$, there exists $\alpha \in \Gamma$ such that $B_o(x, \varepsilon_0) \subset \mathcal{O}_\alpha$. For such ε_0, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subset B_o(x_1, \varepsilon_0) \cup \cdots \cup B_o(x_n, \varepsilon_0).$$

But for any $i = 1, \ldots, n$, there exists $\alpha_i \in \Gamma$ such that $B_o(x_i, \varepsilon_0) \subset \mathcal{O}_{\alpha_i}$, i.e.

$$A \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_n}.$$

This completes the proof that A is compact. □

The above proof suggests the following definition.

Definition 9. The subset A is called sequentially compact if and only if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k\to\infty} x_{n_k} \in A$. Also A is called totally bounded if for any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subset B_o(x_1, \varepsilon) \cup \cdots \cup B_o(x_n, \varepsilon).$$

In the above proof we showed the following result.

Theorem 3.1. *Let (X, D) be a metric type space and τ the topology defined above. Let A be a nonempty subset of X.*

(1) *A is compact if and only if A is sequentially compact.*
(2) *If A is compact, then A is totally bounded.*

It is amazing that the above results hold in metric type spaces when we do not know whether open balls are open and closed balls are closed for τ.

4. KKM maps in metric type spaces

It is well known that the Brouwer fixed point theorem, the KKM theorem (Knaster–Kuratowski–Mazurkiewicz theorem), and other results in nonlinear analysis are equivalent. There are many extensions and many applications of KKM theorem. The most important result for KKM mappings is the famous Fans theorem [12], which has been used as a very versatile tool in modern nonlinear analysis. The first attempt to extend such theorems to metric spaces was done in [13], where the underlying space was the case of hyperconvex metric spaces. Following this extension others (see [14,15] for example) tried to investigate the case of metric spaces. In [16] the authors were able to prove a KKM theorem and its applications in metric spaces, and the new class called $\mathcal{N}R$ which extends nicely the case of hyperconvex metric spaces. Recently an extension to cone metric spaces was done in [7,17]. This extension is stringent since it assumes that the underlying Banach spaces and its associated cones are essential. Here we will show that most of the stated results extends in fact nicely to metric type spaces.

Let X and Y be two topological spaces and $F : X \to 2^Y$ be a multifunction with nonempty values, where 2^Y denotes the set of all subsets of Y. A multifunction $F : X \to 2^Y$ is said to be:

(i) closed if its graph $\mathrm{Gr}(F) = \{(x, y) \in X \times Y, y \in F(x)\}$ is closed;
(ii) compact if the closure $\overline{F(X)}$ is a compact subset of Y.

For a set X, we denote the set of all nonempty finite subsets of X by $\langle X\rangle$. Let A be a nonempty bounded subset of a metric type space (M, D). Then

(i) $co(A) = \bigcap\{B \subset M, B$ is a closed ball in M such that $A \subset B\}$
(ii) $\mathcal{A}(M) = \{A \subset M,\ A = co(A)\}$, i.e. $A \in \mathcal{A}(M)$ if and only if A is an intersection of all closed balls containing A. In this case,we say that A is an admissible set in M.
(iii) A is called subadmissible, if for each $D \subset \langle A\rangle$, $co(D) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

Recall that closed and open balls of M are defined as

$$B(x, r) = \{y \in M,\ D(x, y) \le r\}, \qquad B_o(x, r) = \{y \in M,\ D(x, y) < r\},$$

for any $x \in M$ and $r \ge 0$.

Let M be a metric type space and X a subadmissible subset of M. A multifunction $G : X \to 2^M$ is called a KKM mapping, if for each $A \in \langle X\rangle$, we have $co(A) \subset G(A) = \cup\{G(a), a \in A\}$. More generally, if Y is a topological space and $G : X \to 2^Y$, $F : X \to 2^Y$ are two multifunctions such that for any $A \in \langle X\rangle$, we gave $F(co(A)) \subset G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \to 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \to 2^Y$ with respect to F the family $\{\overline{G(x)}, x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$\mathrm{KKM}(X, Y) = \{F : X \to 2^Y, F \text{ has the KKM property}\}.$$

Let X be a nonempty subset of a metric type space M. Then $F : X \to M$ is said to have the approximate fixed point property if for any $\varepsilon > 0$, there exists an $x_\varepsilon \in X$ such that $F(x_\varepsilon) \cap B_o(x_\varepsilon, \varepsilon) \ne \emptyset$, i.e. there exists $y \in F(x_\varepsilon)$ such that $D(x_\varepsilon, y) < \varepsilon$. We now establish the approximate fixed point property of KKM-type mapping on a subadmissible subset of a metric type space similar to the ones obtained in [16,7].

Theorem 4.1. *Let (M, D) be a metric type space and X a nonempty subadmissible subset of M. Let $F \in \mathrm{KKM}(X, X)$ be such that $\overline{F(X)}$ is totally bounded. Then F has the approximate fixed point property.*

Proof. Set $Y = \overline{F(X)} \subset \overline{X}$. Since Y is totally bounded, then for any $\varepsilon > 0$, there exists a finite subset $A \subset X$ such that $Y \subset \bigcup_{x\in A} B_o(x, \varepsilon)$. Define $G : X \to 2^X$ by

$$G(x) = Y \cap \overline{B_o(x, K\varepsilon)^c},$$

where Z^c is the complement of Z in M. Clearly $G(x)$ is closed. Note that for any $x \in M$, we have

$$B_o(x, \varepsilon) \subset \overline{B_o(x, K\varepsilon)^c}^{\,c} \subset B_o(x, K\varepsilon).$$

Indeed, let $y \in B_o(x, \varepsilon)$. Assume that $y \notin \overline{B_o(x, K\varepsilon)^c}^{\,c}$, i.e. $y \in \overline{B_o(x, K\varepsilon)^c}$. From the properties of the closure in metric type spaces, there exists a sequence $\{y_n\} \in B_o(x, K\varepsilon)^c$ such that $\lim_{n\to\infty} y_n = y$. Hence

$$K\varepsilon \le D(x, y_n) \le K\Big(D(x, y) + D(y, y_n)\Big).$$

If we let $n \to \infty$, we get $K\varepsilon \le KD(x, y)$, or $\varepsilon \le D(x, y)$. This is a contradiction to $y \in B_o(x, \varepsilon)$. Hence

$$B_o(x, \varepsilon) \subset \overline{B_o(x, K\varepsilon)^c}^{\,c}.$$

Next let $y \in \overline{B_o(x, K\varepsilon)^c}^{\,c}$. Let us prove that $y \in B_o(x, K\varepsilon)$. Assume not, i.e. $y \notin B_o(x, K\varepsilon)$. Hence $y \in B_o(x, K\varepsilon)^c$ which implies $y \in \overline{B_o(x, K\varepsilon)^c}$. This is a contradiction with $y \in \overline{B_o(x, K\varepsilon)^c}^{\,c}$. Therefore we have

$$\overline{B_o(x, K\varepsilon)^c}^{\,c} \subset B_o(x, K\varepsilon).$$

On the other hand, since $Y \subset \bigcup_{x\in A} B_o(x, \varepsilon)$, then we have $\bigcap_{x\in A} G(x) = \emptyset$. So G is not a generalized KKM mapping with respect to F. Since $F \in \mathrm{KKM}(X, X)$, there exists a finite nonempty subset $B \subset X$ such that $F(co(B)) \not\subset \bigcup_{x\in B} G(x)$. So there exists $x_o \in F(co(B))$ such that $x_o \notin G(x)$ for any $x \in B$. In other words, we have $x_o \in \overline{B_o(x, K\varepsilon)^c}^{\,c}$, for any $x \in B$. Hence $x_o \in B_o(x, K\varepsilon)$ for any $x \in B$ or $B \subset B_o(x_o, K\varepsilon)$. By the definition of $co(B)$ we deduce that $co(B) \subset B(x_o, K\varepsilon)$. Since $x_o \in F(co(B))$, there exists $x_\varepsilon \in co(B)$ such that $x_o \in F(x_\varepsilon)$. But $x_\varepsilon \in co(B) \subset B(x_o, K\varepsilon)$, gives $D(x_o, x_\varepsilon) \le K\varepsilon$. Therefore we have proved

$$F(x_\varepsilon) \cap B(x_\varepsilon, K\varepsilon) \neq \emptyset.$$

Since ε was arbitrary, the proof of the theorem is complete. □

As a direct consequence of this result, we get the following fixed point result.

Theorem 4.2. *Let (M, D) be a metric type space and X a nonempty subadmissible subset of M. Let $F \in \mathrm{KKM}(X, X)$ be closed and compact. Then F has a fixed point, i.e. there exists $x \in X$ such that $x \in F(x)$.*

Proof. Since F is compact, then $\overline{F(X)}$ is compact. Hence $\overline{F(X)}$ is totally bounded. The previous theorem implies the existence of $x_\varepsilon \in X$ such that

$$F(x_\varepsilon) \cap B(x_\varepsilon, \varepsilon) \neq \emptyset,$$

for any $\varepsilon > 0$. In particular, for any $n \ge 1$, there exists $x_n \in X$ such that

$$F(x_n) \cap B(x_n, 1/n) \neq \emptyset.$$

Hence there exists $y_n \in F(x_n)$ such that $D(x_n, y_n) < 1/n$, for any $n \ge 1$. Since F is compact, there exists a subsequence $\{y_{n_k}\}$ which is convergent to y. Clearly we have $\{x_{n_k}\}$ is also convergent to y. Since $\{(x_n, y_n)\} \in \mathrm{Gr}(F)$ and $\mathrm{Gr}(F)$ is closed, then $(y, y) \in \mathrm{Gr}(F)$, i.e. $y \in F(y)$. □

The following lemma will be useful to prove Schauder's type fixed point theorem for metric type spaces.

Lemma 4.1. *Let (M, D) be a metric type space and X a nonempty subadmissible subset of M. Suppose that Y is a topological space, $F \in \mathrm{KKM}(X, Y)$ and $f : Y \to X$ is continuous, then $f \circ F \in \mathrm{KKM}(X, X)$.*

Proof. Let $G : X \to 2^X$ be generalized KKM mappings with respect to $f \circ F$ such that $G(x)$ is closed for each $x \in X$. Then, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, since G is a generalized KKM mapping with respect to $f \circ F$ we have $f \circ F(co\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{1\le i\le n} G(x_i)$. Hence

$$F(co\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{1\le i\le n} f^{-1}\Big(G(x_i)\Big).$$

Therefore, $f^{-1}(G)$ is a generalized KKM mapping with respect to F. Since $F \in \mathrm{KKM}(X, Y)$, then the family $\{f^{-1}(G(x)), x \in X\}$ has the finite intersection property since f is continuous. This will imply that the family $\{G(x), x \in X\}$ has the finite intersection property. This shows that $f \circ F \in \mathrm{KKM}(X, X)$. □

Theorem 4.3. *Let (M, D) be a metric type space and X a nonempty subadmissible subset of M. Suppose that the identity mapping $I : X \to X$ belongs to $\mathrm{KKM}(X, X)$, then any continuous mapping $f : X \to X$ such that $\overline{f(X)}$ is compact, has a fixed point.*

More results similar to the ones found in [7] can be proved in this context.

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