Nonlinear Analysis 74 (2011) 141–145

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Extension of Caristi's fixed point theorem to vector valued metric spaces

R.P. Agarwal [a,b], M.A. Khamsi [c,b,*]

[a] Department of Mathematical Sciences, Florida Institute of Technology, 150 W. University Blvd, Melbourne, FL 32901, United States

[b] Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, P.O. Box 411, Dhahran 31261, Saudi Arabia

[c] Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968-0514, United States

ARTICLE INFO

Article history:
Received 23 June 2010
Accepted 13 August 2010

MSC:
primary 06F30
46B20
47E10

Keywords:
Ordered vector spaces
Banach lattice
Order continuous
Minimal element
Positive operator
Contraction and nonexpansive mappings
Fixed point
Subadditive functions

ABSTRACT

The paper deals with the classical Caristi fixed point theorem in vector valued metric spaces. The results obtained seem to be new in this setting.

1. Introduction

In recent years, Caristi's fixed point theorem [1–3] has been the subject of intensive research. Recall that this theorem states that any map $T : M \to M$ has a fixed point provided that M is complete and there exists a lower semi-continuous map ϕ mapping M into the nonnegative numbers such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

for every $x \in M$. This general fixed point theorem has found many applications in nonlinear analysis. It is shown, for example, that this theorem yields essentially all the known inwardness results [4] of geometric fixed point theory in Banach spaces. Recall that inwardness conditions are the ones which assert that, in some sense, points from the domain are mapped toward the domain. Possibly the weakest of the inwardness conditions, the Leray–Schauder boundary condition is the assumption that a map points x of ∂M anywhere except to the outward part of the ray originating at some interior point of M and passing through x.

The proofs given for Caristi's result vary and use different techniques (see [1,5,2,6]). It is worth mentioning that because of Caristi's result's close connection to Ekeland's [7] variational principle, many authors refer to it as the Caristi–Ekeland fixed point result. For more on Ekeland's variational principle and the equivalence between the Caristi–Ekeland fixed point result and the completeness of metric spaces, the reader is advised to read [8].

* Corresponding author at: Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968-0514, United States. Tel.: +1 915 747 6763; fax: +1 915 747 6502.

E-mail addresses: agarwal@fit.edu, agarwal@kfupm.edu.sa (R.P. Agarwal), mohamed@utep.edu, mkhamsi@kfupm.edu.sa (M.A. Khamsi).

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.08.025

In this work we prove a vector version of this theorem in vector valued metric spaces. The approach was used intensively in [9] where a Banach Contraction Principle was proved in this setting and used to obtain a result more general than the one obtained in [10] concerning isolated solutions of multi-point boundary value problems.

2. Caristi's fixed point theorem

The main motivation behind this work is the main point raised in [9] that the domain of existence of a solution to a system of first-order differential equations may be increased by considering vector valued distances. The example therein strengthens this point. In fact this point was noted by Bernfeld and Lakshmikantham [11] who indicated that "there is more flexibility working with generalized spaces" (meaning vector valued metric spaces).

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V}; \theta \preceq v\}$, where θ is the zero-vector of $\mathcal{V}$, satisfies the usual properties:

(1) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$,
(2) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$,
(3) $\alpha\, \mathcal{V}_+ \subset \mathcal{V}_+$ for all $\alpha \geq 0$.

The concept of vector valued metric spaces relies on the following definition.

Definition 1. Let M be a set. A map $d : M \times M \to \mathcal{V}$ defines a distance if:

(i) $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for any $x, y \in M$,
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for any $x, y, z \in M$.

The pair (M, d) is called a vector valued metric space (vvms for short).

In [9] the following theorem, seen as a generalization of the Banach Contraction Principle, is proved.

Theorem 1. *Let (M, d) be a complete vvms, where $\mathcal{V} = \mathbb{R}^N$. Let $T : M \to M$. Assume there exists a positive operator $A : \mathbb{R}^N \to \mathbb{R}^N$, i.e. $A(\mathbb{R}^N_+) \subset \mathbb{R}^N_+$ which satisfies $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A, such that*

$$d(T(x), T(y)) \preceq A\, d(x, y),$$

for any $x, y \in M$. Then:

(1) *there exists $\omega \in M$ such that for any $x_0 \in M$, the orbit $\{T^n(x_0)\}$ converges to ω, and moreover we have*

$$d(T^n(x_0), \omega) \preceq A^n(I - A)^{-1} d(x_0, T(x_0)) = \left(\sum_{k=n}^{\infty} A^k\right) d(x_0, T(x_0)),$$

for any $n \geq 1$;
(2) *the point ω is the only fixed point of T in M.*

As Caristi did for the case of the classical Banach Contraction Principle, let us discuss his ideas for the vvms. Under the assumptions of Theorem 1, where $\mathcal{V}$ is no longer the finite dimensional space $\mathbb{R}^N$, we have

$$d(T(x), T^2(x)) \preceq Ad(x, T(x))$$

for any $x \in M$, which implies

$$d(x, T(x)) + d(T(x), T^2(x)) \preceq d(x, T(x)) + Ad(x, T(x)).$$

Hence

$$d(x, T(x)) - Ad(x, T(x)) \preceq d(x, T(x)) - d(T(x), T^2(x))$$

or

$$(I - A)d(x, T(x)) \preceq d(x, T(x)) - d(T(x), T^2(x)).$$

Set $d_A(x, y) = (I - A)d(x, y)$, for any $x, y \in M$. Then it is easy to check that if $I - A$ is a positive one-to-one operator, then d_A is a vector valued distance defined on M. So if we set $F(x) = d(x, T(x))$, then we have

$$d_A(x, T(x)) \preceq F(x) - F(T(x)).$$

As Caristi did, one may wonder under what assumptions on any vvms (M, d) and $F : M \to \mathcal{V}_+$, any map $T : M \to M$ which satisfies

$$d(x, T(x)) \preceq F(x) - F(T(x)),$$

for any $x \in M$, has a fixed point. We will approach this question through Brønsted order. Indeed the map F defines an order on M as follows:

$$x \preceq y \Longleftrightarrow d(x, y) \preceq F(y) - F(x)$$

for any $x, y \in M$. Using this order, any map $T : M \to M$ which satisfies

$$d(x, T(x)) \preceq F(x) - F(T(x)),$$

for any $x \in M$, will fix any minimal point. Therefore the fixed point problem shifts to the existence of minimal points of the order $\preceq$ in M. Throughout the remainder of this work we will assume that $\mathcal{V}$ is a complete Banach lattice which is order continuous. Let $\{x_\alpha; \alpha \in \Gamma\}$ be a chain in M. Then $\{F(x_\alpha); \alpha \in \Gamma\}$ is a chain in $\mathcal{V}_+$. Since $\mathcal{V}$ is order complete, then $v = \inf\{F(x_\alpha); \alpha \in \Gamma\}$ exists in $\mathcal{V}_+$. Assume there exists $\alpha_0 \in \Gamma$ such that $F(x_{\alpha_0}) = v$. Then it is easy to see that $\alpha_0 \preceq \alpha$, for any $\alpha \in \Gamma$. Assume then that $F(x_\alpha) \neq v$, for any $\alpha \in \Gamma$. Since $\mathcal{V}$ is order continuous then we have $\inf_{\alpha\in\Gamma} \|F(x_\alpha) - v\| = 0$. Then for any $n \geq 1$, there exists $\alpha_n \in \Gamma$ such that $\|F(x_{\alpha_n}) - v\| < \frac{1}{n}$. Since $\{x_\alpha; \alpha \in \Gamma\}$ is a chain in M, there exists β_n such that $x_{\beta_n} = \min\{x_{\alpha_i}, i = 1, 2, \ldots, n\}$, for any $n \geq 1$. Clearly $\{F(x_{\beta_n})\}$ is a decreasing sequence which converges in norm to v. Since

$$d(x_{\beta_{n+1}}, x_{\beta_n}) \preceq F(x_{\beta_{n+1}}) - F(x_{\beta_n}), \quad n = 1, 2, \ldots,$$

then we have

$$d(x_{\beta_{n+h}}, x_{\beta_n}) \preceq \sum_{k=1}^{h-1} d(x_{\beta_{n+k}}, x_{\beta_{n+k+1}}) \preceq \sum_{k=1}^{h-1} F(x_{\beta_{n+k}}) - F(x_{\beta_{n+k+1}})$$

or

$$d(x_{\beta_{n+h}}, x_{\beta_n}) \preceq F(x_{\beta_n}) - F(x_{\beta_{n+h}})$$

for any $n, h \geq 1$. Hence the series of positive vectors $\sum d(x_{\beta_{n+1}}, x_{\beta_n})$ is order bounded. Since $\mathcal{V}$ is order continuous, then $\sum d(x_{\beta_{n+1}}, x_{\beta_n})$ is convergent in norm. This forces the sequence $\{x_{\beta_n}\}$ to be Cauchy. Therefore if we assume $(M, \mathcal{V})$ to be complete, then $\{x_{\beta_n}\}$ will converge to some point $x \in M$. Next we show that x is a lower bound of $\{x_\alpha; \alpha \in \Gamma\}$. In order to prove that, we will need the function F to be lower semi-continuous (lsc for short). Then we must have

$$F(x) \preceq \liminf_{n\to\infty} F(x_{\beta_n}).$$

In particular, we have $x \preceq x_{\beta_n}$, for any $n \geq 1$. Next we fix $\alpha \in \Gamma$. If there exists $n_0 \geq 1$ such that $x_{n_0} \preceq x_\alpha$, then we will have $x \preceq x_\alpha$. Assume otherwise that for any $n \geq 1$, we have $x_\alpha \preceq x_{\beta_n}$, which implies $F(x_\alpha) \preceq F(x_{\beta_n})$, for any $n \geq 1$. By the definition of v, we can easily deduce that $F(x_\alpha) = v$, which contradicts our assumption above. Hence $\{x_\alpha; \alpha \in \Gamma\}$ has a lower bound. Therefore Zorn's lemma will ensure the existence of a minimal element for $\prec$ in M. This proves the following:

Theorem 2. *Let (M, d) be a complete vvms over an order complete and order continuous Banach lattice $\mathcal{V}$. Let $F : M \to \mathcal{V}_+$ be a lsc map. Then any $T : M \to M$ such that*

$$d(x, T(x)) \leq F(x) - F(T(x)),$$

for any $x \in M$, has a fixed point.

Note that if (M, d) is a complete vvms, then the vector valued distance $d_A(x, y) = (I - A)d(x, y)$, where $I - A$ is a positive one-to-one operator, is also complete. As a corollary we obtain the following:

Corollary 1. *Let (M, d) be a complete vvms over an order complete and order continuous Banach lattice $\mathcal{V}$. Let $T : M \to M$ be a continuous map such that*

$$d(T(x), T^2(x)) \leq Ad(x, T(x))$$

for any $x \in M$, where $I - A$ is a positive one-to-one operator. Then T has a fixed point.

Proof. Indeed we proved that

$$d_A(x, T(x)) = (I - A)d(x, T(x)) \preceq d(x, T(x)) - d(T(x), T^2(x))$$

holds for any $x \in M$. Since (M, d_A) is complete and $F(x) = d(x, T(x))$ is continuous, then Theorem 2 implies the existence of a fixed point of T. □

The above corollary may be seen as an improvement to the main result of [9]. Indeed if we take $\mathcal{V} = l_2$, the Hilbert space, and $A : \mathcal{V} \to \mathcal{V}$ such that

$$A(x_n) = ((1 - \varepsilon_n)x_n),$$

where $\varepsilon_n \in (0, 1)$ and $\lim_{n\to\infty} \varepsilon_n = 0$, then $(I - A)$ is a positive nonsingular operator such that its spectral radius $\rho(A) = 1$.

Next we prove a result similar to the main theorem of [12] which generalizes many known extensions to Caristi's fixed point theorem (see the references therein).

Theorem 3. *Let (M, d) be a complete vvms over an order complete and order continuous Banach lattice $\mathcal{V}$. Let $F : M \to \mathcal{V}_+$ be a lsc map. Let $T : M \to M$ be such that*

$$d(x, T(x)) \preceq \phi(x)(F(x) - F(T(x))),$$

for any $x \in M$, where $\phi(x)$ is an operator which satisfies $\theta \preceq \phi(x)(v)$ if and only if $\theta \preceq v$. Moreover we assume that there exist $x_0 \in M$ and a positive operator A of $\mathcal{V}$ such that $\phi(x) \preceq A$ whenever $F(x) \preceq F(x_0)$. Then T has a fixed point.

Proof. Set

$$M_0 = \{x \in M; F(x) \preceq F(x_0)\}.$$

Clearly M_0 is not empty. Since F is lsc, then M_0 is a closed subset of M. Hence M_0 is a complete vvms. It is clear also that we must have $F(T(x)) \preceq F(x)$, for any $x \in M$. Therefore we must have $T(M_0) \subset M_0$. If we denote by T_0 the restriction of T to M_0, then T_0 satisfies

$$d(x, T(x)) \preceq A(F(x) - F(T(x))),$$

for any $x \in M_0$. The new vector function $F^* : M_0 \to \mathcal{V}_+$ defined by $F^*(x) = AF(x)$ is lsc. Theorem 2 ensures the existence of a fixed point of T_0 in M_0, which happens to be also a fixed point of T. □

3. Kirk's problem

The next result is similar to the main theorem found in [13] which extends Caristi's classical fixed point theorem and gives a partial positive answer to Kirk's problem [3,6]. Indeed Kirk asked whether a map $T : M \to M$ such that

$$\eta(d(x, Tx)) \leq \phi(x) - \phi(Tx),$$

for all $x \in M$, for some positive function η, has a fixed point. In fact the original question of Kirk was stated for when $\eta(t) = t^p$, for some $p > 1$. This was discussed in [13]. In particular, a negative answer to this problem is given. Also it is shown that if $\eta : [0, \infty) \to [0, \infty)$ is nondecreasing and continuous, such that there exist $c > 0$ and $\delta_0 > 0$ such that for any $t \in [0, \delta_0]$ we have $\eta(t) \geq c\ t$, then Kirk's problem has a positive answer. Note that under the above assumptions, since η is continuous, there exists $\varepsilon_0 > 0$ such that $\eta^{-1}([0, \varepsilon_0]) \subset [0, \delta_0]$.

The extension of this result to vvms is the main drive of this section. Indeed let (M, d) be a complete vvms, over a Banach lattice $\mathcal{V}$ which is order complete and order continuous. Let $\eta : \mathcal{V}_+ \to \mathcal{V}_+$ be continuous and such that there exist a one-to-one positive operator A of $\mathcal{V}$ and $\delta_0 > 0$ such that

$$A\,v \preceq \eta(v),$$

for any $v \in \mathcal{V}$ such that $\|v\| < \delta_0$. Since η is continuous, there exists $\varepsilon_0 > 0$ such that

$$\eta^{-1}(B_0(0, \varepsilon_0)) \subset B_0(0, \delta_0),$$

where $B_0(0, r) = \{x \in \mathcal{V}, \|x\| < r\}$ is the open ball in the Banach space $\mathcal{V}$. We have the following result.

Theorem 4. *Let M be a complete vvms, over a Banach lattice $\mathcal{V}$ which is order complete and order continuous. Define the relation $\prec_\eta$ by*

$$x \prec_\eta y \iff \eta(d(x, y)) \preceq \phi(y) - \phi(x)$$

where η satisfies all the above assumptions, and $\phi : M \to \mathcal{V}_+$ is lsc. Then $(M, \prec_\eta)$ has a minimal element x_*, i.e. if $x \prec_\eta x_*$ then we must have $x = x_*$.

Proof. Since $\mathcal{V}$ is order complete, then $v = \inf\{\phi(x); x \in M\}$ exists in $\mathcal{V}_+$. Since $\mathcal{V}$ is order continuous then we have $\inf_{x \in M} \|\phi(x) - v\| = 0$. Therefore there exists $x_0 \in M$ such that $\|\phi(x_0) - v\| < \varepsilon_0$. Set

$$M_0 = \{x \in M; \phi(x) \preceq \phi(x_0)\}.$$

Clearly M_0 is a nonempty closed subset of M. Hence (M_0, d) is complete. Let $x, y \in M_0$, with $x \prec_\eta y$. Hence

$$v \preceq \phi(x) \preceq \phi(y) \preceq \phi(x_0)$$

which implies $\phi(y) - \phi(x) \preceq \phi(x_0) - v$. So

$$\|\phi(y) - \phi(x)\| \leq \|\phi(x_0) - v\| < \varepsilon_0.$$

Since $\eta(d(x, y)) \preceq \phi(y) - \phi(x)$, we obtain that $\|\eta(d(x, y))\| < \varepsilon_0$. Our assumptions on η imply

$$Ad(x, y) \preceq \eta(d(x, y)) \preceq \phi(y) - \phi(x).$$

On M_0 we define the new relation $\prec_*$ by

$$x \prec_* y \iff Ad(x, y) \leq \phi(y) - \phi(x).$$

Clearly $(M_0, \prec_*)$ is a partially ordered set with all assumptions necessary to secure the existence of a minimal element x_* for $\prec_*$. Let us see that x_* is also a minimal element for the relation $\prec_\eta$ in M. Indeed let $x \in M$ be such that $x \prec_\eta x_*$. Then we have $\eta(d(x, x_*)) \preceq \phi(x_*) - \phi(x)$. In particular we have $\phi(x) \preceq \phi(x_*)$ which implies $\phi(x) \leq \phi(x_0)$; i.e. $x \in M_0$. As before, we have $\|\eta(d(x, x_*))\| < \varepsilon_0$ which implies

$$Ad(x, x_*) \preceq \eta(d(x, x_*)) \preceq \phi(x_*) - \phi(x)$$

which in turn implies $x \prec_* x_*$. Since x_* is minimal in $(M_0, \prec_*)$ we get $x = x_*$. This completes the proof of Theorem 4. □

This is an amazing result because the relation $\prec_\eta$ is not a partial order. Of course if one assumes η to be subadditive, i.e. $\eta(v_1 + v_2) \preceq \eta(v_1) + \eta(v_2)$ for any $v_1, v_2 \in \mathcal{V}$, then $\prec_\eta$ is a partial order on M.

Using Theorem 4 we are ready to state the vvms version of the main result of [13].

Theorem 5. *Let M be a complete vvms, over a Banach lattice $\mathcal{V}$ which is order complete and order continuous. Let $T : M \to M$ be a map such that for all $x \in M$*

$$\eta(d(x, Tx)) \preceq \phi(x) - \phi(Tx),$$

where the functions η and ϕ satisfy the assumptions described above, then T has a fixed point.

Proof. Define the relation $\prec_\eta$ as in Theorem 4. Obviously we have $T(x) \prec_\eta x$ for any $x \in M$. Theorem 4 ensures the existence of x_*, a minimal element for $\prec_\eta$. Clearly we must have $T(x_*) = x_*$. □

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