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NORMAL STRUCTURE FOR BANACH SPACES WITH SCHAUDER DECOMPOSITION

BY

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ABSTRACT. We introduce a new constant in Banach spaces which implies, in certain cases, the weak- or weak*-normal structure.

Introduction. M. S. Brodskii and D. P. Mil'man [3] have introduced a geometric property, called normal structure, for convex subsets of Banach spaces. This property was introduced into fixed point theory by W. A. Kirk [10]. In this paper we associate to every Banach space, with Schauder finite decomposition, a constant which may be easily calculated. This constant allows us to decide, in certain cases, if the Banach space has weak-normal structure (alternatively weak*-normal structure in the dual case). For details on normal structure and its generalizations in Banach spaces, we suggest the survey of S. Swaminathan [17].

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Definitions and notations. In this paper X will always denote a real Banach space with F.D.D. (see definition 6). For terms not explicitly defined reference may be made to Day's book [6].

DEFINITION 1. *A point x of a convex bounded set C is called a non-diametral point for C if*

$$\sup\{\|x-y\| : y \in C\} < \operatorname{diam}(C).$$

DEFINITION 2. *A sequence (x_n) is called diametral if $c = \operatorname{diam}(x_i) > 0$ and*

$$\lim_{n\to\infty} d(x_n, \operatorname{conv}\{(x_i)) : i < n\} = c$$

It follows that for every $x \in \operatorname{conv}(x_n)$, we have $\lim \|x_n - x\| = c$. Every $x \in \operatorname{conv}(x_n)$ is a diametral point, any subsequence of (x_n) is also diametral and is not convergent.

DEFINITION 3. *We say that X has normal structure if every bounded convex subset K of X, which contains more than one point, has a non-diametral point. Restricting*

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K to be weak (respectively weak) compact, we similarly define weak (respectively weak*) normal structure.*

Let us recall the most useful characterization of normal structure [3].

THEOREM 1. *A bounded convex subset K of X contains a non-diametral point if and only if K does not contain a diametral sequence.*

DEFINITION 4. *An ultrapower $\tilde{X}$ of X is defined by $\tilde{X} = \ell_\infty(X)/\mathcal{N}$ with $\ell_\infty(X) = \{(x_n \in X$ and $\sup \|x_n\| < \infty\}$ and $\mathcal{N} = \{(x_n) \in \ell_\infty(X); \lim_{\mathcal{U}} \|x_n\| = 0\}$, where $\mathcal{U}$ is a free ultrafilter over $\mathbf{N}$. Let (P) be a property on X. We say that X has the property super-(P) if every ultrapower of X has the property (P).*

For more details about ultrapower spaces, we refer to [15].

DEFINITION 5. *A Banach space F is called a spreading model for X, generated by the bounded sequence (x_n) if the following hold: (a) There exists a sequence (e_n) such that F is the closed linear space of $\{e_n; n \in \mathbf{N}\}$. (b) There exists a subsequence (y_m) of (x_n) so that (y_m) has no norm convergent subsequence and for all $n \in \mathbf{N}$ and scalars $(\alpha_i)_{1 \leqq i \leqq n}$, we have*

$$\left\| \sum_{i=1}^{i=n} \alpha_i e_i \right\|_F = \lim \left\{ \left\| \sum_{i=1}^{i=n} \alpha_i y_{m_i} \right\| : m_1 < \ldots < m_n, m_1 \longrightarrow \infty \right\}.$$

Let (P) be a property on X. We say that X has the property $M-(P)$ if every spreading model for X has the property (P).

For more information about the spreading model we refer to [2,4,7].

DEFINITION 6. *A sequence (X_n) of finite dimensional subspaces of X is called a Schauder finite dimensional decomposition (F.D.D) of X, if every $x \in X$ has unique representation of the form $x = \Sigma x_i$ with $x_i \in X_i$ for every $i \in \mathbf{N}$.*

Let $x \in X$, we defined $\text{supp}(x)$ to be the set of integers $i \in \mathbf{N}$ such that $x_i \neq 0$. Let A and B be two subsets of $\mathbf{N}$ and $k \in \mathbf{N}$, we will write $A < B$(resp. $A < B + k$) if for every $(a, b) \in A \times B$, we have: $a < b$ (resp. $a < b + k$).

For more details about F.D.D. and Schauder bases, we suggest [14].

Examples and basic results.

DEFINITION 7. *Let X be a Banach space, with a F.D.D. Define $\beta_p(X)$, for $p \in [1, \infty]$, to be the infinimum of the set of numbers λ such that*

$$\left(\|x\|^p + \|y\|^p\right)^{1/p} \leqq \lambda \|x + y\|$$

for every x and y in X which verify $\text{supp}(x) < \text{supp}(y)$.

EXAMPLES: 1. The space $\ell_{p,q}$ defined in [5], which is simply ℓ_p renormed by:

$$|x| = \left(\|x^+\|_{\ell_p}^q + \|x^-\|_{\ell_p}^q\right)^{1/q}$$

where x^+ and x^- are the positive and negative parts of x in the lattice structure of ℓ_p. W. Bynum has shown that $\ell_{p,1}$, using the constant WCS (see definition in [5]) which seems to be more difficult to compute, has normal structure.

2. The James space J which consists of sequences $x = (x_n) \in c_0$ such that

$$\|x\| = \sup\left\{(x_{p_1} - x_{p_1})^2 + (x_{p_2} - x_{p_3})^2 + \cdots + (x_{p_{n-1}} - x_{p_n})^2\right\} \text{ is finite,}$$

where the supremum is taken for every n and every increasing sequence of integers (p_i). Recall that J fails normal structure. Nevertheless it is shown in [1] that for every u and v which verify $\text{supp}(u) + 1 < \text{supp}(v)$, we have

$$\left(\|u\|^2 + \|v\|^2\right)^{1/2} \leqq \|u + v\|.$$

We deduce that $\beta_2(J) = 1$.

REMARK. Sometimes the term $(x_{p_m} - x_{p_1})$ is added in the definition of $\|x\|$, and then we obtain a new space J_1 which is isomorphic to J. In [9] the author proves that any weakly-compact convex subset K of J_1 has the fixed point property, i.e. any selfmapping T defined on K has a fixed point, provided that T satisfies $\|T(x) - T(y)\| \leqq \|x - y\|$ for every x, y in K. It is not clear whether J_1 has weak-normal structure.

PROPOSITION 1. a) *Let $p \in [1, \infty]$ and $q > p$, we have*

$$1 \leqq \beta_q(X) \leqq \beta_p(X) \leqq 2^{(q-p)/pq}\beta_q(X).$$

b) *We have $\beta_p(\ell_p) = 1$ and $\beta_1(\ell_{p,1}) = 2^{1-1/p}$.*

PROOF. a) For every pair of scalars (α, β) we have:

$$\left(|\alpha|^q + |\beta|^q\right)^{1/q} \leqq \left(|\alpha|^p + |\beta|^p\right)^{1/p} \leqq 2^{(q-p)/pq}\left(|\alpha|^q + |\beta|^q\right)^{1-q}$$

from which we obtain our inequalities on β_p and β_q.

b) Let $(u, v) \in \ell_p$ be such that $\text{supp}(u) < \text{supp}(v)$. Then $\text{supp}(u^+) < \text{supp}(v^+)$ and $\text{supp}(u^-) < \text{supp}(v^-)$. This implies that (*)$\|u + v\| = (\|u\|^p + \|v\|^p)^{1/p}$, $\|u^+ + v^+\|^p = \|u^+\|^p + \|v^+\|^p$ and $\|u^- + v^-\|^p = \|u^-\|^p + \|v^-\|^p$. On the other hand we have, by definition of the norm $|\cdot|$, $|u| + |v| = (\|u^+\| + \|u^-\|) + (\|v^+\| + \|v^-\|)$. Thus $|u| + |v| \leqq 2^{1-1/p}[(\|u^+\|^p + \|v^+\|^p)^{1/p} + (\|u^-\|^p + \|v^-\|^p)^{1/p}] = 2^{1-1/p}(\|u^+ + v^+\| + \|u^- + v^-\|)$. As the hypothesis on u and v implies that $(u + v)^+ = u^+ + v^+$ and $(v + u)^- = u^- + v^-$, we have (**) $|u| + |v| \leqq 2^{1-1/p}|u + v|$. We conclude by (*) and (**) that $\beta_p(\ell_p) = 1$ and $\beta_1(\ell_{p,1}) \leqq 2^{1-1/p}$. If we take $u = e_1$ and $v = e_2$, we obtain $2^{1-1/2} \leqq \beta_1(\ell_{p,1})$. □

The following theorem studies the relation between the constant β_p and the Banach-Mazur distance (for isomorphic Banach spaces X and Y, the Banach-Mazur distance from X to Y, denoted $d(X, Y)$, is defined to be the infinimum of $\|U\|\,\|U^{-1}\|$ taken over all bicontinuous linear operators U from X onto Y).

THEOREM 2. *Let X and Y be two isomorphic Banach spaces. Suppose that X has a F.D.D. Then Y also has a F.D.D. and we have*

$$\beta_p(Y) \leqq d(X,Y)\beta_p(X).$$

PROOF. Our theorem will be deduced from the following obvious lemma.

LEMMA 1. *Let $(X, \|\cdot\|)$ be as in Theorem 2, and $|\cdot|$ be an equivalent norm on X such that $\|\cdot\| \leqq |\cdot| \leqq \lambda\|\cdot\|$, then $\beta_p(X, |\cdot|) \leqq \lambda\beta_p(X, \|\cdot\|)$.*

Indeed let $\epsilon > 0$ and $U : X \longrightarrow Y$ be an isomorphism such that $\|U^{-1}\| = 1$ and $\|U\| = d_\epsilon$ where $d_\epsilon \leqq d + \epsilon$. Therefore we have for every $x \in X$

$$(^*) \quad \|x\|_X \leqq \|U(x)\|_Y \leqq d_\epsilon \|x\|_X.$$

Let (X_1) be the F.D.D. of X, if we put $Y_i = U(X_i)$, we obtain a F.D.D. for Y which satisfies the following property (**) $\operatorname{supp}(y) = \operatorname{supp}(U^{-1}(y))$ for every y in Y. Given (*) and (**) and the lemma the conclusion of Theorem 2 is deduced.

We obtain the following result as an application of Theorem 2.

COROLLARY 1. (a) *Let $|\cdot|$ be an equivalent norm on c_0, we have $2^{1/p} \leqq \beta_p(c_0, |\cdot|)$.* (b) *Let X be a Banach space which contains a subspace isomorphic to c_0, then $2^{1/p} \leqq \beta_p(X)$.*

PROOF. Let us first show how (b) can be deduced from (a). Let (e_i) be the basis of c_0. Then (e_i) converges weakly to 0 in X. By a classical argument, for $\epsilon > 0$ there exists in X a sequence (u_i) of blocks such that $d([u_i], c_0) \leqq 1+\epsilon$, where $[u_i]$ is the closed linear span of $\{u_i; i \in \mathbf{N}\}$. Therefore by (a) we deduce that $2^{1/p} \leqq \beta_p(c_0) \leqq (1+\epsilon)\beta_p([u_i])$. As (u_i) are successive blocks in X, we have: $\beta_p([u_i]) \leqq \beta_p(X)$. This gives the conclusion of (b), since ϵ is arbitrary. Let us complete the proof by proving (a). In [8] it is shown that if $\|\cdot\|$ is an equivalent norm on c_0, then given $\epsilon > 0$, there exists u and v in c_0 which satisfy $\operatorname{supp}(u) < \operatorname{supp}(v)$ and $\operatorname{Max}(|\alpha|, |\gamma|) \leqq \|\alpha u + \gamma v\| \leqq \operatorname{Max}(|\alpha|, |\gamma|)(1+\epsilon)$ for all scalars α and γ. The definition of β_p implies $(\|u\|^p + \|v\|^p)^{1/p} \leqq \beta_p(c_0, \|\cdot\|)\|u+v\|$. Therefore we obtain $2^{1/p}(1+\epsilon)^{-1} \leqq \beta(c_0, \|\cdot\|)$. As ϵ is arbitrary, the conclusion of (a) holds. □

Main result.

THEOREM 3. *Let X be a Banach space, with a finite codimensional subspace Y such that $\beta_p(Y) < 2^{1/p}$ for some $p \in [1, \infty]$. Then X has weak-normal structure.*

PROOF. Suppose that X fails weak-normal structure. Then there exists a weakly convergent diametral sequence (x_n). Since diametral property is invariant by translation, we may suppose that (x_n) converges weakly to 0. Our hypothesis on Y implies that there exists Z, a finite dimensional subspace of X, such that $X = Y \oplus Z$. Therefore

$x_n = y_n + z_n$ with $y_n \in Y$ and $z_n \in Z$ for every $n \in \mathbf{N}$. Since (x_n) converges weakly to 0, we deduce that both (y_n) and (z_n) converge weakly to 0. Using the fact that Z has a finite dimension, (z_n) converges (in norm) to 0 in X. Therefore there exists a sequence (u_n) of successive blocks and a subsequence (y_{m_n}) such that $\lim \|y_{m_n} - u_n\| = 0$. The definition of $\beta_p(Y)$ implies that

$$\left(\|u_{n+1}\|^p + \|u_n\|^p\right)^{1/p} \leqq \beta_p(Y)\|u_{n+1} - u_n\|.$$

But $\lim \|u_n\| = \lim \|y_{m_n}\| = \lim \|x_{m_n} - z_{m_n}\| = \lim \|x_{m_n}\| = \dim(x_i) = c$ and $\lim \|u_{n+1} - u_n\| = \lim \|y_{m_{n+1}} - y_{m_n}\| = \lim \|x_{m_{n+1}} - x_{m_n}\| = \operatorname{diam}(x_i) = c$. We deduce that $2^{1/p} \leqq \beta_p(Y)$ since $c > 0$. This yields a contradiction to our hypothesis on Y. □

COROLLARY 2. *Let X be a Banach space isomorphic to ℓ_p. Assume that $d(X, \ell_p) < 2^{1/p}$, then X has normal structure.*

PROOF. Since X is reflexive, the weak-normal structure and normal structure are the same. Theorem 2 implies that $\beta_p(X) \leqq d(X, \ell_p)\beta_p(\ell_p)$. Since $\beta_p(\ell_p) = 1$ we obtain $\beta_p(X) \leqq d(X, \ell_p) < 2^{1/p}$. The conclusion follows from Theorem 2.

REMARK. Let us note that the number $2^{1/p}$ is optimum in the sense that for every $p \in]1, \infty[$ there exists X_p such that $d(X_p, \ell_p) = 2^{1/p}$ and X_p fails normal structure. Indeed let $\epsilon > 0$ and define an equivalent norm $|\cdot|_\epsilon$ on ℓ_p by $|x|_\epsilon = \mathrm{Max}(\|x\|_p, \epsilon\|x\|_1)$. Consider the space $X_\epsilon = (\ell_p, |\cdot|_\epsilon)$. It is clear that $d(X_\epsilon, \ell_p) = \mathrm{Max}(1, \epsilon)$. Let us notice that if $2^{1/p} \geqq \epsilon$ then the canonical basis of ℓ_p is diametral. The desired space is $X_{2^{1/p}}$.

The next result gives a partial positive answer to T. Landes' problem [12].

PROPOSITION 2. *Let (X_i) be a sequence of Banach spaces such that*

$$\beta = \sup_{i \in \mathbf{N}} \beta_1(X_i) < 2.$$

Then $X = \otimes_1 X_i$ has weak-normal structure.

PROOF. Suppose that X fails weak-normal structure. Then there exists a weakly convergent diametral sequence (x_n) in X. We may suppose, with loss of generality, that (x_n) converges weakly to 0. By definition of X we have $x_n = \Sigma x_i(n)$. Since (x_n) converges weakly to 0., we deduce that $(x_i(n))$ converges weakly to 0 in X_i for every $i \in \mathbf{N}$. By passing to a subsequence, we may assume that $(x_i(n))$ is a sequence of blocks, related to the F.D.D. of X_i, for every $i \in \mathbf{N}$. Our hypothesis on (X_i) implies that

$$\|x_i(n+1)\|_{X_i} + \|x_i(n)\|_{X_i} \leqq \beta\|x_i(n+1) - x_i(n)\|_{X_i} \quad \text{for every } i \in \mathbf{N}.$$

Then by definition of the norm of X, we get

$$\|x_{n+1}\|_X + \|x_n\|_X \leqq \beta\|x_{n+1} - x_n\|_X.$$

Since (x_n) is diametral and convergent weakly to 0 (this implies that $0 \in \overline{\text{conv}}(x_i)$ and therefore $\lim \|x_n\| = \text{diam}(x_i)$), we get $2 \leqq \beta$. This is a contradiction. □

Our last theorem concerns the M-(normal structure)-property (see definition 5).

THEOREM 4. *Let X be a reflexive Banach space. Assume that $\beta_p(X) < 2^{1/p}$, for some $p \in [1, \infty[$. Then X has M-(weak-normal structure).*

PROOF. Let (x_n) be a bounded sequence and F be the spreading model generated by (x_n). Since X is reflexive we may suppose that (x_n) is weakly convergent to x. *First case:* $x = 0$. By a classical argument, there exists a sequence (u_i) of successive blocks and a subsequence $(x_{n(i)})$ such that $\lim \|u_i - x_{n(i)}\| = 0$. This implies that the spreading model generated by (u_i) is F. Since (x_n) converges weakly to 0, the sequence (e_i) is a Schauder basis of F, (see [7]). Let u and v be in F which satisfy: $\text{supp}(u) < \text{supp}(v)$. We write

$$u = \sum_{i \leqq a} c_i e_i \text{ and } v = \sum_{a < i \leqq b} b_i e_i.$$

Therefore

$$\|u+v\|_F = \lim \left\{ \| \sum c_i u_{n_i} + \sum b_j u_{m_j} \|_X : n_1 < \ldots < n_a < m_1 < \ldots < m_b, n_1 \longrightarrow \infty \right\}.$$

Hence $(\|u\|_F^p + \|v\|_F^p)^{1/p} \leqq \beta_p(X)\|u + v\|_F$, which implies that $\beta_p(F) \leqq \beta_p(X)$. The hypothesis on $\beta_p(X)$ implies that F has weak-normal structure. *Second case:* $x \neq 0$. It is shown in [7] that in this case F is isometric to a subspace of G with $G = F_0 \oplus \mathbf{R}$, where F_0 is the spreading model generated by the sequence $(x_n - x)$. The first case implies that $\beta_p(F_0) < 2^{1/p}$ and by Theorem 3 we deduce that G has weak-normal structure. Since F is isometric to a subspace of G, we conclude that F has weak normal structure.

COROLLARY 3. *The M-(normal structure) is not a self dual property.*

PROOF. Indeed let $X = \ell_{p,1}$ with $p \in]1, \infty[$. We have shown that $\beta_1(X) = 2^{1/q}$ where q is the conjugate to p (i.e. $1/p + 1/q = 1$). By Theorem 4, X has M-weak-normal structure. Since X is superreflexive, X has M-(normal structure) whereas $X^* = \ell_{q,\infty}$ fails to have normal structure.

REMARK. We don't know if $\ell_{p,1}$ has super-normal structure.

The constant β_p and weak*-normal structure. Let X be a Banach space with a shrinking Schauder basis (x_i). Recall that X^* has a basis (x_i^*) which is boundedly complete ((x_i^*) are the biorthogonal functionals associated to the basis (x_i)). Let Y be a Banach space, isomorphic to X^*. In this paper, the weak*-topology considered on Y is the weak*-topology defined by the couple (X, X^*). This implies, in particular, that if (y_i) converges weak* to 0 (in Y), then there exists a sequence (u_i) of successive blocks and a subsequence $(y_{n(i)})$ such that $\lim_{i \to \infty} \|y_{n(i)} - u_i\| = 0$. The same proof, as in Theorem 3, gives us the following result:

THEOREM 5. *Let X be a Banach space with a shrinking basis. Assume that $\beta_p(X^*) < 2^{1/p}$. Then X^* has weak*-normal structure.*

An application of Theorem 4 is Lim's theorem [13] which says that ℓ_1 has weak*-normal structure. Let us recall that P. Soardi [16] proved that if X is a Banach space which satisfies $d(X, \ell_1) < 2$, then X has the weak*-fixed point property, i.e., any weak*-compact convex subset K of X and any selfmapping T defined on K has a fixed point, provided that T satisfies $\|T(x) - T(y)\| \leqq \|x - y\|$ for every x, y in K. Kirk [11] asked whether X has weak*-normal structure.

COROLLARY 4. *Let Y be a Banach space which is isomorphic to ℓ_1. Assume that $d(Y, \ell_1) < 2$. Then Y has weak*-normal structure.*

REMARK. Here, we consider on Y, isomorphic to ℓ_1, the weak*-topology given by the couple (c_o, ℓ_1). Recall that there exists a Banach space X such that X^* is linearly isometric to ℓ_1 and fails the weak*-fixed point property and therefore cannot have weak*-normal structure (in this case the weak*-topology is related to the couple (X, X^*)). The space X is the Banach space c which consists of all convergent sequences.

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