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3 authors:

Nawab Hussain
King Abdulaziz University

Mohamed A Khamsi
University of Texas at El Paso

Abdul Latif
King Abdulaziz University

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Common fixed points for $\mathcal{JH}$-operators and occasionally weakly biased pairs under relaxed conditions

N. Hussain [a,*], M.A. Khamsi [b,c], A. Latif [a]

[a] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[b] Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA

[c] Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, P.O. Box 411, Dhahran 31261, Saudi Arabia

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ABSTRACT

Some common fixed point theorems due to Bhatt et al. [A. Bhatt, et al., Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions, Nonlinear Anal. 73 (2010) 176–182], Jungck and Rhoades [G. Jungck and B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory 7 (2) (2006) 287–296. Fixed Point Theory 9 (2008) 383–384 (erratum)] and Imdad and Soliman [M. Imdad, A.H. Soliman, Some common fixed point theorems for a pair of tangential mappings in symmetric spaces, Appl. Math. Lett. 23 (2010) 351–355] are extended to two new classes of non-commuting selfmaps which contain the occasionally weakly compatible and weakly biased selfmaps as proper subclasses. Some illustrative examples are also provided to highlight the realized improvements.

1. Introduction and preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been the focus of vigorous research activity. In 1976, Jungck [1], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. Sessa [2] introduced the notion of weakly commuting maps. Jungck [3] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Pant [4] defined pointwise R-weakly commuting maps and proved common fixed point theorems, assuming the continuity of at least one of the mappings. Jungck [5] defined a pair of selfmappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. Jungck and Pathak [6] defined the concept of the weakly biased maps in order to generalize the concept of weak compatibility.

The set of fixed points of T (resp. f) is denoted by $F(T)$ (resp. $F(f)$). A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx (x = fx = Tx)$. Maps $f, T : X \to X$ are called (1) commuting if $Tfx = fTx$ for all $x \in X$, (2) R-weakly commuting [4] if for all $x \in X$, there exists $R > 0$ such that $\|fTx - Tfx\| \le R\|fx - Tx\|$. If $R = 1$, then the maps are called weakly commuting; (3) compatible [3] if $\lim_n \|Tfx_n - fTx_n\| = 0$ when $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$

* Corresponding author. Tel.: +966 2 6331085.
E-mail addresses: nhusain@kau.edu.sa, hussainjam@hotmail.com (N. Hussain), mohamed@math.utep.edu, mkhamsi@kfupm.edu.sa (M.A. Khamsi), alatif@kau.edu.sa (A. Latif).

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for some t in X; (4) weakly compatible [5] if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$; (5) occasionally weakly compatible (owc) [7,8] if $fTx = Tfx$ for some $x \in X$ with $fx = Tx$.

The pair (T, f) is called a Banach operator pair, if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but the converse is not true, in general; see [9–11]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair. For more on metric fixed point theory, the reader may consult the book [12].

In this paper, we introduce two new and different classes of non-commuting selfmaps. These classes contain the occasionally weakly compatible and weakly biased selfmaps as proper subclasses. For these new classes, we establish common fixed point results on the space (X, d) which is more general than symmetric space. Our results unify, extend, and complement recent fixed point theorems due to Aliouchie [13], Bhatt et al. [14], Imdad and Soliman [15] and Jungck and Rhoades [7] and many others. The application of our results in a dynamical system is also given in this paper.

2. $\mathcal{JH}$-operators and generalized contractions

Definition 2.1 (*[16,7]*)**.** Let X be a set and f, g selfmaps of X. A point x is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g. Let $C(f, g)$ and $PC(f, g)$ denote the sets of coincidence points and points of coincidence, respectively, of the pair (f, g).

Definition 2.2 (*[8]*)**.** Two selfmaps f and g of a metric space (X, d) are called *$\mathcal{P}$-operators* iff there is a point u in X such that $u \in C(f, g)$ and

$$d(u, fu) \le \delta(C(f, g)).$$

Clearly, occasionally weakly compatible and nontrivial weakly compatible maps f and g which do have a coincidence point are $\mathcal{P}$-operators.

Definition 2.3. Two selfmaps f and g of a metric space (X, d) are called *$\mathcal{JH}$-operators* iff there is a point $w = fx = gx$ in $PC(f, g)$ such that

$$d(w, x) \le \delta(PC(f, g)).$$

Example 2.4. Let $X = \mathbb{R}$ with usual norm and $M = [0, \infty)$. Define $f, g : M \to M$ by $gx = 2x$ and $fx = x^2$, for all $x \ne 0$ and $g0 = f0 = 1$. Then $C(f, g) = \{0, 2\}$ and $PC(f, g) = \{1, 4\}$. Obviously f and g are $\mathcal{P}$- and $\mathcal{JH}$-operators but not occasionally weakly compatible and not weakly compatible. Further note that $F(f) = \{1\}$ and $g(1) = 2 \notin F(f)$ which imply that (g, f) is not a Banach operator pair.

Definition 2.5. Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.1}$$

For a space (X, d) satisfying (2.1) and $A \subseteq X$, the diameter of A is defined by

$$\delta(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Here we extend the concept of $\mathcal{JH}$-operators to the space (X, d) satisfying condition (2.1).

Definition 2.6. Two selfmaps f and g of a space (X, d) satisfying (2.1) are called *$\mathcal{JH}$-operators* iff there is a point $w = fx = gx$ in $PC(f, g)$ such that

$$d(w, x) \le \delta(PC(f, g)) \quad \text{and} \quad d(x, w) \le \delta(PC(f, g)).$$

Example 2.7. Consider $X = [0, 1]$ and define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} e^{x-y} - 1 & \text{if } x \ge y \\ e^{y-x} & \text{if } x < y. \end{cases}$$

Define $f, g : X \to X$, by $fx = x^2$ and $gx = \frac{x}{2}$, for all $x \ne 0$ and $g0 = f0 = 1$. Then $C(f, g) = \left\{0, \frac{1}{2}\right\}$ and $PC(f, g) = \left\{\frac{1}{4}, 1\right\}$. Obviously f and g are $\mathcal{JH}$-operators but not occasionally weakly compatible. Further note that $F(f) = \{1\}$ and $g(1) = \frac{1}{2} \notin F(f)$ which imply that (g, f) is not a Banach operator pair.

In this section, we prove some fixed point theorems for *$\mathcal{JH}$-operators* on space (X, d), without putting the restriction of triangle inequality or symmetry on d. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the condition $\phi(t) < t$ for each $t > 0$. We now prove the following theorem.

Theorem 2.8. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying condition (2.1). Suppose f and g are $\mathcal{JH}$-operators on X satisfying the following condition:*

$$d(fx, fy) \le ad(gx, gy) + b\max\{d(fx, gx), d(fy, gy)\} + c\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}, \tag{2.2}$$

for each $x, y \in X$ where a, b, c are real numbers such that $0 < a + c < 1$. Then f and g have a unique common fixed point.

Proof. By hypothesis there exists a point x in X such that $w = fx = gx$. Suppose there exists another point $y \in X$ for which $z = fy = gy$. Then from (2.2), we have

$$d(fx, fy) \le (a + c)d(fx, fy).$$

Since $a + c < 1$, the above inequality implies that $d(fx, fy) = 0$, which, in turn implies that $w = fx = fy = z$. Therefore, there exists a unique element w in X such that $w = fx = gx$. Thus $\delta(PC(f, g)) = 0$ implies that $d(x, w) = 0$ and hence x is a unique common fixed point of f and g.
Assume that $F : [0, \infty) \to R$ satisfies:

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, \infty)$ and
(ii) F is nondecreasing on $[0, \infty)$.

Define, $\mathcal{F}[0, \infty) = \{F : F$ satisfies (i)–(ii) above$\}$.
Let $\psi : [0, \infty) \to R$ satisfy:

(iii) $\psi(t) < t$ for each $t \in (0, \infty)$ and
(iv) ψ is nondecreasing on $[0, \infty)$.

Define, $\Psi[0, \infty) = \{\psi : \psi$ satisfies (iii)–(iv) above$\}$.
For some examples of mappings F which satisfy (i)–(ii), we refer to [17,18]. □

Theorem 2.9. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying condition (2.1). Suppose f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each $\mathcal{JH}$-operators. If*

$$d(w, z) = d(z, w), \tag{2.3}$$

whenever w and z are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, and

$$F(d(fx, gy)) \le \psi(F(M(fx, gy))), \tag{2.4}$$

for each $x, y \in X$ for which $fx \ne gy$, where

$$M(fx, gy) := \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\}, \tag{2.5}$$

then f, g, S and T have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx = Sx = w$ and $gy = Ty = z$. Therefore by (2.3)–(2.5) we have

$$\begin{aligned} M(fx, gy) &= \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\} \\ &= d(fx, gy). \end{aligned}$$

Now we claim that $gy = fx$. For, otherwise, by (2.4),

$$\begin{aligned} F(d(fx, gy)) &\le \psi(F(M(fx, gy))) \\ &\le \psi(F(d(fx, gy))) < F(d(fx, gy)), \end{aligned}$$

a contradiction and hence $gy = fx$. Moreover, if there is another point z such that $fz = Sz$, then, using (2.4) and (2.5) it follows that $fz = Sz = gy = Ty$, or $fx = fz$. Hence $w = fx = Sx$ is the unique point of coincidence of f and S. Thus $\delta(PC(f, S)) = 0$ implies that $d(x, fx) = 0$ and hence $x = w$ is a unique common fixed point of f and S. Similarly, $y = z$ is a unique common fixed point of g and T. Suppose that $w \ne z$. Using (2.4) and (2.5) as above we get,

$$\begin{aligned} M(fw, gz) &= \max\{d(Sw, Tz), d(Sw, fw), d(Tz, gz), d(Sw, gz), d(Tz, fw)\} \\ &= d(fw, gz) \end{aligned}$$

$$F(d(w, z)) = F(d(fw, gz)) < F(d(w, z))$$

which is a contradiction. Thus w is the unique common fixed point of f, g, S and T. □

Theorem 2.10. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying condition (2.1). Suppose f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each $\mathcal{JH}$-operators. Suppose*

$$d(w, z) = d(z, w), \tag{2.6}$$

whenever w and z are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, and

$$(d(fx, gy))^p < a(d(fx, Ty))^p + (1-a)\max\Big\{(d(fx, Sx))^p, (d(gy, Ty))^p, (d(fx, Sx))^{\frac{p}{2}}(d(fx, Ty))^{\frac{p}{2}}, (d(Ty, fx))^{\frac{p}{2}}(d(Sx, gy))^{\frac{p}{2}}\Big\}, \tag{2.7}$$

for each $x, y \in X$ for which $fx \neq gy$, where $0 < a$, and $p \geq 1$. Then f, g, S and T have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $fx = Sx$ and $gy = Ty$. Therefore by (2.7) we have,

$$\begin{aligned} d(fx, gy)^p &< a(d(fx, gy))^p + (1-a)\max\left\{0, 0, 0, (d(gy, fx))^{\frac{p}{2}}(d(fx, gy))^{\frac{p}{2}}\right\} \\ &= a(d(fx, gy))^p + (1-a)\left((d(gy, fx))^{\frac{p}{2}}(d(fx, gy))^{\frac{p}{2}}\right). \end{aligned}$$

Since $fx = Sx = w$ and $gy = Ty = z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, condition (2.6) implies that

$$d(fx, gy)^p < a(d(fx, gy))^p + (1-a)((d(fx, gy))^p) = d(fx, gy)^p,$$

which is a contradiction and hence $gy = fx$. Suppose that there exists another point u such that $fu = Su$. Then, using (2.7) one obtains $fu = Su = gy = Ty = fx = Sx$. Hence $w = fx = fu$ is the unique point of coincidence of f and S. Thus $\delta(PC(f, S)) = 0$ implies that $d(x, fx) = 0$ and hence $x = w$ is a unique common fixed point of f and S. Similarly, $y = z$ is a unique common fixed point of g and T. Suppose that $w \neq z$. Using (2.6) and (2.7) as above we get,

$$d(w, z)^p = d(fx, gy)^p < d(fx, gy)^p = d(w, z)^p,$$

which is a contradiction. Thus w is the unique common fixed point of f, g, S and T. □

Let the control function $\Phi : R^+ \to R^+$ be a continuous nondecreasing function such that $\Phi(2t) \leq 2\Phi(t)$ and, $\Phi(0) = 0$ iff $t = 0$.
Let $\Psi : R^+ \to R^+$ be another function such that $\Psi(t) < t$ for each $t > 0$.

Theorem 2.11. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying condition (2.1). Suppose f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each $\mathcal{JH}$-operators. Suppose*

$$d(w, z) = d(z, w), \tag{2.8}$$

whenever w and z are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, and

$$\Phi(d(fx, gy)) \leq \Psi(M_\Phi(x, y)), \tag{2.9}$$

where

$$M_\Phi(x, y) = \max\left\{\Phi(d(Sx, Ty)), \Phi(d(Sx, fx)), \Phi(d(gy, Ty)), \frac{1}{2}[\Phi(d(fx, Ty)) + \Phi(d(Sx, gy))]\right\}, \tag{2.10}$$

for each $x, y \in X$. Then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $w = fx = Sx$ and $z = gy = Ty$. We claim that $fx = gy$. Suppose that $fx \neq gy$. Then, from (2.8) and (2.9), we get,

$$\begin{aligned} 0 &< \Phi(d(x, y)) \leq \Psi(M_\Phi(x, y)) \\ &= \Psi(\Phi(d(fx, gy))) < \Phi(d(fx, gy)), \end{aligned}$$

which is a contradiction. Therefore $\Phi(d(fx, gy)) = 0$, which further implies that, $d(fx, gy) = 0$. Hence the claim follows i.e. $w = fx = gy = z$. Now from the repeated use of condition (2.9) we can show that f, g, S, and T have a unique common fixed point.
Define $G = \{\phi : \mathbb{R}^5 \to \mathbb{R}^5\}$ such that

(g_1) ϕ is nondecreasing in the 4th and 5th variables,
(g_2) If $u, v \in \mathbb{R}^+$ are such that

$$u \leq \phi(v, v, u, u+v, 0) \quad \text{or} \quad u \leq \phi(v, u, v, u+v, 0) \quad \text{or} \quad u \leq \phi(u, u, v, u+v, 0) \quad \text{or} \quad u \leq \phi(v, u, v, u, u+v),$$

then $u \leq hv$ where $0 < h < 1$ is a constant,
(g_3) If $u \in \mathbb{R}^+$ is such that

$$u \leq \phi(u, 0, 0, u, u) \quad \text{or} \quad u \leq \phi(0, u, 0, u, u) \quad \text{or} \quad u \leq \phi(0, 0, u, u, u),$$

then $u = 0$. □

Theorem 2.12. *Let X be a non-empty set and $d : X \times X \mapsto [0, \infty)$ be a function satisfying condition (2.1). Suppose f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each $\mathcal{JH}$-operators. Suppose*

$$d(w, z) = d(z, w), \tag{2.11}$$

whenever w and z are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, and

$$d(fx, gy) \le \phi(d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)), \tag{2.12}$$

for all $x, y \in X$, where $\phi \in G$, then f, g, S and T have a unique common fixed point.

Proof. By hypothesis there exist points x, y in X such that $w = fx = Sx$ and $z = gy = Ty$. We claim that $fx = gy$. Suppose that $fx \ne gy$. Then condition (2.12) implies that

$$d(fx, gy) \le \phi(d(fx, gy), 0, 0, d(fx, gy), d(gy, fx)).$$

Since $fx = Sx = w$ and $gy = Ty = z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively, condition (2.11) implies that

$$d(fx, gy) \le \phi(d(fx, gy), 0, 0, d(fx, gy), d(fx, gy)).$$

Therefore, from (g_3), we get that, $d(fx, gy) = 0$. Hence the claim follows. Suppose that there exists another point u such that $fu = Su$. Then, using (2.11) one obtains $fu = Su = gy = Ty = fx = Sx$. Hence $w = fx = fu$ is the unique point of coincidence of f and S. Thus $\delta(PC(f, S)) = 0$ implies that $d(x, fx) = 0$ and hence $x = w$ is a unique common fixed point of f and S. Similarly, $y = z$ is a unique common fixed point of g and T. Suppose that $w \ne z$. Using (2.11) and (2.12) as above we get,

$$\begin{aligned} d(w, z) &= d(fx, gy) \\ &\le \phi(d(fx, gy), 0, 0, d(fx, gy), d(fx, gy)) \\ &= \phi(d(w, z), 0, 0, d(w, z), d(w, z)), \end{aligned}$$

which, from (g_3), implies that, $d(w, z) = 0$ and hence $w = z$. Thus w is the unique common fixed point of f, g, S and T. □

3. Occasionally weakly biased pairs and generalized contractions

Definition 3.1 (*[6]*)**.** The ordered pair (f, g) of two selfmaps of a metric space (X, d) is called *weakly g-biased*, if and only if $d(gfx, gx) \le d(fgx, fx)$ whenever $fx = gx$.

Definition 3.2. The ordered pair (f, g) of two selfmaps of a metric space (X, d) is called *occasionally weakly g-biased*, if and only if there exists some $x \in X$ such that $fx = gx$ and $d(gfx, gx) \le d(fgx, fx)$.

Clearly, an occasionally weakly compatible and a nontrivial weakly g-biased pair (f, g) are occasionally weakly g-biased pairs, but the converse does not hold, in general, as the following examples show.

Example 3.3. Let $X = \mathbb{R}$ with usual norm and $M = [0, 1]$. Define $f, g : M \to M$ by

$$fx = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$gx = \begin{cases} 1 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Here $C(f, g) = \{1/4\}$ (see [6]) and

$$|gf(1/4) - g(1/4)| = |0 - 1/2| = 1/2 \le |fg(1/4) - f(1/4)| = |1 - 1/2| = 1/2$$

implies that (f, g) is an occasionally weakly g-biased pair. Further, $fg(1/4) \ne gf(1/4)$. Hence $\{f, g\}$ is not an occasionally weakly compatible pair. Also, $|1/4 - f(1/4)| = |1/4 - 1/2| = 1/4 > 0 = \delta(C(f, g))$ implies that $\{f, g\}$ is not a $\mathcal{P}$-operator pair. Also, it is not a $\mathcal{JH}$-operator pair. Further note that $F(g) = \left\{\frac{1}{3}\right\}$ and $f\left(\frac{1}{3}\right) = \frac{2}{3} \notin F(g)$ which imply that (f, g) is not a Banach operator pair.

Example 3.4. Let $X = \mathbb{R}$ with usual norm and $M = [0, 1]$. Define $f, g : M \to M$ by

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{4}\right], \\ 1 - 2x & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$gx = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Here $C(f, g) = \left\{\frac{1}{4}\right\}$ and

$$\left|gf\left(\frac{1}{4}\right) - g\left(\frac{1}{4}\right)\right| = \left|1 - \frac{1}{2}\right| = \frac{1}{2} \leq \left|fg\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| = \left|0 - \frac{1}{2}\right| = \frac{1}{2}$$

imply that (f, g) is an occasionally weakly g-biased pair. Further note that $fg(1/4) \neq gf(1/4)$. Hence $\{f, g\}$ is not an occasionally weakly compatible pair.

Example 3.5. Let $X = \mathbb{R}$ with usual norm and $M = [0, \infty)$. Define $f, g : M \to M$ by $gx = 2x$ and $fx = 2x^2$, for all $x \neq 0$ and $f0 = g0 = \frac{1}{2}$. Then $C(f, g) = \{0, 1\}$. Obviously (f, g) is neither an occasionally weakly compatible nor a weakly g-biased pair. Further note that

$$|gf(1) - g(1)| = |4 - 2| = 2 \leq |fg(1) - f(1)| = |8 - 2| = 6$$

implies that (f, g) is an occasionally weakly g-biased pair.

Definition 3.6. A symmetric on a set X is a mapping $d : X \times X \to [0, \infty)$ such that

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y, \quad \text{and}$$
$$d(x, y) = d(y, x).$$

A set X, together with a symmetric d is called a *symmetric space*.

Let $\phi : R^+ \to R^+$ be a nondecreasing function satisfying the condition $\phi(t) < t$ for each $t > 0$. We now prove the following theorem which extends Theorem 2.1 in [14] and corresponding results in [19].

Theorem 3.7. *Let f, g be selfmaps of symmetric space X and the pair (f, g) be occasionally weakly g-biased. If for the control function ϕ, we have*

$$d(fx, fy) \leq \phi(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}), \tag{3.1}$$

for each $x, y \in X$, then f, g have a unique common fixed point.

Proof. By hypothesis there exist point u in X such that $fu = gu$ and $d(gfu, gu) \leq d(fgu, fu)$. We claim that fu is the unique common fixed point of f and g. We first assert that fu is a fixed point of f. If $ffu \neq fu$ then by using (3.1) and $d(gfu, gu) \leq d(fgu, fu)$, we get

$$\begin{aligned} d(ffu, fu) &\leq \phi(\max\{d(gfu, gu), d(gfu, fu), d(gu, ffu), d(gu, fu)\}) \\ &= \phi(\max\{d(gfu, gu), d(ffu, fu)\}) \\ &\leq \phi(\max\{d(fgu, fu), d(ffu, fu)\}) \\ &= \phi(\max\{d(ffu, fu), d(ffu, fu)\}) \\ &= \phi(d(ffu, fu)) \\ &< d(ffu, fu) \end{aligned}$$

which is a contradiction. Therefore $ffu = fu = fgu$. Hence $d(gfu, gu) \leq d(fgu, fu) = 0$ which further implies that $gfu = gu = fu = ffu$. Thus, fu is a common fixed point of f and g. For uniqueness, suppose that $u, v \in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then (3.1) gives

$$\begin{aligned} d(u, v) = d(fu, fv) &\leq \phi(\max\{d(gu, gv), d(gu, fv), d(gv, fu), d(gv, fv)\}) \\ &= \phi(d(u, v)) \\ &< d(u, v). \end{aligned}$$

This is a contradiction. Therefore, $u = v$. Thus, the common fixed point of f and g is unique. □

Corollary 3.8. *Let f, g be selfmaps of symmetric space X and the pair (f, g) be occasionally weakly g-biased. If for the control function ϕ, we have*

$$d(fx, fy) \leq \phi(d(gx, gy)),$$

for each $x, y \in X$, then f, g have a unique common fixed point.

The proof of the following theorem can be easily obtained by replacing condition (3.1) by condition (3.2) in the proof of Theorem 3.7.

Theorem 3.9. *Let f, g be selfmaps of symmetric space X and the pair (f, g) be occasionally weakly g-biased. Suppose that*

$$d(fx, fy) < \max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}, \tag{3.2}$$

for each $x, y \in X$, $x \neq y$, then f, g have a unique common fixed point.

Example 3.10. Consider $X = [0, \infty)$ equipped with the symmetric defined by $d(x, y) = e^{|x-y|} - 1$ for all $x, y \in X$. Define $f, g : X \to X$, by $fx = 2x + 1$ and $gx = x + 2$, for all $x \in X$. Then $C(f, g) = \{1\}$. Further note that

$$d(gf(1), g(1)) = d(5, 3) = e^2 - 1 \leq d(fg(1), f(1)) = d(7, 3) = e^4 - 1$$

implies that (f, g) is an occasionally weakly g-biased pair which is neither a $\mathcal{JH}$- nor a $\mathcal{P}$-operator pair. Notice that the pair has no common fixed point as condition (3.2) is not satisfied for $x = 1$ and $y = 2$.

4. Banach operator pairs and generalized contractions

In this section, we prove some common fixed point theorems for a *Banach operator pair* on space (X, d), without assuming the restriction of triangle inequality or symmetry on d. It is worth to mention here that the class of Banach operator pairs is different from the classes of occasionally weakly compatible, $\mathcal{P}$- and $\mathcal{JH}$-operators and occasionally weakly g-biased pairs (see Examples 2.2, 2.5 and 3.3 and Refs. [10,11]).

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the condition $\phi(t) < t$ for each $t > 0$. We now prove the following theorem.

Theorem 4.1. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying condition (2.1). Suppose (f, g) is a nontrivial Banach operator pair on X and satisfying the following condition:*

$$d(fx, fy) \leq ad(gx, gy) + b\max\{d(fx, gx), d(fy, gy)\} + c\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}, \tag{4.1}$$

for each $x, y \in X$ where a, b, c are real numbers such that $0 < a + c < 1$. Then f and g have a unique common fixed point.

Proof. By hypothesis $F(g) \neq \phi$ and $f(F(g)) \subset F(g)$. From (4.1), we have for any $x, y \in F(g)$

$$d(fx, fy) \leq ad(x, y) + b\max\{d(fx, x), d(fy, y)\} + c\max\{d(x, y), d(x, fx), d(y, fy)\}.$$

By Theorem 2.8 (with g an identity map on X), f has a unique fixed point on $F(g)$ and hence f and g have a unique common fixed point. □

Theorem 4.2. *Let X, d, ψ and F be as in Theorem 2.9. Suppose f, S are selfmaps of X and that $\{f, S\}$ is a nontrivial Banach operator pair. If*

$$F(d(fx, fy)) \leq \psi(F(M(fx, fy))),$$

for each $x, y \in X$ for which $fx \neq fy$, where

$$M(fx, fy) := \max\{d(Sx, Sy), d(Sx, fx), d(Sy, fy), d(Sx, fy), d(Sy, fx)\},$$

then f and S have a unique common fixed point.

Proof. Proof follows as in Theorem 4.1 instead of applying Theorem 2.8, we apply Theorem 2.9 to get the conclusion. □

Remark 4.3. More results similar to the ones found in [14,15] can be proved in this context.

Remark 4.4. As an application of Corollary 3.8, the existence and uniqueness of a common solution of the functional equations arising in dynamic programming can be established which extends Theorem 4.1 [14] and Theorem 5.2 [11] to more general class of occasionally weakly g-biased pairs.

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