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The AR-property in linear metric spaces

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Abstract

Roberts constructed a linear metric space which contains a compact convex set without any extreme points. The space constructed by Roberts is complicated and special.

We investigate the topological property of Roberts' example and demonstrate that the linear metric space constructed by Roberts is an AR, therefore is homeomorphic to Hilbert space. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the most difficult problems in infinite-dimensional topology is the problem of identifying the AR-property among linear metric spaces. This problem is of special importance because infinite-dimensional separable complete linear metric spaces with the AR-property are homeomorphic to Hilbert space, see [3].

Observe that Cauty [2] constructed a σ-compact linear metric space which is not an AR. By a theorem of Torunczyk [13], the completion of any non-AR-linear metric space is still a non-AR-space. Therefore the completion of Cauty's example provides a separable complete linear metric space which is not an AR.

It should also be observed that while Cauty showed the existence of non-AR-linear metric spaces, it is difficult to use his argument to obtain an intuitive picture of such a space. In fact, Cauty's example is based on some rather deep facts from infinite-dimensional

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topology and a more self-contained example of a non-AR-linear metric space would be much appreciated. Naturally, it is hoped that such an example should be found among pathological objects in linear metric spaces.

It is also hoped that an investigation for the AR-property of pathological objects in linear metric spaces will shed light on the following question which is one of the most outstanding open problems in infinite-dimensional topology:

Question 1. Is every compact convex set in a linear metric space an AR? Does every compact convex set have the fixed point property?

The second part of the above question, known as "Schauder's Conjecture", was posed by Schauder in early 1930's, but is still open today.

In this context, we investigate the AR-property of the famous example due to Roberts [12] of a linear metric space containing a compact convex set with no extreme points. Roberts' example contrasts with the classical theorem of Krein and Milman [5] stating that any compact convex set in a locally convex space is the closure of the convex hull of its extreme points. Therefore, the Krein–Milman theorem does not hold for non-locally convex spaces.

In [10], see also [11], it was proved that the compact convex set with no extreme points constructed by Roberts is an AR, therefore is homeomorphic to Hilbert cube. The following question was posed in [10, Question 1]:

Question 2. Is every convex set in the linear metric space constructed by Roberts an AR?

In this paper, we provide a partial answer to the above question by demonstrating that the whole space constructed by Roberts is an AR, therefore is homeomorphic to Hilbert space.

Our result provides a new example of a pathological space homeomorphic to Hilbert space. Some other pathological linear metric spaces possessing the topological structure of Hilbert space were also obtained in [9,4].

2. Roberts' construction and our result

We are going to describe Roberts' construction [12] of a linear metric space containing a compact convex set with no extreme points.

We recall the following definitions in [12]: A *paranorm* N on a vector space X is a function $N : X \to [0, \infty)$ with the following properties:

(1) $N(\theta) = 0$,
(2) $N(x) = N(-x)$ for every $x \in X$,
(3) $N(x + y) \leqslant N(x) + N(y)$ for every $x, y \in X$, and
(4) $\lim_{\alpha \to 0} N(\alpha x) = 0$ for every $x \in X$.

A paranorm N on X is said to be:

(1) *total* if $x \neq \theta$ implies $N(x) > 0$;

(2) *monotone* if $N(\alpha x) \leqslant N(x)$ for every $x \in X$ and $\alpha \in [0,1]$;
(3) *norm bounded* if there exists a norm $\|.\|$ on X such that

$$N(x) \leqslant \|x\| \quad \text{for every } x \in X.$$

Let X be a finite-dimensional space with a basis $B = \{e_1, \ldots, e_m\}$, let $\varepsilon > 0$, $h > 0$, and let $e = e_1 + \cdots + e_m$. We say that e is an ε*-needle point* of X with height h, briefly an (ε, h)-needle point, with respect to the paranorm N and the basis B if:

(1) N is monotone, total and norm bounded.
(2) If $x \in \operatorname{conv}\{\theta, me_1, \ldots, me_m\}$, then there exists an $\alpha \in [0,1]$ such that $N(x - \alpha e) < \varepsilon$.
(3) $N(me_i) < \varepsilon$ for $i = 1, \ldots, m$.
(4) $N(\alpha e) = \alpha h$ for every $\alpha \in [0,1]$.

The following fact [12, Proposition 2.6] is the key to Roberts' construction:

Proposition 1. *Given $\varepsilon > 0$ and $h > 0$, there exists an m-dimensional space V with basis $B = \{e_1, \ldots, e_m\}$ and a paranorm N on V such that $e = e_1 + \cdots + e_m$ is an (ε, h)-needle point with respect to the paranorm N and the basis B.*

Moreover the paranorm N is bounded by the norm $|\,.\,|$ defined by

$$|x| = \sum_{i=1}^{m} |\alpha_i| \quad \textit{for every } x = \sum_{i=1}^{m} \alpha_i e_i \in V. \tag{2.1}$$

Proof. The fact that the paranorm N is bounded by the norm $|\,.\,|$ defined by (2.1) was not stated explicitly in Proposition 2.6 [12], however this fact can be obtained by analyzing the proofs of Propositions 2.1 through 2.6 given in [12].

In fact, observe that in the proof of Proposition 2.5 [12] one can take the norm N' on V given by

$$N'(x) = \frac{1}{m} \sum_{i=1}^{m} |\alpha_i| \quad \text{for every } x = \sum_{i=1}^{m} \alpha_i e_i \in V.$$

(We adopt the notation used in the proofs of Propositions 2.1 to 2.6 of [12], and therefore we assume that the reader has access to the paper [12] while reading this proof.) Then we have $N'(e) = 1$. Let N_0 denote the paranorm on V defined in the proof of Proposition 2.6 [12]. Then $N_0(e) \geqslant M - 1 \geqslant 1$, and $Q = \inf\{2N_0, N'\}$ is a monotone total norm bounded paranorm with $Q(e) = 1$. Let $N_1 = \inf\{P, 2Q\}$, where P is the paranorm on the one-dimensional space $Re = \{\lambda e\colon\ \lambda \in R\}$ defined by $P(\alpha e) = |\alpha|$ for $\alpha \in R$. Finally observe that in the proofs of Propositions 2.5 and 2.6 [12] the paranorm N on V was given by

$$N(x) = hN_1(x) \quad \text{for } x \in V.$$

Therefore for $m \geqslant 2h$ we have

$$N(x) = hN_1(x) \leqslant 2hQ(x) \leqslant 2hN'(x) = \frac{2h}{m}|x| \leqslant |x|,$$

for every $x \in V$, and the claim is proved. $\square$

Now we going to describe the linear metric space E constructed by Roberts [12]. Let $\mathcal{N}$ denote the set of all positive integers. For a sequence $\{d(n)\} \subset \mathcal{N}$ we put $m(1) = 1$ and inductively define $m(n+1) = d(n)m(n)$. Let $\pi_1 = \{[0,1)\}$. Assume that π_n is a partition of $[0,1)$ into $m(n)$ equal length intervals of the form $S = [a,b)$. For each $S \in \pi_n$, let $\pi_{n+1}(S)$ denote the partition of S into $d(n)$ equal length subintervals. Define $\pi_{n+1} = \bigcup_{S\in\pi_n} \pi_{n+1}(S)$.

Consider the vector space consisting of all functions on $[0,1)$ which are finite linear combinations of characteristic functions of the form $\chi_{[a,b)}$. Let

$$E_n = \operatorname{span}\{\chi_S\colon\ S \in \pi_n\}, \quad \text{and} \quad E_\infty = \bigcup_{n=1}^{\infty} E_n, \tag{2.2}$$

where χ_S denotes the characteristic function of S. For every $S \in \pi_n$, let

$$B_{n+1}(S) = \bigl\{e_{n+1}(T)\colon\ T \in \pi_{n+1}(S)\bigr\}, \quad \text{and} \quad E_{n+1}(S) = \operatorname{span}\bigl\{B_{n+1}(S)\bigr\}, \tag{2.3}$$

where $e_{n+1}(T) = m(n+1)\chi_T$. Then we have

$$E_{n+1} = \bigoplus_{S\in\pi_n} E_{n+1}(S) \quad \text{and} \quad E_1 \subset E_2 \subset \cdots \subset E_\infty = \bigcup_{n=1}^{\infty} E_n. \tag{2.4}$$

Roberts' construction can be summarized in the following theorem, see [12, Section 3]:

Theorem 1. *For suitable sequences $\{d(n)\} \subset \mathcal{N}$ and $\{m(n+1)\} = \{d(n)m(n)\} \subset \mathcal{N}$, with $m(1) = 1$, there exist sequences $\{N_n\}$, $\{N_{n+1}^S\colon\ S \in \pi_n\}$ of paranorms on E_n and on $E_{n+1}(S)$, respectively, with $\dim E_n = m(n)$ and $\dim E_{n+1}(S) = d(n)$ such that the following conditions hold*:

(i) $N_1(x) = \int_0^1 |x(t)|\,\mathrm{d}t$ *for every* $x \in E_1$.

(ii) *For any $n \in \mathcal{N}$ and $S \in \pi_n$, the paranorm N_{n+1}^S on $E_{n+1}(S)$ is constructed as in Proposition* 1 *with $\varepsilon_1 = 4$, $\varepsilon_{n+1} < [m(n)]^{-1}2^{-n-1}$ for $n \geqslant 2$, and $h_n \in [4,5]$ for every $n \in \mathcal{N}$. Therefore*

(ii-a) *$N_{n+1}^S(x) \leqslant |x|$ for every $x \in E_{n+1}(S)$, where $|\,.\,|$ is the norm on $E_{n+1}(S)$ given by* (2.1);

(ii-b) *$m(n+1)\chi_S = \sum_{T\in\pi_{n+1}(S)} m(n+1)\chi_T$ is an $(\varepsilon_{n+1}, h_{n+1})$-needle point of $E_{n+1}(S)$ with respect to the paranorm N_{n+1}^S, and the basis $B_{n+1}(S)$, see* (2.3).

(iii) *For every $n > 1$ the paranorm N_{n+1} on E_{n+1} is given by*

$$N_{n+1}(x) = \inf\left\{N_n(y) + \sum_{S\in\pi_n} N_{n+1}^S(x(S))\right\} \quad \text{for } x \in E_{n+1}, \tag{2.5}$$

where the infimum is taken over all the expressions of x of the form

$$x = y + \sum_{S\in\pi_n} x(S), \quad \text{where } y \in E_n,\ \text{and } x(S) \in E_{n+1}(S).$$

(iv) *The formula*

$$N(x) = \lim_{n\to\infty} N_n(x) \quad \text{for } x \in E_\infty \tag{2.6}$$

defines a monotone F-norm on E_∞.

(v) *If* $x \in E_n$ *and* $N_n(x) < 4$, *then* $N_m(x) = N_n(x)$ *for every* $m > n$.

(vi) *Let* $(E, \|\,.\,\|)$ *denote the completion of* (E_∞, N) *with respect to the* F*-norm* N *defined by* (2.6). *Then*

$$C = \overline{\bigcup_{n=1}^{\infty}[A_n]} \subset E, \quad \text{where } [A_n] = \operatorname{conv}\bigl\{\pm m(n)\chi_S\colon\ S \in \pi_n\bigr\}, \tag{2.7}$$

is a compact convex set with no extreme points.

In [10] it was shown that the compact convex set C defined by (2.7) is an AR, therefore is homeomorphic to the Hilbert cube. Our result in this paper is the following.

Main Theorem. *E is an* AR, *therefore is homeomorphic to Hilbert space.*

The proof of our Main Theorem will be based on Theorem 2 below, which is an immediate consequence of Michael's selection theorem, see for instance [1], stating that if X is a complete linear metric space and Y is a locally convex closed subspace of X, then there exists a continuous selection $g : X/Y \to X$, i.e., $g(x) \in \pi^{-1}(x)$ for every $x \in X/Y$, where $\pi : X \to X/Y$ denotes the quotient map.

Theorem 2. *Let Y be a closed locally convex linear subspace of a complete linear metric space X. If X is an* AR, *then the quotient space X/Y is also an* AR.

Our proof also uses the following characterization of ANR-spaces established in [6]: Let $\{\mathcal{U}_n\}$ be a sequence of open covers of a metric space X. For a given cover $\mathcal{U}_n$, let

$$\operatorname{mesh}(\mathcal{U}_n) = \sup\{\operatorname{diam} U\colon\ U \in \mathcal{U}_n\}.$$

We say that $\{\mathcal{U}_n\}$ is a zero-approaching sequence if $\operatorname{mesh}(\mathcal{U}_n) \to 0$ as $n \to \infty$.

For a given cover $\mathcal{V}$ let $\mathcal{N}(\mathcal{V})$ denote the *nerve* of $\mathcal{V}$, that is, the simplicial complex whose simplices are the finite nonempty subsets in $\mathcal{V}$ with nonempty intersection. Note that the elements of $\mathcal{V}$ are the vertices of $\mathcal{N}(\mathcal{V})$. Let

$$\mathcal{U} = \bigcup_{n=1}^{\infty}\mathcal{U}_n \quad \text{and} \quad \mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty}\mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})$$

and for $\sigma \in \mathcal{K}(\mathcal{U})$, write

$$n(\sigma) = \max\bigl\{n \in \mathcal{N}\colon\ \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\bigr\}. \tag{2.8}$$

The following characterization of ANR-spaces was established in [6], see also [8].

Theorem 3. *A metric space with no isolated points is an* ANR *if and only if exists a zero-approaching sequence $\{\mathcal{U}_n\}$ of open covers of X and a map $g : \mathcal{K}(\mathcal{U}) \to X$ such that $g|\mathcal{U} \to X$ is a selection*; *i.e., $g(U) \in U$ for every $U \in \mathcal{U}$, and for any sequence of simplices $\{\sigma_k\}$ in $\mathcal{K}(\mathcal{U})$ with $n(\sigma_k) \to \infty$, we have* $\operatorname{diam} g(\sigma_k) \to 0$.

The Main Theorem will be proved by demonstrating that the space E constructed in Theorem 1 is the quotient of a linear metric AR-space X over a closed locally convex subspace $Y \subset X$. Then from Theorem 2 the result follows.

The remainder of this paper will be divided into three parts: In Section 3 we describe the construction of X whose AR-property will be verified by Theorem 3. A closed subspace $Y \subset X$ for which $X/Y \cong E$ will be constructed in Section 4. Finally the local convexity of the space Y constructed in Section 4 will be demonstrated in Section 5.

3. The construction of X

Observe that the space E constructed in Theorem 1 is the completion of an algebraic sum of the family of finite-dimensional spaces $\{E_n\colon n \in \mathcal{N}\}$ defined by (2.2). Each space E_n consists of linear combinations of characteristic functions $\{\chi_S\colon S \in \pi_n\}$. Our space X will be the completion of the direct sum of $\{X_n\colon n \in \mathcal{N}\}$, where each space X_n will be obtained by shifting the space E_n from the interval $I = [0, 1)$ into the interval $[n-1, n)$.

Precisely speaking, the space X will be defined as follows: Let $X_1 = E_1$ and for $n \geqslant 1$ define

$$X_{n+1}(S) = \operatorname{span}\big\{\chi_{[n+T]}\colon T \in \pi_{n+1}(S)\big\}, \tag{3.1}$$

$$X_{n+1} = \bigoplus_{S\in\pi_n} X_{n+1}(S) \quad \text{and} \quad X_\infty = \bigoplus_{n=1}^{\infty} X_n, \tag{3.2}$$

where $[n+T]$ represents the translation of the interval T by n; that is, $[n+T] = \{n+x\colon x \in T\}$.

For every $S \in \pi_n$ let P^S_{n+1} denote the translation of N^S_{n+1} from $E_{n+1}(S)$ to $X_{n+1}(S)$, that is

$$P^S_{n+1}\bigg(\sum_{T\in\pi_{n+1}(S)} \alpha(T)\chi_{[n+T]}\bigg) = N^S_{n+1}\bigg(\sum_{T\in\pi_{n+1}(S)} \alpha(T)\chi_T\bigg). \tag{3.3}$$

Let $P_1 = N_1$ and since $X_{n+1} = \bigoplus_{S\in\pi_n} X_{n+1}(S)$ for $n \geqslant 1$, we can define the paranorm P_{n+1} on X_{n+1} by

$$P_{n+1} = \sum_{S\in\pi_n} P^S_{n+1}. \tag{3.4}$$

Finally let P denote the F-norm on X_∞ induced by the family of paranorms $\{P_n\colon n \in \mathcal{N}\}$, that is

$$P(x) = \sum_{n=1}^{\infty} P_n(x_n) \quad \text{for every } x = \sum_{n=1}^{\infty} x_n \in X_\infty, \tag{3.5}$$

where $x_n \in X_n$ with $x_n \neq \theta$ for only finitely many $n \in \mathcal{N}$.

Let $(X, \|\,.\,\|)$ denote the completion of (X_∞, P). The following theorem is the first step in the proof of our result.

Theorem 4. *X is an* AR.

Proof. We are going to verify the conditions of Theorem 3. Our proof uses an idea of [7, Theorem 3.1]. Since $\dim X_n < \infty$, for each $n \in \mathcal{N}$ there exists a $\delta_n \in (0, 2^{-n})$ such that for any finite set $A \subset X_n$,

$$\operatorname{diam} A < 2\delta_n \text{ imples } \operatorname{diam}(\operatorname{conv} A) < 2^{-n}. \tag{3.6}$$

Let $r_n : X \to X_n$ denote the projection onto the X_n-component. For every $x \in X$ and $n \in \mathcal{N}$ take $k(x, n) > n$ such that $\|r_{k(n,x)}(x) - x\| < 2^{-n}$. Let $V(x)$ be an open neighborhood of $r_{k(n,x)}(x)$ in $X_{k(n,x)}$ with $\operatorname{diam} V(x) < \delta_{k(n,x)}$. Then from (3.6) we get

$$\operatorname{diam}\bigl(\operatorname{conv} V(x)\bigr) < 2^{-k(n,x)} < 2^{-n}.$$

Denote

$$U(x) = \bigl\{y \in r^{-1}_{k(n,x)}(V(x)):\ \|r_{k(n,x)}(y) - y\| < 2^{-n}\bigr\}, \tag{3.7}$$

$$\mathcal{U}_n = \bigl\{U(x):\ x \in X\bigr\} \quad \text{and} \quad \mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n.$$

We claim that the sequence $\{\mathcal{U}_n\}$ satisfies the conditions of Theorem 3. First, since $\operatorname{diam} V(x) < \delta_{k(n,x)} < 2^{-n}$, from (3.7) we have

$$\operatorname{diam} U < 3(2^{-n}) \quad \text{for every } U \in \mathcal{U}_n,$$

and so $\{\mathcal{U}_n\}$ is a zero-approaching sequence.

Observe that for every $U \in \mathcal{U}$, $U \in \mathcal{U}_n$ for some $n \in \mathcal{N}$, hence $U = U(x)$ for some $x \in X$. We define $g(U) = r_{k(n,x)}(x) \in V(x) \subset X_{k(n,x)}$, and extending g over $\mathcal{K}(\mathcal{U})$ by the convexity we get a map $g : \mathcal{K}(\mathcal{U}) \to X$, with $g|\mathcal{U} : \mathcal{U} \to X$ a selection.

Now, for every $\sigma = \langle U_1, \dots, U_m\rangle \in \mathcal{K}(\mathcal{U})$, $U_i \in \mathcal{U}_{n(\sigma)} \cup \mathcal{U}_{n(\sigma)+1}$ for $i = 1, \dots, m$, where $n(\sigma)$ was defined by (2.8). We are going to compute $\operatorname{diam} g(\sigma)$. By (3.7)

$$U_i = \bigl\{y \in r^{-1}_{k(n,x_i)}(V(x_i)):\ \|r_{k(n,x_i)}(y) - y\| < 2^{-n}\bigr\}, \tag{3.8}$$

where $n = n(\sigma)$ or $n = n(\sigma) + 1$. Denote

$$k_i = k(n, x_i);\quad V_i = V(x_i),\quad \text{and}\quad y_i = r_{k_i}(x_i) \in X_{k_i} \quad \text{for } i = 1, \dots, m, \tag{3.9}$$
$$m_0 = \min\{k_i:\ i = 1, \dots, m\},\quad \text{and}\quad m_j = \min\{k_i:\ k_i > m_{j-1}\} \quad \text{for } j \geqslant 1.$$

Then we get a finite sequence $\{m_0, \dots, m_p\} \subset \mathcal{N}$ with

$$m_0 < \cdots < m_p, \quad \text{and} \quad \{m_i:\ i = 0, \dots, p\} = \{k_i:\ i = 1, \dots, m\}.$$

By the definition of g,

$$g(U_i) = r_{m_i}(x_i) = y_i \in X_{k_i} \quad \text{for every } i = 1, \dots, m.$$

Then $g(\sigma) = \operatorname{conv}\{y_i:\ i = 1, \dots, m\}$, and for every $x \in g(\sigma)$ we have

$$x = \sum_{i=1}^{m} \lambda_i y_i = \sum_{i=0}^{p-1} \sum_{j=m_i+1}^{m_{i+1}} \lambda_j y_j, \quad \text{where } \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

Let

$$\alpha_i = \sum_{j=m_i+1}^{m_{i+1}} \lambda_j \quad \text{for } i = 0, \ldots, p-1,$$

and for $j = m_i + 1, \ldots, m_{i+1}$, denote

$$\mu_{ij} = \begin{cases} (\alpha_i)^{-1}\lambda_j & \text{if } \alpha_i > 0; \\ 0 & \text{if } \alpha_i = 0. \end{cases}$$

Then we have

$$x = \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij} y_j,$$

where

$$\sum_{i=0}^{p-1} \alpha_i = 1 \quad \text{and} \quad \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij} = \begin{cases} 1 & \text{if } \alpha_i > 0; \\ 0 & \text{if } \alpha_i = 0. \end{cases}$$

From (3.8) and (3.9) we get

$$U_i = \left\{ y \in r_{k_i}^{-1}(V_i) \colon \| r_{k_i}(y) - y \| < 2^{-n} \right\} \quad \text{for } i = 1, \ldots, m. \tag{3.10}$$

Take $a \in \bigcap_{i=1}^m U_i$, and denote

$$a_i = r_{m_i}(a) \in \bigcap_{j=m_i+1}^{m_{i+1}} V_j \subset X_{m_i} \quad \text{for } i = 0, \ldots, p-1.$$

Observe that for every $x \in g(\sigma)$ we have

$$x - a = \sum_{i=0}^{p-1} \alpha_i \sum_{j=m_i+1}^{m_{i+1}} \mu_{ij}(y_j - a_i) + \sum_{i=0}^{p-1} \alpha_i (a_i - a). \tag{3.11}$$

Claim 1. $\| \sum_{i=0}^{p-1} \alpha_i (a_i - a) \| < 2^{-n}$.

Proof. Observe that

$$\left\| \sum_{i=0}^{p-1} \alpha_i (a_i - a) \right\| = \left\| \sum_{i=0}^{p-1} \alpha_i [r_{m_i}(a) - a] \right\| = \left\| \sum_{i=0}^{p-1} [r_{m_i}(\alpha_i a) - \alpha_i a] \right\|. \tag{3.12}$$

Write

$$\begin{aligned}
r_{m_0}(\alpha_0 a) - \alpha_0 a &= \left[r_{m_0}(\alpha_0 a) - r_{m_1}(\alpha_0 a) \right] + \cdots \\
&\quad + \left[r_{m_{p-2}}(\alpha_0 a) - r_{m_{p-1}}(\alpha_0 a) \right] + \left[r_{m_{p-1}}(\alpha_0 a) - \alpha_0 a \right]; \\
r_{m_1}(\alpha_1 a) - \alpha_1 a &= \left[r_{m_1}(\alpha_1 a) - r_{m_2}(\alpha_1 a) \right] + \cdots \\
&\quad + \left[r_{m_{p-2}}(\alpha_1 a) - r_{m_{p-1}}(\alpha_1 a) \right] + \left[r_{m_{p-1}}(\alpha_1 a) - \alpha_1 a \right]; \\
&\vdots \\
r_{m_{p-2}}(\alpha_{p-2} a) - \alpha_{p-2} a &= \left[r_{m_{p-2}}(\alpha_{p-2} a) - r_{m_{p-1}}(\alpha_{p-2} a) \right] \\
&\quad + \left[r_{m_{p-1}}(\alpha_{p-2} a) - \alpha_{p-2} a \right].
\end{aligned}$$

Since $\sum_{i=0}^{j}\alpha_i \leqslant 1$ for $j=0,\dots,p-1$, from (3.12) we get

$$\begin{aligned}\left\|\sum_{i=0}^{p-1}\alpha_i(a_i-a)\right\| \leqslant{}& \left\|\alpha_0[r_{m_0}(a)-r_{m_1}(a)]\right\| + \left\|(\alpha_0+\alpha_1)[r_{m_1}(a)-r_{m_2}(a)]\right\| \\ &+\cdots+\left\|(\alpha_0+\cdots+\alpha_{p-2})[r_{m_{p-2}}(a)-r_{m_{p-1}}(a)]\right\| \\ &+\left\|(\alpha_0+\cdots+\alpha_{p-1})[r_{m_{p-1}}(a)-a]\right\| \\ \leqslant{}& \left\|r_{m_0}(a)-r_{m_1}(a)\right\|+\cdots+\left\|r_{m_{p-2}}(a)-r_{m_{p-1}}(a)\right\| \\ &+\left\|r_{m_{p-1}}(a)-a\right\|.\end{aligned}$$

Observe that $r_{m_i}(a)-r_{m_i+1}(a)\in X_{m_i}$ for $i=0,\dots,p-1$, hence from (3.5) we obtain

$$\begin{aligned}\left\|r_{m_0}(a)-a\right\| = \left\|r_{m_0}(a)-r_{m_1}(a)\right\| + \cdots \\ + \left\|r_{m_{p-2}}(a)-r_{m_{p-1}}(a)\right\| + \left\|r_{m_{p-1}}(a)-a\right\|.\end{aligned}$$

Since $\|r_{m_0}(a)-a\|<2^{-n}$, see (3.8), the claim is proved. □

Now, observe that $a_i = r_{m_i}(a)\in\bigcap_{j=m_i+1}^{m_{i+1}}V_j$ for $i=0,\dots,p-1$ and $y_j\in V_j$ for $j=m_i+1,\dots,m_{i+1}$. Therefore

$$\begin{aligned}&\operatorname{diam}\{a_i-y_j,\ j=m_i+1,\dots,m_{i+1}\} \\ &\quad\leqslant 2\max\{\operatorname{diam}V_j,\ j=m_i+1,\dots,m_{i+1}\}<2\delta_{m_i},\end{aligned}$$

for $i=0,\dots,p-1$, which by (3.6) yields

$$\operatorname{diam}\bigl(\operatorname{conv}\{a_i-y_j,\ j=m_i+1,\dots,m_{i+1}\}\bigr)<2^{-m_i}\quad\text{for } i=0,\dots,p-1.$$

Since $m_0<m_1<\cdots<m_{p-1}$, from (3.10) we obtain

$$\begin{aligned}\|x-a\| &\leqslant \sum_{i=0}^{p-1}\left\|\sum_{j=m_i+1}^{m_{i+1}}\mu_{ij}(y_j-a_i)\right\|+2^{-n} \\ &\leqslant 2^{-n}+\sum_{i=0}^{p-1}2^{-m_i}<2^{-n}+2^{-m_0+1},\end{aligned}$$

for every $x\in g(\sigma)$.

Observe that $m_0>n$ and $n=n(\sigma)+1$. Hence

$$\|x-a\|\leqslant 2^{-n(\sigma)+1}+2^{-n(\sigma)+1}=2^{-n(\sigma)+2},$$

for every $x\in g(\sigma)$. Therefore

$$\operatorname{diam} g(\sigma)\leqslant 2^{-n(\sigma)+3}\quad\text{for every }\sigma\in\mathcal{K}(\mathcal{U}).$$

Consequently X is an AR by Theorem 3, and the proof of Theorem 4 is complete. □

4. The construction of Y and proof of the Main Theorem

We are going to construct a closed locally convex subspace Y of X for which $X/Y\cong E$.

For every $n \in \mathcal{N}$ and $S \in \pi_n$, denote

$$Y_n(S) = \left\{\chi_{[n-1+S]} - \chi_{[k+S]}\colon\ k = n, n+1, \ldots\right\},$$

and define Y by

$$Y_\infty = \operatorname{span}\left\{\bigcup_{n=1}^{\infty}\bigcup_{S\in\pi_n} Y_n(S)\right\} \quad \text{and} \quad Y = \overline{Y_\infty} \subset X. \tag{4.1}$$

The following theorem, proved in Section 5, is a crucial step in our proof.

Theorem 5. *Y is a locally convex space.*

We are going to show that $X/Y \cong E$. For every $n \in \mathcal{N}$ let $f_n : X_n \to E_n$ denote the map that moves X_n back to E_n, i.e., the linear map induced by

$$f_n\bigl(\chi_{[n-1+S]}\bigr) = \chi_S \quad \text{for every } S \in \pi_n.$$

Finally, let $f_\infty : X_\infty \to E_\infty$ denote the linear map induced by $\{f_n\colon\ n \in \mathcal{N}\}$. Then for every $x \in X_\infty$,

$$x = \sum_{n=1}^{\infty}\sum_{S\in\pi_n} x_n(S)a_n(S) \quad \text{where } a_n(S) = m(n)\chi_{[n-1+S]},$$

we have

$$f_\infty(x) = \sum_{n=1}^{\infty}\sum_{S\in\pi_n} x_n(S)e_n(S) \in E_\infty \quad \text{where } e_n(S) = m(n)\chi_S.$$

By definition,

$$N\bigl(f_\infty(x)\bigr) \leqslant P(x) \quad \text{for every } x \in X_\infty.$$

Since X_∞ is dense in X, the map $f_\infty : X_\infty \to E_\infty$ can be extended to a linear continuous map $f : X \to E$ such that

$$N\bigl(f(x)\bigr) \leqslant P(x) \quad \text{for every } x \in X. \tag{4.2}$$

We claim that

Proposition 2. *$f^{-1}(\theta) = Y$ and $f(X) = E$.*

Proof. Observe that for every $x \in Y_\infty$,

$$x = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\sum_{S\in\pi_n} x_n^k(S)\bigl(\chi_{[n-1+S]} - \chi_{[k+S]}\bigr); \quad \text{see (4.1)}$$

where only finitely many $x_n^k(S)$ are non-zero. By definition,

$$f_k(\chi_{[k+S]}) = \chi_S \quad \text{for every } k \geqslant n,$$

which yields $f(x) = \theta$. Since Y_∞ is dense in Y we get $Y \subset f^{-1}(\theta)$.

Conversely, to verify $f^{-1}(\theta) \subset Y$, it suffices to establish the following fact.

Claim 2. $f^{-1}(\theta) \cap X_\infty \subset Y_\infty$.

Proof. First observe that every element $x \in X_\infty$ can be written uniquely in the form

$$x = \sum_{n=1}^{\infty} x_n \quad \text{where } x_n \in X_n \text{ with } x_n \neq 0 \text{ for only finitely many } n \in \mathcal{N}.$$

We say that x is *of length* m, denoted $\ell(x) = m$, if $x_m \neq \theta$ and $x_n = \theta$ for any $n > m$. We let $\ell(\theta) = 0$.

We prove the claim by induction on the length of x. If $\ell(x) = 0$, then $x = \theta$, hence the claim holds. Assume that the claim has been proved for $\ell(x) \leqslant m$. Let $x \in f^{-1}(\theta) \cap X_\infty$ with $\ell(x) \leqslant m + 1$. Then

$$x = \sum_{n=1}^{m+1} \sum_{S \in \pi_n} x_n(S) \chi_{[n-1+S]}, \quad \text{and} \quad f(x) = \sum_{n=1}^{m+1} \sum_{S \in \pi_n} x_n(S) \chi_S = \theta. \tag{4.3}$$

Thus, for $x_{m+1} = \sum_{S \in \pi_{m+1}} x_{m+1}(S) \chi_{[m+S]}$ we have

$$f(x_{m+1}) = \sum_{S \in \pi_{m+1}} x_{m+1}(S) \chi_S = -\sum_{n=1}^{m} \sum_{S \in \pi_n} x_n(S) \chi_S \in E_m. \tag{4.4}$$

Hence $f(x_{m+1})$ can be rewritten in the form

$$f(x_{m+1}) = \sum_{S \in \pi_m} \hat{x}_m(S) \chi_S. \tag{4.5}$$

Let

$$y_k(S) = \begin{cases} x_k(S) & \text{if } k < m; \\ x_m(S) + \hat{x}_m(S) & \text{if } k = m, \end{cases} \tag{4.6}$$

$$y = \sum_{n=1}^{m} \sum_{S \in \pi_n} y_n(S) \chi_{[n-1+S]}.$$

From (4.4)–(4.6) we have

$$\begin{aligned} f(y) &= \sum_{n=1}^{m} \sum_{S \in \pi_n} y_n(S) \chi_S = \sum_{n=1}^{m} \sum_{S \in \pi_n} x_n(S) \chi_S + \sum_{S \in \pi_m} \hat{x}_m(S) \chi_S \\ &= \sum_{n=1}^{m} \sum_{S \in \pi_n} x_n(S) \chi_S - \sum_{n=1}^{m} \sum_{S \in \pi_n} x_n(S) \chi_S = \theta. \end{aligned}$$

Hence $y \in f^{-1}(\theta)$ and $\ell(y) \leqslant m$. Therefore by inductive assumption we have $y \in Y_\infty$. Consequently by (4.1),

$$\begin{aligned} x &= y - \sum_{S \in \pi_m} \hat{x}_m(S) \chi_{[m-1+S]} + \sum_{S \in \pi_m} \hat{x}_m(S) \chi_{[m+S]} \\ &= y - \sum_{S \in \pi_m} \hat{x}_m(S) \big(\chi_{[m-1+S]} - \chi_{[m+S]} \big) \in Y_\infty. \end{aligned}$$

The claim is proved. $\square$

We now show that the quotient map $f^*: X/Y = X/f^{-1}(\theta) \to E$ is an isometry; that is

$$\|x+Y\| = \inf\bigl\{\|x-y\|:\ y\in Y\bigr\} = \bigl\|f(x)\bigr\| \quad \text{for every } x\in X.$$

It suffices to show that

$$\|x+Y_\infty\| = \inf\bigl\{\|x-y\|:\ y\in Y_\infty\bigr\} = \bigl\|f(x)\bigr\| \quad \text{for every } x\in X_\infty.$$

Since $y\in Y_\infty$, we have $f(x-y)=f(x)$. Hence from (4.2) we get

$$\bigl\|f(x)\bigr\| \leqslant \|x-y\| \quad \text{for every } y\in Y_\infty,$$

which yields

$$\bigl\|f(x)\bigr\| \leqslant \|x+Y_\infty\| \quad \text{for every } x\in X_\infty.$$

To prove that the above inequality must be an equality, we assume on the contrary that

$$\bigl\|f(x)\bigr\| < \|x+Y_\infty\| \quad \text{for some } x\in X_\infty.$$

Then for each $n\in\mathcal{N}$, $S\in\pi_n$ and $T\in\pi_{n+1}(S)$ there exists $x_{n+1}(T)\in R$ such that

$$f(x) = \sum_{n=1}^{\infty}\sum_{S\in\pi_n}\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T)e_{n+1}(T) \quad \text{where } e_{n+1}(T) = m(n+1)\chi_T,$$

and

$$\sum_{n=1}^{\infty}\sum_{S\in\pi_n} N_{n+1}^S\Biggl(\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T)e_{n+1}(T)\Biggr) < \|x+Y_\infty\|. \tag{4.7}$$

Observe that, for

$$z = \sum_{n=1}^{\infty}\sum_{S\in\pi_n}\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T)a_{n+1}(T) \quad \text{where } a_{n+1}(T) = m(n+1)\chi_{[n+T]},$$

we have $f(z-x)=\theta$, and so $z-x\in f^{-1}(\theta)\cap X_\infty = Y_\infty$. Hence $z\in x+Y_\infty$, and from (4.7) we get

$$\begin{aligned}\|z\| &= \sum_{n=1}^{\infty}\sum_{S\in\pi_n} P_{n+1}^S\Biggl(\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T)a_{n+1}(T)\Biggr)\\ &= \sum_{n=1}^{\infty}\sum_{S\in\pi_n} N_{n+1}^S\Biggl(\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T)e_{n+1}(T)\Biggr) < \|x+Y_\infty\|,\end{aligned}$$

a contradiction.

Finally we claim that $f(X)=E$. In fact, since $f^*: X/f^{-1}(\theta)\to E$ is an isometry, $f^*(X/f^{-1}(\theta))$ is complete, therefore $f^*(X/f^{-1}(\theta)) = E$. Hence

$$f(X) = f^*\bigl(X/f^{-1}(\theta)\bigr) = E.$$

Consequently the proof of Proposition 2 is complete. □

Now we are able to complete the proof of our Main Theorem: From Proposition 2 we get $E\cong X/Y$. By Theorem 5, $Y=f^{-1}(\theta)$ is locally convex, and by Theorem 4, X is an AR. Hence X/Y is an AR by Theorem 2. Consequently E is an AR and the Main Theorem is demonstrated.

5. Proof of Theorem 5

In this section we prove Theorem 5, the final step in the proof of our result in this paper. Observe that for any $x \in X$, the expression

$$x = \sum_{n=1}^{\infty} \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} x_{n+1}(T) a_{n+1}(T) \quad \text{where } a_{n+1}(T) = m(n+1)\chi_{[n+T]}, \tag{5.1}$$

is unique. Therefore as in (2.1) we can define

$$|x| = \sum_{n=1}^{\infty} \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} \bigl|x_{n+1}(T)\bigr|. \tag{5.2}$$

There is no guarantee that $|x| < \infty$ even if $\|x\| < \infty$.

The proof of Theorem 5 will be divided into two steps. The first is the following:

Lemma 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| < \varepsilon$ whenever $|x| < \delta$.*

Proof. We first consider the following special case:

$$x_{n+1}(T) \geqslant 0 \quad \text{for every } T \in \pi_{n+1}(S),\ S \in \pi_n \text{ and } n \in \mathcal{N}. \tag{5.3}$$

Now given $\varepsilon > 0$, take $n_0 \in \mathcal{N}$ such that $2^{-n_0} < \varepsilon/4$. For any $x \in X$ of the form (5.1) we define

$$x(n_0) = \sum_{n=1}^{n_0} \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} x_{n+1}(T) a_{n+1}(T);$$

$$x^{\perp}(n_0) = \sum_{n=n_0+1}^{\infty} \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} x_{n+1}(T) a_{n+1}(T).$$

Take $\delta_0 > 0$ such that $\|x(n_0)\| < \varepsilon/2$ whenever $|x(n_0)| < \delta_0$. We claim that $\delta = \min\{\delta_0, \varepsilon/20\}$ satisfies the required condition.

In fact, let $x \in X$ with $|x| < \delta$. Then, since $|x(n_0)| \leqslant |x| < \delta$, we have $\|x(n_0)\| < \varepsilon/2$. Therefore it suffices to show that $\|x^{\perp}(n_0)\| < \varepsilon/2$ whenever $|x^{\perp}(n_0)| \leqslant |x| < \delta \leqslant \varepsilon/20$.

For every $n > n_0$, denote

$$a_{n+1}^S = \sum \bigl\{a_{n+1}(T)\colon\ T \in \pi_{n+1}(S)\bigr\} = m(n+1)\chi_{[n+S]}.$$

Since $(E_{n+1}(S), N_{n+1}^S)$ and $(X_{n+1}(S), P_{n+1}^S)$ are isometrically isomorphic, see (3.3), from Theorem 1(ii-b) it follows that a_{n+1}^S is an $(\varepsilon_{n+1}, h_{n+1})$-needle point of $X_{n+1}(S)$ with respect to the paranorm P_{n+1}^S and to the basis $\{a_{n+1}(T) = m(n+1)\chi_{[n+T]}\colon\ T \in \pi_{n+1}(S)\}$. Let $x_{n+1}^S = \sum_{T \in \pi_{n+1}(S)} x_{n+1}(T)$, and

$$\alpha_{n+1}^T = \begin{cases} x_{n+1}(T)[x_{n+1}^S]^{-1} & \text{if } x_{n+1}^S > 0; \\ 0 & \text{if } x_{n+1}^S = 0. \end{cases} \tag{5.4}$$

Then we have

$$\sum_{T \in \pi_{n+1}(S)} \alpha_{n+1}^T = \begin{cases} 1 & \text{if } x_{n+1}^S > 0; \\ 0 & \text{if } x_{n+1}^S = 0. \end{cases}$$

Therefore there exists an $\alpha_{n+1}^S \in [0,1]$ such that

$$P_{n+1}^S\left(\sum_{T\in\pi_{n+1}(S)} \alpha_{n+1}^T a_{n+1}(T) - \alpha_{n+1}^S a_{n+1}^S\right) < \varepsilon_{n+1}. \tag{5.5}$$

Since $|x_{n+1}^S| \leqslant |x| < \delta \leqslant \varepsilon/20 < 1$ and by Theorem 1(ii) $h_{n+1} \leqslant 5$, we obtain

$$\begin{aligned}
&P_{n+1}^S\left(\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T) a_{n+1}(T)\right) = P_{n+1}^S\left(x_{n+1}^S \sum_{T\in\pi_{n+1}(S)} \alpha_{n+1}^T a_{n+1}(T)\right)\\
&\quad\leqslant P_{n+1}^S\left(x_{n+1}^S\left[\sum_{T\in\pi_{n+1}(S)} \alpha_{n+1}^T a_{n+1}(T) - \alpha_{n+1}^S a_{n+1}^S\right]\right) + P_{n+1}^S\left(x_{n+1}^S \alpha_{n+1}^S a_{n+1}^S\right)\\
&\quad\leqslant P_{n+1}^S\left(\sum_{T\in\pi_{n+1}(S)} \alpha_{n+1}^T a_{n+1}(T) - \alpha_{n+1}^S a_{n+1}^S\right) + P_{n+1}^S\left(x_{n+1}^S a_{n+1}^S\right)\\
&\quad< \varepsilon_{n+1} + |x_{n+1}^S| h_{n+1} \leqslant \varepsilon_{n+1} + 5|x_{n+1}^S|.
\end{aligned}$$

Since

$$\varepsilon_{n+1} < \left[m(n)\right]^{-1} 2^{-n-1} \quad \text{and} \quad \left|x^{\perp}(n_0)\right| \leqslant |x| < \delta \leqslant \varepsilon/20,$$

we have

$$\begin{aligned}
\left\|x^{\perp}(n_0)\right\| &= \sum_{n=n_0+1}^{\infty} \sum_{S\in\pi_n} P_{n+1}^S\left(\sum_{T\in\pi_{n+1}(S)} x_{n+1}(T) a_{n+1}(T)\right)\\
&< \sum_{n=n_0+1}^{\infty} \sum_{S\in\pi_n} \left(\varepsilon_{n+1} + 5|x_{n+1}^S|\right)\\
&< \sum_{n=n_0+1}^{\infty} m(n)\varepsilon_{n+1} + 5\sum_{n=n_0+1}^{\infty} \sum_{S\in\pi_n} \left|x_{n+1}^S\right|\\
&< \sum_{n=n_0+1}^{\infty} 2^{-n-1} + 5\sum_{n=n_0+1}^{\infty} \sum_{S\in\pi_n} \sum_{T\in\pi_{n+1}(S)} \left|x_{n+1}(T)\right|\\
&= 2^{-n_0} + 5\left|x^{\perp}(n_0)\right|\\
&< \varepsilon/4 + 5(\varepsilon/20) = \varepsilon/2.
\end{aligned}$$

Consequently the lemma is proved in the special case of (5.3). □

To see how the general case follows from the special case we denote

$$x^+ = \sum_{n=1}^{\infty} \sum_{S\in\pi_n} \sum_{x_{n+1}(T)\geqslant 0} x_{n+1}(T) a_{n+1}(T);$$

$$x^- = \sum_{n=1}^{\infty} \sum_{S\in\pi_n} \sum_{x_{n+1}(T)< 0} x_{n+1}(T) a_{n+1}(T).$$

Then by the special case we have $\|x^+\| < \varepsilon/2$ and $\|x^-\| < \varepsilon/2$ whenever $|x| < \delta$. Since $x = x^+ + x^-$, we have $\|x\| < \varepsilon$. The lemma is demonstrated.

Lemma 2. *For every $y \in Y$ with $\|y\| < 1$ we have $|y| \leqslant \|y\|$.*

Before proving Lemma 2 let us observe that the following reformulation of Theorem 5 is an easy consequence of Lemmas 1 and 2.

Theorem 5. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\sum_{i=1}^k \alpha_i y^i\| < \varepsilon$ whenever $y^i \in Y$ with $\|y^i\| < \delta$, and $\alpha_i \geqslant 0$ with $\sum_{i=1}^k \alpha_i = 1$.*

Proof. Given $\varepsilon > 0$, first take $\delta \in (0, 1)$ satisfying the condition of Lemma 1, and let

$$y^i \in Y \text{ with } \|y^i\| < \delta < 1, \quad \text{and} \quad \alpha_i \geqslant 0, \text{ with } \sum_{i=1}^k \alpha_i = 1.$$

Then by Lemma 2, $|y^i| \leqslant \|y^i\| < \delta$ for every $i = 1, \ldots, k$. Since

$$\left|\sum_{i=1}^k \alpha_i y^i\right| \leqslant \sum_{i=1}^k \alpha_i |y^i| < \sum_{i=1}^k \alpha_i \delta = \delta,$$

from Lemma 1 we get $\|\sum_{i=1}^k \alpha_i y^i\| < \varepsilon$. Theorem 5 is proved. □

Accordingly our final step is to prove Lemma 2. Clearly it suffices to verify Lemma 2 for $y \in Y_\infty$. We will proceed with a proof by induction on the length $\ell(y)$ of y.

Observe that if $\ell(y) = 1$, then $y = \theta$, and Lemma 2 holds. Assume that Lemma 2 has been proved for $\ell(y) \leqslant k$. Let $y \in Y_\infty$ with $\|y\| < 1$ and $\ell(y) \leqslant k + 1$. Then

$$y = \sum_{n=1}^k \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} y_{n+1}(T) a_{n+1}(T), \tag{5.6}$$

and

$$f(y) = \sum_{n=1}^k \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} y_{n+1}(T) e_{n+1}(T) = \theta. \tag{5.7}$$

Observe that, for

$$y_{k+1} = \sum_{S \in \pi_k} \sum_{T \in \pi_{k+1}(S)} y_{k+1}(T) a_{k+1}(T) \in X_{k+1}, \tag{5.8}$$

from (5.8) we have

$$\begin{aligned} f(y_{k+1}) &= \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} y_{k+1}(T) e_{k+1}(T) \\ &= -\sum_{n=1}^k \sum_{S \in \pi_n} \sum_{T \in \pi_{n+1}(S)} y_{n+1}(T) e_{n+1}(T) \in E_k. \end{aligned} \tag{5.9}$$

Hence $f(y_{k+1})$ can be rewritten in the form

$$f(y_{k+1}) = \sum_{T\in\pi_k} \overline{y}_k(T)e_k(T) \quad \text{where } e_k(T) = m(k)\chi_T, \tag{5.10}$$

which implies

$$y_{k+1} = \sum_{T\in\pi_k} \overline{y}_k(T)a_k^T \quad \text{where } a_k^T = m(k)\chi_{[k+T]}. \tag{5.11}$$

Let

$$\hat{y}_i(T) = \begin{cases} y_i(T) & \text{if } i < k; \\ y_k(T) + \overline{y}_k(T) & \text{if } i = k, \end{cases} \tag{5.12}$$

$$\hat{y} = \sum_{n=1}^{k-1} \sum_{S\in\pi_n} \sum_{T\in\pi_{n+1}(S)} \hat{y}_{n+1}(T)a_{n+1}(T) \quad \text{where } a_{n+1}(T) = m(n+1)\chi_{[n+T]}.$$

Then $\ell(\hat{y}) \leqslant k$, $\hat{y} \in Y_\infty$, and from Theorem 1(v) and from (3.3), (3.5), (5.13) we get $|\hat{y}\| \leqslant \|y\| < 1$. Hence by inductive assumption,

$$\|\hat{y}\| \geqslant |\hat{y}| = \sum_{n=1}^{k-1} \sum_{S\in\pi_n} \sum_{T\in\pi_{n+1}(S)} \left|\hat{y}_{n+1}(T)\right|.$$

Since

$$\left|\hat{y}_k(T)\right| \geqslant \left|y_k(T)\right| - \left|\overline{y}_k(T)\right| \quad \text{for every } T \in \pi_k(S), \text{ see (5.13)}$$

and $\pi_k = \bigcup_{S\in\pi_{k-1}} \pi_k(S)$, we have

$$\|\hat{y}\| \geqslant \sum_{n=1}^{k-1} \sum_{S\in\pi_n} \sum_{T\in\pi_{n+1}(S)} \left|y_{n+1}(T)\right| - \sum_{S\in\pi_{k-1}} \sum_{T\in\pi_k(S)} \left|\overline{y}_k(T)\right|. \tag{5.13}$$

From (5.7), (5.9), (5.12) and (5.13) we get

$$\begin{aligned} y &= \hat{y} - \sum_{S\in\pi_{k-1}} \sum_{T\in\pi_k(S)} \overline{y}_k(T)a_k(T) + y_{k+1} \\ &= \sum_{n=1}^{k-2} \sum_{S\in\pi_n} \sum_{T\in\pi_{n+1}(S)} \hat{y}_{n+1}(T)a_{n+1}(T) \\ &\quad + \sum_{S\in\pi_{k-1}} \sum_{T\in\pi_k(S)} \left[\hat{y}_k(T) - \overline{y}_k(T)\right]a_k(T) + \sum_{S\in\pi_k} \overline{y}_k(S)a_k^S. \end{aligned}$$

From (3.5) we have

$$\begin{aligned} \|y\| &= \sum_{n=1}^{k-2} \sum_{S\in\pi_n} P_{n+1}^S\left(\sum_{T\in\pi_{n+1}(S)} \hat{y}_{n+1}(T)a_{n+1}(T)\right) \\ &\quad + \sum_{S\in\pi_{k-1}} P_k^S\left(\sum_{T\in\pi_k(S)} [\hat{y}_k(T) - \overline{y}_k(T)]a_k(T)\right) \\ &\quad + \sum_{S\in\pi_k} P_{k+1}^S\left(\overline{y}_k(S)a_k^S\right). \end{aligned} \tag{5.14}$$

From Theorem 1(ii-a) and from (3.3) we obtain

$$P_k^S\biggl(\sum_{T\in\pi_k(S)} \overline{y}_k(T)a_k(T)\biggr) \leqslant \sum_{T\in\pi_k(S)} \bigl|\overline{y}_k(T)\bigr| \quad \text{for every } S\in\pi_k.$$

It follows that

$$\begin{aligned}
&\sum_{S\in\pi_{k-1}} P_k^S\biggl(\sum_{T\in\pi_k(S)} [\hat{y}_k(T)-\overline{y}_k(T)]a_k(T)\biggr)\\
&\quad\geqslant \sum_{S\in\pi_{k-1}}\Biggl(P_k^S\biggl(\sum_{T\in\pi_k(S)} \hat{y}_k(T)a_k(T)\biggr) - P_k^S\biggl(\sum_{T\in\pi_k(S)} \overline{y}_k(T)a_k(T)\biggr)\Biggr)\\
&\quad\geqslant \sum_{S\in\pi_{k-1}}\Biggl(P_k^S\biggl(\sum_{T\in\pi_k(S)} \hat{y}_k(T)a_k(T)\biggr) - \sum_{T\in\pi_k(S)}\bigl|\overline{y}_k(T)\bigr|\Biggr). \qquad (5.15)
\end{aligned}$$

Observe that for every $S\in\pi_k$, $a_k^S = m(k)\chi_{[k+S]}$ is an $\varepsilon_{k+1}, h_{k+1}$-needle point of $X_{k+1}(S)$ with respect to the paranorm P_{k+1}^S, and by Theorem 1(ii), $h_n \geqslant 4$ for every $n\in\mathcal{N}$. Moreover since $\|y\| < 1$, from (3.3), (3.5) we have $|\overline{y}_k(S)| \leqslant 1$ for every $S\in\pi_k$. Consequently

$$P_{k+1}^S\bigl(\overline{y}_k(S)a_k^S\bigr) = \bigl|\overline{y}_k(S)\bigr|h_{k+1} \geqslant 4\bigl|\overline{y}_k(S)\bigr| \quad \text{for every } S\in\pi_k.$$

Now, since

$$\sum_{S\in\pi_k}\bigl|\overline{y}_k(S)\bigr| = \sum_{S\in\pi_{k-1}}\sum_{T\in\pi_k(S)}\bigl|\overline{y}_k(T)\bigr|, \qquad (5.16)$$

from (5.15), (5.16) and from (3.4), (3.5) we get

$$\begin{aligned}
\|y\| \geqslant{}& \sum_{n=1}^{k-2}\sum_{S\in\pi_n} P_{n+1}^S\biggl(\sum_{T\in\pi_{n+1}(S)} \hat{y}_{n+1}(T)a_{n+1}(T)\biggr)\\
&+\sum_{S\in\pi_{k-1}}\Biggl(P_k^S\biggl(\sum_{T\in\pi_k(S)} \hat{y}_k(T)a_k(T)\biggr) - \sum_{T\in\pi_k(S)}\bigl|\overline{y}_k(T)\bigr|\Biggr)\\
&+4\sum_{S\in\pi_{k-1}}\sum_{T\in\pi_k(S)}\bigl|\overline{y}_k(T)\bigr| = \|\hat{y}\| + 3\sum_{S\in\pi_{k-1}}\sum_{T\in\pi_k(S)}\bigl|\overline{y}_k(T)\bigr|.
\end{aligned}$$

Therefore from (5.14), (5.17) we obtain

$$\begin{aligned}
\|y\| &\geqslant \sum_{n=1}^{k-1}\sum_{S\in\pi_n}\sum_{T\in\pi_{n+1}(S)}\bigl|y_{n+1}(T)\bigr| + 2\sum_{S\in\pi_k}\sum_{T\in\pi_{k+1}(S)}\bigl|y_{k+1}(T)\bigr|\\
&\geqslant \sum_{n=1}^{k}\sum_{S\in\pi_n}\sum_{T\in\pi_{n+1}(S)}\bigl|y_{n+1}(T)\bigr| = |y|.
\end{aligned}$$

The proof of Lemma 2 is complete. □

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