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On the Fixed Points of Commuting Nonexpansive Maps in Hyperconvex Spaces

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Let $\{T_1, ..., T_N\}$ be a finite commuting family of nonexpansive maps of a hyperconvex space such that each T_i has bounded orbits. We show: (i) Each point has a bounded orbit under the semigroup generated by $\{T_i\}$; (ii) There is a common fixed point for the family if (and only if) $T = T_1 T_2 \cdots T_N$ has a fixed point; (iii) For each $\varepsilon > 0$, there is a nonempty set of common ε-approximate fixed points for the family. Some additional related results are also given. © 1992 Academic Press, Inc.

A metric space M is called *hyperconvex* if for any collection of closed balls, $\{B(x_\alpha; r_\alpha) : \alpha \in A\}$, satisfying $d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta$ for any pair of indices, we have $\bigcap \{\beta(x_\alpha; r_\alpha) : \alpha \in A\}$ is nonempty.

The Nachbin–Kelley–Goodner theorem [La, p. 92] asserts: A Banach space is hyperconvex if and only if it is linearly isometric to $C(\mathscr{S})$ for some Stonian (extremally disconnected) compact Hausdorff space $\mathscr{S}$. Thus, $l_\infty(I)$ for any set I, and $L_\infty(\mu)$ for a finite measure μ, are examples of such spaces. Any hyperconvex space, indeed, any metric space, embeds isometrically in some $l_\infty(I)$.

Aronszajn and Panitchpakdi [AP] introduced hyperconvex spaces and proved that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. (This is analogous to the P_1 property of hyperconvex Banach spaces.)

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A map T on a metric space M is called *nonexpansive* if $d(Tx, Ty) \leqslant d(x, y)$ holds for any pair of points. Sine [S_1] and Soardi [So] proved independently that a nonexpansive map on a bounded hyperconvex space has a common fixed point. Lin and Sine [LS] showed that if $\mathscr{S}$ is a commuting family of nonexpansive maps on any hyperconvex space which has a nonempty set of common fixed points, then there is a nonexpansive retraction onto $\mathrm{Fix}(\mathscr{S})$ which commutes with every member of $\mathscr{S}$.

DEFINITION. If $\mathscr{S}$ is a semigroup of maps of M, the *$\mathscr{S}$-orbit* of x in M is the set $\{Tx : T \in \mathscr{S}\}$. In the case $\mathscr{S} = \{T^n : n \geqslant 1\}$ we will simply refer to this as the orbit of x.

It is obvious that if a nonexpansive map has a fixed point, then all orbits are bounded. The following example of Prus [P] shows that the boundedness of the space assumed in the Sine–Soardi fixed point theorem cannot be replaced by the weaker assumption of bounded orbits.

EXAMPLE [Prus]. There exists a fixed point free nonexpansive map of l_∞ with bounded orbits.

Let λ denote a Banach limit and define for each $a = (a_1, a_2, ...)$ in $l_\infty(N)$

$$T(a_1, a_2, ...) = (1 + \lambda(a), a_1, a_2, ...).$$

Then the orbit of 0 is bounded since

$$T^n(0, 0, 0, ...) = (1, ..., 1, 0, 0, ...)$$

with 1 in the first n coordinates. It is easily checked that T is fixed point free. It is also easily checked that T is nonexpansive, in fact, is an order preserving, affine isometry.

Remark. Any probability measure supported on $\beta N\text{-}N$ will work as well here as a Banach limit.

LEMMA 1. *Let $\mathscr{S}$ be a commutative family of nonexpansive maps on a hyperconvex metric space M. Let τ be a topology on M for which balls are closed. If there exists a bounded set A so that A is contained in the τ-closure of $T(A)$ for every T in $\mathscr{S}$, then* $\mathrm{Fix}(\mathscr{S}) \neq \varnothing$.

Proof. Let $\delta = \mathrm{dia}(A)$ and set $J = \bigcap \{B(x; \delta) : x \in A\}$. Then J is a bounded hyperconvex set which contains A. Let y be in J and T in $\mathscr{S}$. Then for x in A we have $d(Ty, Tx) \leqslant d(y, x) \leqslant \delta$. Hence $T(A) \subset B(Ty; \delta)$. Since balls are τ-closed we have

$$A \subset \tau\text{-closure } T(A) \subset B(Ty; \delta).$$

Now J is invariant under $\mathscr{S}$ so Baillon's result implies the existence of a common fixed point in J.

Remark. If A is bounded and $T(A)=A$ for every T in $\mathscr{S}$ then we can take the metric topology for τ.

For completeness we include the following result of [KR]. Convexity, assumed there, is superfluous.

THEOREM 2. *Let T be nonexpansive on (all of) a hyperconvex dual Banach space (e.g., $L_\infty(\mu)$ or $l_\infty(I)$). Suppose there is a nonempty w^*-compact set C which is T-invariant. Then* $\mathrm{Fix}(T) \neq \varnothing$.

Proof. The assumption and Zorn's lemma imply that there is a minimal w^*-compact T-invariant set A. Let A' be the w^*-closure of $T(A)$. Then $A' \subset A$ since A is w^*-closed. Thus $T(A') \subset T(A) \subset A'$ which shows A' is T invariant. By minimality $A'=A$ and we can now apply Lemma 1 with the w^*-topology playing the role of τ.

Remark. The condition in the Theorem is clearly necessary, for any ball centered at a fixed point is both w^*-compact and T invariant.

THEOREM 3. *Let $\mathscr{S}$ be an abelian semigroup of isometries on a hyperconvex space M. Then* $\mathrm{Fix}(\mathscr{S}) \neq \varnothing$ *if (and only if) $\mathscr{S}$ has bounded orbits and $\bigcap \{T(M) : T \in \mathscr{S}\} \neq \varnothing$.*

Proof. Clearly $\mathrm{Fix}(\mathscr{S})$ is contained in $\bigcap \{T(M) : T \in \mathscr{S}\}$ so necessity is obvious. To show sufficiency we let x be a point of $\bigcap \{T(M) : T \in \mathscr{S}\}$. Then for each T in $\mathscr{S}$ there exists a point x_T with $Tx_T = x$. Since T is one-to-one, x_T is unique. Let δ be the diameter of $\{Tx : T \in \mathscr{S}\}$. Then, since T is an isometry,

$$d(x_T, x) = d(Tx_T, Tx) = d(x, Tx) \leqslant \delta.$$

Now for T and S in $\mathscr{S}$ we obtain $Tx_{TS} = x_S$, since $S(Tx_{TS}) = x$. Also

$$d(Tx_S, x) = d(STx_S, Sx) = d(Tx, Sx) \leqslant \delta.$$

For the convenience of presentation we may assume that the identity map, I, is in $\mathscr{S}$. Then $x_I = x$. Let $A = \{Tx_S : T, S \in \mathscr{S}\}$. Then $T(A) \subset A$ for T in $\mathscr{S}$. By the above computations, $A \subset B(x; \delta)$.

Let $y = T_1(x_s)$ be in A. Then $Tx_{TS} = x_S$ implies

$$y = T_1 x_S = T_1 T x_{TS} = T(T_1 x_{TS}),$$

so y is in $T(A)$. Thus $T(A) = A$ for every T in $\mathscr{S}$. Lemma 1 now implies $\mathrm{Fix}(\mathscr{S}) \neq \varnothing$.

COROLLARY 4. *Let $\mathscr{S}$ be an abelian group of nonexpansive maps on a hyperconvex space M, with $\mathscr{S}$-bounded orbits. Then* $\mathrm{Fix}(\mathscr{S}) \neq \varnothing$.

Proof. The group identity is a nonexpansive retract U. Apply Theorem 3 to the restriction of $\mathscr{S}$ to $U(M)$, which is hyperconvex. (A direct proof from Lemma 1 can be obtained by taking $A = \{Tx : T \in \mathscr{S}\}$ for some x in $U(M)$.)

Remark. The example of translations on $\mathbb{R}$ shows that boundedness of the orbits cannot be dropped from the hypothesis of the corollary nor from the sufficiency part of Theorem 3.

THEOREM 5. *Let $\{T_1, ..., T_N\}$ be commuting nonexpansive maps of a hyperconvex space and assume that each T_i has a bounded orbit. Then:*

(i) *Each point x has a bounded orbit under the semigroup $\mathscr{S}$ generated by $\{T_1, ..., T_N\}$.*

(ii) $\bigcap \{\mathrm{Fix}(T_i) : 1 \leqslant i \leqslant N\} \neq \varnothing$ *if (and only if)* $\mathrm{Fix}(T_1 T_2 \cdots T_N) \neq \varnothing$.

Proof. (i) The argument is by induction on the number of maps. Suppose that for n maps, the generated semigroup, $\mathscr{S}_n$, has bounded orbits. Hence there exists $\alpha > 0$ so that $d(Wx, x) \leqslant \alpha$ for every map W in $\mathscr{S}_n$. By assumption, the orbit $\{T_{n+1}^k x : k \geqslant 0\}$ is also bounded, so $d(T_{n+1}^k x, x) \leqslant \beta$ for all $k \geqslant 0$ and some β. With W in $\mathscr{S}_n$ and $k \geqslant 0$ we have

$$\begin{aligned} d(T_{n+1}^k Wx, x) &\leqslant d(T_{n+1}^k Wx, Wx) + d(Wx, x) \\ &\leqslant d(T_{n+1}^k x, x) + d(Wx, x) \leqslant \alpha + \beta \end{aligned}$$

using commutivity and nonexpansiveness.

(ii) Let x_0 be a fixed point of the map $T = T_1 T_2 \cdots T_N$ and let A be the orbit of x_0 under $\mathscr{S}$. Then A is bounded by (i). Now $A = \{T_1^{k_1} \cdots T_N^{k_N} x_0 : k_i \geqslant 0\}$. For $x = T_1^{k_1} T_2^{k_2} \cdots T_N^{k_N} x_0$ we have $x = T_1^{k_1+1} T_2^{k_2+1} \cdots T_N^{k_N+1} x_0 = T_1 [T_1^{k_1} k_2^{k_2+1} \cdots T_N^{k_N+1} x_0]$ since x_0 is $T = T_1 T_2 \cdots T_N$ fixed. Hence $T_i A = A$. Now apply Lemma 1.

EXAMPLE. There is a sequence $\{T_i : i \geqslant 1\}$ of commuting nonexpansive maps on l_∞ satisfying

(i) The semigroup generated by $\{T_i\}$ has bounded orbits.

(ii) For each n, $\bigcap \{\mathrm{Fix}(T_i) : 1 \leqslant i \leqslant n\} \neq \varnothing$.

(iii) $\bigcap \{\mathrm{Fix}(T_i) : i \geqslant 1\} = \varnothing$.

Let T be a nonexpansive fixed point free map with bounded orbits (e.g., Prus' example). Let $H_n = \{x \in l_\infty : d(x, Tx) \leqslant 1/n\}$. It is shown in [KR] (see also Theorem 8 below) that there exists a closed, nonempty, convex, bounded set C which is T-invariant. We can apply the well known technique of replacing T with a strict contradiction which also leaves C invariant. The fixed point for the strict contraction is an ε-fixed point for T. Thus for each n, $H_n \cap C \neq \emptyset$. By [$S_2$] the set H_n is hyperconvex. Let P_k be the nonexpansive retract of l_∞ onto H_n and then define $T_n = P_n P_{n-1} \cdots P_1$. It is easily seen that $\{T_n\}$ is a commuting family of nonexpansive maps and T_n is a nonexpansive retraction onto H_n. Thus $\bigcap \{\text{Fix}(T_i) : 1 \leqslant i \leqslant n) = H_n \neq \emptyset$. Since T is fixed point free $\bigcap \{\text{Fix}(T_i) : i \geqslant 1\} = \emptyset$.

Now take x_n in $H_n \cap C$ so $\delta = \text{dia}\{x_n\} < \infty$. Then $d(T_n x_1, x_n) \leqslant d(x_1, x_n)$. Also $d(T_n x_1, x_1) \leqslant d(T_n x_1, x_n) + d(x_n, x_1) \neq 2d(x_1, x_n) \leqslant 2\delta$. Hence $\{T_n x_1\}$ is bounded. Since $\{T_n\}$ is a semigroup it has bounded orbits.

THEOREM 6. *Let $\mathscr{S} = \{T_t : t = (t_1, ..., t_N) \in \mathbb{R}_+^N\}$ be an n-parameter abelian semigroup of nonexpansive maps of a hyperconvex space. Assume that $\mathscr{S}$ has bounded orbits. Then* $\text{Fix}(\mathscr{S}) \neq \emptyset$ *if (and only if) there exists $r \in R_+^N$ with $r_i > 0$ for $1 \leqslant i \leqslant N$ so that* $\text{Fix}(T_r) \neq \emptyset$.

Proof. Let x_0 be in $\text{Fix}(T_r)$ for some r with strictly positive components. Let $A = \{T_t x_0 : t \in \mathbb{R}_+^N\}$ be the orbit of x_0. For any s in R_+^N we have $T_s(A) \subset A$. But by taking $n > 0$ so that $nr - s$ is in $\mathbb{R}_+^N$ (as we may do as the components of r are strictly positive) we obtain

$$T_s(T_{nr-s+t})\, x_0 = T_t T_{nr} x_0 = T_t x_0.$$

So $T_s(A) = A$ for every s in $\mathbb{R}_+^N$ and Lemma 1 now applies.

Remark. The case $N = 1$ is proved in [KR].

Theorems 5 and 6 can be applied in a hyperconvex dual Banach space if there exists a nonempty w^*-compact set C which is invariant under the semigroup $\mathscr{S}$, by applying Theorem 2.

THEOREM 7. *Let $\mathscr{S}$ be an abelian semigroup of nonexpansive maps on a hyperconvex dual Banach space. Assume there exists a nonempty w^*-compact set C which is $\mathscr{S}$-invariant. If* $\text{Fix}(T)$ *is w^*-closed for each T in $\mathscr{S}$ then* $\text{Fix}(\mathscr{S}) \neq \emptyset$.

Proof. Let K be a minimal nonempty w^*-compact $\mathscr{S}$-invariant set (which exists by the assumptions on C). Let δ be the diameter of K. Fix T in $\mathscr{S}$ and let K_T be a minimal w^*-compact, T-invariant, nonempty subset of K of diameter δ_T. By the construction of Lemma 1 as applied in

Theorem 2 we see that T has a fixed point z in $\bigcap \{B(x; \delta_T) : x \in K_T\}$. For a point y in K we have

$$d(z, y) \leqslant d(z, x) + d(x, y) \leqslant \delta_T + \delta \leqslant 2\delta$$

when x is in K_T. Thus $\mathrm{Fix}(T) \cap B(y; 2\delta) \neq \emptyset$ for any T in $\mathscr{S}$.

Fix y in K and let $E = \bigcup \{B(Ty; 2\delta) : T \in \mathscr{S}\}$. Then E is bounded and $\mathscr{S}$-invariant. Also for any T in $\mathscr{S}$ the set $\mathrm{Fix}(T) \cap E$ is nonemtpy. Let $T_1, ..., T_N$ be in $\mathscr{S}$ and let $T = T_1 T_2 \cdots T_N$. Then the set $\mathrm{Fix}(T) \cap E$ is nonempty and the constructions of Theorem 5 and Lemma 1 show that the set

$$\bigcap \{\mathrm{Fix}(T_i) : 1 \leqslant i \leqslant N\} \cap (E + \beta)$$

is nonempty where β is the diameter of E (and $E + \beta$ is the β-parallel set of E). Thus there is a bounded set D so that D meets $\bigcap \{\mathrm{Fix}(T_i) : 1 \leqslant i \leqslant N\}$ for each N. We can replace D with a larger closed ball. The ball is w^*-compact so the w^*-closed assumption now yields $\bigcap \{\mathrm{Fix}(T) : T \in \mathscr{S}\} \neq \emptyset$ by the finite intersection property.

Remark. $\mathrm{Fix}(T)$ will be w^*-closed under the (very unreasonable) assumption that T is w^*-continuous.

Problem. Can the assumption that $\mathrm{Fix}(T)$ be w^*-closed be removed from Theorem 7? The situation is analogous to common fixed points for commuting families gives some normal structure. Belluce and Kirk [BK] showed if X is an arbitrary Banach space and C is a weakly compact, nonempty, convex set with normal structure, invariant under a finite commuting family of nonexpansive maps, then there is a common fixed point. The question for arbitrary commuting families remained open for some time until resolved in the affirmative by Lim [Li]. This last result, together with Baillon's result [B], is included as special cases of a recent abstract approach to the problem [KP]. Another result on common fixed points under different hypotheses was given by Bruck [Br].

DEFINITION. Let T be a map on a metric space M. A point x is an ε-approximate fixed point for T if $d(Tx, x) \leqslant \varepsilon$. We will denote the set of ε-approximate fixed points of T by $F_\varepsilon(T)$.

If T is nonexpansive on a hyperconvex space M and $F_\varepsilon(T) \neq \emptyset$ then $F_\varepsilon(T)$ is itself hyperconvex (and hence a nonexpansive retract of M) [S_2].

THEOREM 8. *Let T be a nonexpansive map with bounded orbits on a hyperconvex space. For x in M, set $\delta_x = \mathrm{dia}\{T^n x : n \geqslant 0\}$. Then*

$$F_\varepsilon(T) \cap B(x; \tfrac{3}{2}\delta_x) \neq \emptyset$$

for every $\varepsilon > 0$.

Proof. It is well known that M is isometric to a subset of $l_\infty(M)$. Since the assertion and the assumptions of the theorem are invariant under isometries we may assume M is contained in $l_\infty(I)$ for some index set I. Since M is hyperconvex there is a nonexpansive retraction R of $l_\infty(I)$ onto M. Let $\hat{T} = TR$. Then $\hat{T}: l_\infty(I) \to M \subset l_\infty(I)$ is nonexpansive with bounded orbits (since $\hat{T}^n = T^n R$). Let $A_m = \bigcap \{B(T^n x; 1/2\,\delta_x) : m \leqslant n < \infty\}$. By hyperconvexity of $l_\infty(I)$ we see that $A_m \neq \emptyset$ and clearly this is a closed convex set in $l_\infty(I)$. Now $\{A_m\}$ is an increasing sequence, and $\hat{T}(A_m) \subset A_{m+1}$, so $\hat{T}(\bigcup \{A_m : 0 \leqslant m < \infty\}) \subset \{A_m : 0 \leqslant m < \infty\}$. By continuity of $\hat{T}$, the set K defined by closure $\bigcup \{A_m : 0 \leqslant m < \infty\}$ is $\hat{T}$ invariant and is clearly closed and convex. But $A_m \subset B(x; 3/2\,\delta_x)$ so K must be contained in this ball as well. Again we can use the Banach contraction principle to see that for each $\varepsilon > 0$ there is a y_ε in K with $\|\hat{T}y_\varepsilon - y_\varepsilon\| \leqslant \varepsilon$. Let $x_\varepsilon = \hat{T}y_\varepsilon$ so that x_ε is in both M and K. Thus x_ε is in $F_\varepsilon(T) \cap [B(x; (3/2)\,\delta) \cap M]$ which is the assertion of the theorem.

Remark. For T nonexpansive on a closed convex set in an arbitrary Banach space a similar result holds but the estimate is not as good. We give that result next for comparison purposes. The argument is quite different.

PROPOSITION 9. *Let $\mathscr{S}$ be an abelian semigroup of nonexpansive maps on a closed convex set C of a Banach space having $\mathscr{S}$-bounded orbits. For x in C let $\delta_x = \operatorname{dia}\{T_x : T \in \mathscr{S}\}$. Then $B(x; 2\delta_x)$ contains a nonempty closed convex set K which is $\mathscr{S}$-invariant.*

Proof. Let λ be an invariant mean on the semigroup $\mathscr{S}$. Define K to be $\{y \in C : \lambda(\|Tx - y\|) \leqslant \delta\}$ with $\delta = \delta_x$. Clearly x is in K. If y is in K then $\inf\{\|Tx - y\| : T \in \mathscr{S}\} \leqslant \delta$ by the positivity of λ. Hence $\|y - x\| \leqslant \inf\{\|y - Tx\| + \|Tx - \alpha\| : T \in \mathscr{S}\} \leqslant 2\delta$. It is not difficult to check that K is closed and convex. Let y be in K and T_0 in $\mathscr{S}$. Then, using the translation invariance of λ, we obtain

$$\lambda(\|Tx - T_0 y\|) = \lambda(\|T_0 Tx - T_0 y\|) \leqslant \lambda(\|Tx - y\|) \leqslant \delta.$$

Thus K is $\mathscr{S}$-invariant.

THEOREM 10. *Let $T_1, T_2, \ldots, T_N$ be a commuting family of nonexpansive maps of a hyperconvex space M, such that each T_i has bounded orbits. Then for every $\varepsilon > 0$ we have that $\bigcap \{F_\varepsilon(T_i) : 1 \leqslant i \leqslant N\}$ is a nonempty hyperconvex subset of M.*

Proof. For $N = 1$ we have $F_\varepsilon(T_1) \neq \emptyset$ by Theorem 8 and the hyperconvexity by [S_2]. We proceed by induction on N. Suppose $H = \bigcap \{F_\varepsilon(T_i) : 1 \leqslant i \leqslant N - 1\}$ is nonempty and hyperconvex. But by commutivity, T_N leaves $F_\varepsilon(T_i)$ invariant for $1 \leqslant i \leqslant N-1$. Thus H is invariant under T_N. So we can apply Theorem 8 and [S_2] to the restriction of T_N to H to give the result.

Remark. Theorem 10 cannot be generalized to a sequence $\{T_i : i \geqslant 1\}$ of commuting nonexpansive maps with bounded orbits (under the generated semigroup). For the counterexample we take T on l_∞ to be Prus' map. The diameter of any orbit is at least 1. For if $x = \{x_k\}$ is in l_∞, we take $c = \lambda\{a_k\}$. The first k coordinates of $T^k x$ are $1+c$. Let δ be the diameter of $\{T^n x\}$. Then $\|T^k x - x\| \leqslant \delta$ which implies $|1+c-a_k| \leqslant \delta$. Thus $1+c-\delta \leqslant a_k \leqslant 1+c+\delta$ for every k which, in turn, implies $1+c-\delta \leqslant \lambda(\{a_n\}) = c$. Hence $\delta \geqslant 1$. But then for $\varepsilon < 1$ there are no *common* ε-approximate fixed points of $\{T^n\}$.

EXAMPLE. A modification of the Prus map can be used to answer another open question. If $J = \bigcap B(x_\alpha : r_\alpha)$ is a nonempty ball intersection in a hyperconvex space M and w is a point in M, then there exists a point y in J so that $d(w, J) = d(w, y)$. This is an easy ball intersection argument. The question is whether there is still such a point y if the ball intersection is replaced with a hyperconvex subset H of M. We will show that the answer is no. Recall that the shift operation on the positive integers N induces a homomorphism h of $\beta N \backslash N$. Banach limits correspond to probability measures on $\beta N \backslash N$ which are h-invariant. There will be minimal nonempty closed invariant subsets of $\beta N \backslash N$ for the dynamical system. It is known that each such minimal set, D, supports an uncountable set of invariant probability measures [A1]. If we take λ_1 and λ_2 to be distinct extreme invariant probabilities supported on D then λ_1 and λ_2 have the same closed support in $\beta N \backslash N$ but also have disjoint measurable supports. Let $\sigma = (1/2)(\lambda_1 - \lambda_2)$. We now define T on $l_\infty(N)$ by

$$T(\{x_1, x_2, ...\}) = \{\sigma(x), x_1, x_2, ...\}.$$

Then for $w = \{1, 0, 0, ...\}$ it is not difficult to see that $d(w, Tl_\infty) = 1/2$ but this distance is not achieved for σ is a norm 1 functional which does not achieve its norm. As T is an isometry, the set H, defined to be Tl_∞, is hyperconvex.

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