

ON NORMAL STRUCTURE, FIXED POINT PROPERTY AND CONTRACTIONS OF TYPE (γ)

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ABSTRACT. We prove that a Banach space X has normal structure provided it contains a finite codimensional subspace Y such that all spreading models for Y have normal structure. We show that a Banach space X is strictly convex if the set of fixed points of any nonexpansive map defined in any convex subset $C \subset X$ is convex and give a sufficient condition for uniform convexity of a space in terms of nonexpansive map of type (γ) .

INTRODUCTION

Our aim is to study the normal structure for Banach spaces. We recall that a normed space X is said to have normal structure (N.S.) if and only if every convex bounded set in X which contains more than one point has a nondiametral point.

This property was introduced by Brodskii and Mil'man [3] who also gave the following characterization in terms of sequences: a space X has N.S. if and only if X contains no bounded diametral sequence. For normal type structures and their applications to fixed point property, we refer to the exhaustive survey of Kirk [9].

The first theorem of this paper states that if Y is a finite codimensional subspace of X which has the M -normal structure, then X has the normal structure. We also generalize a theorem of Zizler [13] on spaces which are uniformly convex in every direction and prove a converse to a theorem of Bruck [4] on nonexpansive mappings of type (γ) .

DEFINITIONS AND NOTATIONS

In this paper X will always denote a real Banach space; $\text{sp}(A)$ denotes the subspace generated by A (subset of X) and we will say A is finite dimensional if $\text{sp}(A)$ is a finite dimensional subspace of X . $S(X)$ denotes the unit sphere of X . For terms not explicitly defined reference may be made to Day's book [5].

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Definitions. (1) A point x of a convex bounded set C is called a nondiametral point of C iff $\sup_{y \in C} \|x - y\| < \text{diam } C$.

(2) A sequence (x_n) is called diametral if

$$c = \text{diam}(x_i) > 0 \quad \text{and} \quad \lim_n d(x_{n+1}, \text{conv}_{i \leq n}(x_i)) = c.$$

It follows that

* For every $x \in \text{conv}(x_n)$, we have: $\lim_n \|x_n - x\| = c$.

* If (x_n) is diametral any subsequence of (x_n) is also diametral and is not convergent. The diametral property is invariant by translation.

(3) We say that X is uniformly convex in the direction $z \in S(X)$ if for each $\varepsilon > 0$

$$\delta(\varepsilon, z) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| ; \|x\| = \|y\| = 1 \text{ and } x - y = \varepsilon z \right\} > 0.$$

We say that X is uniformly convex if $\delta_X(\varepsilon) = \inf\{\delta(\varepsilon, z); z \in S(X)\} > 0$ for every $\varepsilon > 0$.

(4) The ultrapower \mathbf{X} of X is defined by $\mathbf{X} = 1_\infty(X)/\mathcal{N}$, where $1_\infty(X) = \{(x_n); x_n \in X \text{ and } \text{Sup}_n \|x_n\| < +\infty\}$ and $\mathcal{N} = \{(x_n) \in 1_\infty(X); \lim_{\mathcal{U}} \|x_n\| = 0\}$ where \mathcal{U} is a free ultrafilter over N .

Let (P) be a property on X . We say that X has the property super-(P) if every subspace of \mathbf{X} has the property (P). For more details about ultrapowers, we refer to [11].

(5) Let $\Gamma = \{\gamma: R_+ \rightarrow R_+; \gamma \text{ continuous and } \gamma(r) = 0 \text{ iff } r = 0\}$. We say that $T: C \rightarrow X$ is a nonexpansive map if $\|T(x) - T(y)\| \leq \|x - y\|$ for every x, y in C . The fixed point set of T will be denoted by F_T . We will say that T is of type (γ) if there exists γ in Γ such that for every x, y in C

$$(**) \quad \gamma \left(\left\| T \left(\frac{x+y}{2} \right) - \frac{T(x) + T(y)}{2} \right\| \right) \leq \|x - y\| - \|T(x) - T(y)\|.$$

Denote $\gamma(T) = \{\gamma \in \Gamma; \text{ such that } (**) \text{ holds}\}$.

(6) A Banach space F is called a spreading model of X , generated by the bounded sequence (x_n) , if the following hold.

(a) $X \subset F$ and there is a sequence (e_n) in F with F the closed linear span of $X \cup \{e_n, n \in N\}$

(b) There exists a subsequence (y_n) of (x_n) so that (y_n) has no norm convergent subsequence and for all $x \in X$, $n \in N$ and scalars $(c_i)_{1 \leq i \leq n}$ we have

$$(*) \quad \left\| x + \sum_1^n c_i e_i \right\| = \lim_{\substack{m_1 < m_2 < \dots < m_n \\ m_i \rightarrow \infty}} \left\| x + \sum_1^n c_i y_{m_i} \right\|.$$

Let (P) be a property on X . We say that X has the property M -(P) if X and every spreading model for X has the property (P). For more information about spreading model and M -properties we refer to [2].

(7) Let X and Y be Banach spaces and consider the space $X \oplus Y$. A norm N on $X \oplus Y$ is said to be of type (L) if:

- (i) The restrictions of N to X and Y are the initial norms of X and Y .
- (ii) The natural projections associated to $X \oplus Y$ have norm 1.
- (iii) For every x, x' in X and y in Y we have $N(x + y) \leq N(x' + y)$ implies $\|x\| \leq \|x'\|$.

NORMAL STRUCTURE AND THE FIXED POINT PROPERTY

It is well known that geometrical properties of finite codimensional subspaces provide for very little information about the normal structure properties of spaces. However we shall see below that some positive results can be obtained.

Theorem 1. *Let X be a Banach space, and Y be a finite codimensional subspace of X . Suppose that Y has M -normal structure. Then X has normal structure.*

Proof. Suppose that X fails to have normal structure. Then X contains a diametral sequence (x_n) . Our assumptions on Y imply that there exists a finite dimensional subspace Z of X such that $X = Z \oplus Y$. Then we can write $x_n = z_n + y_n$ with y_n in Y and z_n in Z . Since (x_n) is bounded, we deduce that (z_n) has a convergent subsequence (z_{n_k}) . It follows that for every positive scalar (c_1, \dots, c_k) which verify $\sum_1^k c_i = 1$, we have

$$\lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| y_{n_{k+1}} - \sum_1^k c_i y_{n_i} \right\| = \lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| x_{n_{k+1}} - z_{n_{k+1}} - \sum_1^k c_i (x_{n_i} - z_{n_i}) \right\|.$$

But

$$\lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| z_{n_{k+1}} - \sum_1^k c_i z_{n_i} \right\| = 0$$

and

$$\lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| x_{n_{k+1}} - \sum_1^k c_i x_{n_i} \right\| = \text{diam}(x_n).$$

Then

$$\lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| y_{n_{k+1}} - \sum_1^k c_i y_{n_i} \right\| = c = \text{diam}(x_n).$$

Let F be the spreading model of Y , generated by (y_{n_k}) , and (e_i) the sequence given by the definition 6. We have

$$\left\| e_{k+1} - \sum_1^k c_i e_i \right\|_F = \lim_{\substack{n_1 < \dots < n_{k+1} \\ n_1 \rightarrow \infty}} \left\| y_{n_{k+1}} - \sum_1^k c_i y_{n_i} \right\| = c$$

for every scalar c_1, \dots, c_k such that $\sum_1^k c_i = 1$.

We conclude that (e_i) is a diametral sequence in F . This implies that F fails to have N.S. Contradiction. \square

Remark 1. If we refine the proof we can get the following. Suppose that $X = Y \oplus Z$ where Z is a Schur space and Y has M -normal structure. Then every weakly compact convex subset C has a nondiametral point.

Our next corollary can also be deduced from [7].

Corollary 1. *Let X be a Banach space, and Y be a finite codimensional subspace of X . Suppose that Y has super normal structure. Then X has super normal structure.*

Proof. It is enough to prove that any ultrapower of X has N.S. Our hypothesis on Y implies that there exists a continuous projection $P: X \rightarrow Y$. If we consider \mathbf{X} and \mathbf{Y} the ultrapowers of X and Y , we define the canonical projection $\mathbf{P}: \mathbf{X} \rightarrow \mathbf{Y}$ by $\mathbf{P}((x_n)) = (P(x_n))$. It is obvious that the kernel of \mathbf{P} is a finite dimensional subspace of \mathbf{X} . Then \mathbf{Y} is a finite codimensional subspace of \mathbf{X} which has the super normal structure. But it is well known [2] that every spreading model is isometric to a subspace of an ultrapower. Therefore according to Theorem 1, \mathbf{X} has normal structure. \square

Proposition 2. *M -normal structure and super normal structure are not equivalent.*

Proof. Consider $E_n = l_1^n$ and X the l_2 -sum of (E_n) . It is proved in [2] that all spreading models of X are isometric to the l_2 -sum of X and l_2 . In the other hand Landes [10] proved that X has normal structure. Again by Landes' result we deduce that all the spreading models of X have N.S. We conclude that X has M -normal structure while every ultrapower \mathbf{X} of X contains l_1 and consequently fails to have normal structure. \square

The next result is a refinement of Zizler's theorem on uniform convexity in every direction.

Theorem 2. *Let X be a Banach space and Y a finite codimensional subspace of X . Suppose that X is uniformly convex in every direction $y \in S(Y)$. Then X has normal structure.*

Proof. Let $Q: X \rightarrow Z$ be a natural projection associated to the decomposition $X = Y \oplus Z$ (where Z is a finite dimensional subspace of X given by the assumption on Y) and C be a bounded convex set of X , not reduced to a single point. Let us prove that C contains a nondiametral point. Without loss of generality we can suppose that $0 \in C$. Let us assume first that there exist x, y in C such that $x \neq y$ and $Q(x) = Q(y)$. Then $x - y$ belongs to Y . Since X is uniformly convex in every direction z in $S(Y)$, we deduce that $1/2(x+y)$ is a nondiametral point of C . Assume now that $Q(x) \neq Q(y)$ for every $x \neq y$ in C . Then the restriction of Q to $\text{sp}(C)$ is one to one. Therefore C is finite dimensional in X and consequently it contains a nondiametral point (see [12]).

Example. Let Y be a Banach space which is uniformly convex in every direction and F be a Banach space. Then $X = F \oplus_p Y$, for $p \in [1, \infty[$, is uniformly convex in every direction $y \in S(Y)$.

We will say that X has the fixed point property if for any nonempty weakly-compact convex subset C and any nonexpansive mapping T defined from C into C , has a nonempty fixed point set. Recall the classical Kirk's theorem

which says that X has the fixed point property provided X has N.S. Let us give the last theorem of this part.

Theorem 3. *Let X and Y be Banach spaces which have the fixed point property. Consider on $X \oplus Y$ a norm of type (L) . Then for every weakly compact convex subsets C_1 and C_2 of X and Y we have $F_T \neq \emptyset$ for every nonexpansive map $T: C_1 \oplus C_2 \rightarrow C_1 \oplus C_2$.*

Proof. Let $T: C_1 \oplus C_2 \rightarrow C_1 \oplus C_2$ be a nonexpansive map. We can write $T = T_1 + T_2$ with $T(x + y) = T_1(x + y) + T_2(x + y)$ for every (x, y) in $C_1 \times C_2$. Fix x in C_1 and consider the map $T_x: C_2 \rightarrow C_2$ defined by $T_x(y) = T_2(x + y)$ for every y in C_2 . The property (ii) in the definition of the norm of type (L) implies that T_x is a nonexpansive map. Our hypothesis on Y implies that the fixed point set of T_x is nonempty. For every x in C_1 let y_x be a fixed point of T_x . Define $T_0: C_1 \rightarrow C_1$ by $T_0(x) = T_1(x + y_x)$. It is clear that T_0 is nonexpansive. Indeed, $\|T(x + y_x) - T(x' + y_{x'})\| \leq \|x + y_x - x' - y_{x'}\|$ but $T(x + y_x) = T_0(x) + y_x$ and $T(x' + y_{x'}) = T_0(x') + y_{x'}$. Using the property (iii) we obtain $\|T_0(x) - T_0(x')\| \leq \|x - x'\|$. Therefore the fixed point set of T_0 is nonempty since X has the f.p.p. Let x_0 be a fixed point of T_0 , then $T(x_0 + y_{x_0}) = T_0(x_0) + y_{x_0}$. Therefore $T(x_0 + y_{x_0}) = x_0 + y_{x_0}$ which implies the conclusion of Theorem 3. \square

Remarks. (1) We don't know if the conclusion of Theorem 3 holds for every convex weakly compact subset of $X \oplus Y$, even if Y is a finite dimensional space.

(2) It is not hard to adapt the proof of Theorem 3 for metric spaces.

A CHARACTERIZATION OF STRICT AND UNIFORM CONVEXITY

In this section we give two main results. The first theorem states a characterization of strict convexity in terms of nonexpansive mappings. The proof can be seen as a refinement of DeMarr's argument [6]. First let us prove one technical lemma.

Lemma 2. *Let X be a Banach space, and $(\beta, \beta_0) \in R_+ \times R_+$ be fixed numbers. Suppose that there exists x and y in $S(X)$ such that: $\|(x + y)/2\| = 1 - \beta$ and $\|(x - y)/2\| > \beta_0$ and $\beta < 1/2$. Then for every a, c in $[-1, 1]$ one has,*

$$|c| \leq \frac{\beta_0 + 2\beta}{(1 - \beta - 2\beta/\beta_0)\beta_0} \left\| \frac{a(x - y)}{2} + \frac{c(x + y)}{2} \right\|.$$

Proof. To simplify the computation we will denote $(x + y)/2$ by \vec{j} and $(x - y)/2$ by \vec{i} . By Hahn-Banach theorem there exists x^* in $S(x^*)$ such that $x^*(\vec{j}) = 1 - \beta$. (&) But $|x^*(x)| \leq 1$ and $|x^*(y)| \leq 1$. Using (&) we deduce that $x^*(x)$ and $x^*(y)$ are positive. Therefore $x^*(x) = 1 - s$ and $x^*(y) = 1 - s'$ with s and s' positive which verify by (&) $s + s' = 2\beta$. Then $|x^*(\vec{j})| = |s - s'|/2 \leq 2\beta$ and $|a| \leq (1/\beta_0)|a\vec{i}|$ for every a in R . Let us prove our inequality. $|c| = |x^*(c\vec{j})|/(1 - \beta) \leq |x^*(a\vec{i} + c\vec{j})|/(1 - \beta) + |x^*(a\vec{i})|/(1 - \beta)$. But

$|x^*(a\vec{i})| \leq |a|2\beta \leq (||a\vec{i} + c\vec{j}|| + |c|)2\beta/\beta_0$. Then $|c| \leq (1/\beta - 1)(1 + 2\beta/\beta_0)||a\vec{i} + c\vec{j}|| + (1/(1 - \beta))(2\beta/\beta_0)|c|$. This implies the desired inequality. \square

Theorem 4. *Let X be a Banach space. T.F.A.E.*

(a) X is strictly convex.

(b) Every nonexpansive map T defined on a convex subset C , has a convex fixed point set.

Proof. It is well known that (a) implies (b). Let us prove that (b) implies (a). Suppose that X is not strictly convex. Then there exist $x \neq y$ in $S(X)$ with $\|x + y\| = 2$. Let $C = \{\beta(x + y)/2 + \beta'(x - y)/2; (\beta', |\beta|) \in [0, 1]\}$ and $T: C \rightarrow C$ defined by

$$T\left(\beta\left(\frac{x+y}{2}\right) + \beta'\left(\frac{x-y}{2}\right)\right) = \beta\left(\frac{x+y}{2}\right) - (1 - |\beta|)\left(\frac{x-y}{2}\right).$$

Lemma 2 shows that T is a nonexpansive mapping. The fixed points of T are $z = \beta(x + y)/2 + (1 - |\beta|)(x - y)/2$ for every β in $[-1, 1]$. We conclude that F_T is not convex. Which gives a contradiction with (b). \square

The notion of contractions of type (γ) was introduced by Baillon [1]. Indeed he proved that any nonexpansive mapping defined on a bounded convex subset of L_p for $p \in [1, \infty)$, is of type (γ) . Therefore he asked if someone can find a characterization of such spaces which have the same property as L_p . Bruck [4] give a partial answer in proving in that if X is a uniformly convex space, then there exists $\gamma_0 \in \Gamma$ such that for every closed bounded convex subset C , with diameter one, and $T: C \rightarrow X$ a nonexpansive map, then T is of type (γ) and $\gamma_0 \in \gamma(T)$. Our main second theorem proves the converse to Bruck's result.

Theorem 5. *Let X be a Banach space. T.F.A.E.*

(a) X is uniformly convex.

(b) There exists $\gamma_0 \in \Gamma$ such that every nonexpansive map T defined on a convex subset C in X , with diameter one, is of type (γ) with $\gamma_0 \in \gamma(T)$.

Proof. Bruck's theorem says exactly that (a) implies (b). Let us prove that (b) implies (a). Suppose that X is not uniformly convex and that (b) holds. Then there exist $x_n, y_n \in S(X)$ and $\beta_0 > 0$ such that $\|x_n - y_n\| \geq \beta_0$ and $\|x_n + y_n\| = 2(1 - \beta_n) \rightarrow 2$ when n tends to infinity (**). Put $\vec{i}_n = (y_n - x_n)/2$ and $\vec{j}_n = (x_n + y_n)/2$. Let $C_n = \{a\vec{i}_n + c\vec{j}_n; (a, |c|) \in [0, 1]\}$ and $T_n: C_n \rightarrow C_n$ be defined by $T_n(a\vec{i}_n + c\vec{j}_n) = (1 - |c|)\vec{i}_n + c\vec{j}_n$. Lemma 2 shows that there exists $r_n > 0$ such that for every x, y in C_n we have $\|T_n(x) - T_n(y)\| \leq r_n\|x - y\|$. From (**) we deduce that $\lim_n r_n = 1$. Consider $L_n = (1/r_n)T_n$. Then L_n is nonexpansive and is still defined on C_n . Therefore (b) implies that L_n is of type (γ) with $\gamma_0 \in \gamma(L_n)$. This means that for every x, y in C_n we have

$$(***) \quad \gamma_0 \left(\left\| L_n\left(\frac{x+y}{2}\right) - \frac{1}{2}(L_n(x) + L_n(y)) \right\| \right) \leq \|x - y\| - \|L_n(x) - L_n(y)\|.$$

Let \mathbf{X} be an ultrapower of X (see Definition 4). We define \mathbf{C} and \mathbf{T} canonically by $\mathbf{C} = \{(x_n) \in \mathbf{X}; x_n \in C_n\}$ and $\mathbf{T}(x_n) = (T_n(x_n))$. It is clear from

(***) that T is a nonexpansive map of type (γ) with $\gamma_0 \in \gamma(T)$. But C and T can be identified with the convex set and the nonexpansive map defined in the proof of the Lemma 2. Therefore the fixed point set of T is not convex and T cannot be of type (γ) . Contradiction. \square

Remark 2. There exist a strictly convex Banach space E , a bounded convex subset C and a nonexpansive map T from C into C which has a nonempty fixed point set, but which is not of type (γ) (See [8]).

Problem. Can we construct a Banach space X which is not uniformly convex and where every nonexpansive mapping, defined on a bounded convex subset, is of type (γ) ? \square

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