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A fixed point theorem for commuting families of relational homomorphisms. Applications to metric spaces, ordered sets and oriented graphs

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ABSTRACT

We extend the notion of compact normal structure to binary relational systems. The notion was introduced by J.P. Penot for metric spaces. We prove that for involutive and reflexive binary relational systems, every commuting family of relational homomorphisms has a common fixed point. The proof is based upon the clever argument that J.B. Baillon discovered in order to show that a similar conclusion holds for bounded hyperconvex metric spaces. This was refined by the first author to metric spaces with a compact normal structure. Since non-expansive mappings are relational homomorphisms, our result includes those of T.C. Lim, J.B. Baillon and the first author. We show that it extends Tarski's fixed point theorem to graphs which are retracts of reflexive oriented zigzags of bounded length. In doing so, we illustrate the fact that the study of binary relational systems and of generalized metric spaces are equivalent.

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1. Introduction

Two results about fixed points are very much related. One is the famous theorem of Tarski ([47], 1955): *every order-preserving map on a complete lattice has a fixed point.* The other is a theorem of R. Sine and P.M. Soardi ([44], [46], 1979): *every non-expansive mapping on a bounded hyperconvex metric space has a fixed point.* Indeed, as was shown by D. Misane and the second author ([32], 1984, see also [37], 1985 and [22], 1986), if one considers a generalization of metric spaces, where, instead of real numbers, the distance values are members of an ordered monoid equipped with an involution, the most natural candidates for spaces with the fixed point property with respect to non-expansive mappings are among the absolute retracts. With a distributivity condition on the monoid, absolute retracts coincide with hyperconvex spaces. In the class of ordered sets, absolute retracts coincide with complete lattices (Banaschewski-Bruns [6]), whereas in the case of ordinary metric spaces, absolute retracts coincide with hyperconvex spaces (Aronszajn-Panitchpakdi, [3]). This explains the relationship between the results of Tarski and Sine-Soardi.

Since A. Tarski obtained, in fact, that every commuting family of order-preserving maps on a complete lattice has a common fixed point, E. Jawhari et al. [22] considered the question whether in this frame every commuting family of non-expansive mappings on a bounded hyperconvex space has a common fixed point, discovering that it was still unsettled in the frame of ordinary metric spaces. They got a positive answer for countable families; J.B. Baillon ([5], 1986) got a positive answer for arbitrary families acting on ordinary hyperconvex metric spaces. The proof of Baillon is based upon a clever compactness argument. At first glance, this argument works with minor changes for generalized hyperconvex spaces considered in [22] (but this was never published). On the other hand, with some extra work, it can be adapted to metric spaces endowed with a compact normal structure –as abstractly defined by J.P. Penot [35], [36]– spaces which include the hyperconvex ones. This extension was done by the first author in [28]. For other results on the existence of a common fixed point for a commuting set of non-expansive maps, see [14], [16], [31].

In this paper, we propose a generalization of Penot's notions in the framework of binary relational systems and their relational homomorphisms. Indeed, on one hand, non-expansive mappings f acting on an ordinary metric space, (or a generalized one), say (E, d), with distance function d from $E \times E$ into the set $\mathbb{R}^+$ of nonnegative reals (or into an ordered monoid V equipped with an involution), are relational homomorphisms of the binary relational system $\mathbf{E} := (E, \{\delta_v : v \in V\})$, where $\delta_v := \{(x, y) \in E \times E : d(x, y) \leq v\}$ for each v belonging to V. On the other hand, Penot's notions are very easy to define in this frame. We prove that *if a reflexive and involutive binary relational system has a compact normal structure then every commuting family of relational homomorphisms has a common fixed point* (Theorem 3.8). As an illustration, we get that *on a graph which is a retract of a product of reflexive oriented zigzags of bounded length, every commuting family of preserving maps has a common fixed point* (Theorem 4.23). Tarski's result corresponds to the case of a retract of a power of a two-element zigzag. Characterizations of reflexive and involutive binary relational systems with a compact normal structure are still open.

This paper is another opportunity to go beyond the analogy between metric spaces and binary relational systems. We consider generalized metric spaces whose distance values belong to an ordered monoid equipped with an involution and satisfying a distributivity condition. These structures, considered in the middle of the eighties [37,22] and then in subsequent papers (e.g. [38,24–26]), were called *involutive Heyting algebras*. They fit in the frame of quantales introduced by Mulvey [33] and are dual to *integral involutive quantales* defined by Kaarli and Radeleczki [23] (Subsection 4.9 has more bibliographical information). In this context, the notion of one-local retract, which is the key in proving our main result, fits naturally with the parent notion of hole-preserving map. We show that *if a generalized metric space* (E, d) *is bounded and hyperconvex then it has a compact normal structure* (Corollary 4.7). And from Theorem 4.5, we obtain that *if a generalized metric space* (E, d) *is bounded and hyperconvex then every commuting family of non-expansive self maps has a common fixed point* (Theorem 4.8). Our fix-point result on graphs fits in this context as well. Graphs with the fixed point property must be oriented (alias antisymmetric). Absolute retracts in the category of oriented graphs coincide with retracts of products of oriented paths and belong to the smaller category of metric spaces with values in the MacNeille completion of the monoid of words over a two-letter alphabet (a result of Bandelt, Saïdane and the second author, proved a longtime ago, see Chapter V of [42] and the forthcoming [8]). The reader will see transition systems as other examples of generalized metric spaces in [38,25,26].

This paper consists of three additional sections. Section 2 contains the notions of compact normal structure for relational systems; an illustration with a fixed-point result is given. Section 3 contains the notion of one-local retract. The main property, Theorem 3.7 is stated; Theorem 3.8 is given as a consequence. Section 4 illustrates our main result. Subsection 4.1 contains the exact relationship between reflexive involutive binary systems and generalized metric spaces over an involutive monoid (e.g. Lemma 4.4 and Theorem 4.5). In Subsection 4.2, the notion of hyperconvexity is recalled. Notions of inaccessibility and boundedness insuring that hyperconvex spaces have a compact normal structure are stated (Corollary 4.7). Spaces over a Heyting algebra with their main properties are presented in Subsection 4.3 (Theorem 4.9 and Theorem 4.10). Subsection 4.4 contains the relationship between one-local retract and hole-preserving maps. Subsection 4.5 brings together the results for ordinary metric spaces. The case of ordered sets is treated in Subsection 4.6. It contains a characterization of posets with a compact structure. The case of directed graphs with the zigzag distance is treated in Subsection 4.7. Subsection 4.8 contains a characterization of graphs isometrically embeddable into a product of oriented zigzags (Theorem 4.20) and our fixed point theorem (Theorem 4.23).

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2. Basic definitions, elementary properties and a fixed-point result

We adapt the basic notions of the theory of metric spaces to binary relations and to binary relational systems. The trick we use for this purpose consists in denoting by $d(x, y) \leq r$ the fact that the pair (x, y) belongs to the binary relation r, to interpret d as a distance, and $d(x, y)$ and r as numbers (a justification is given in Subsection 4.1).

The basic concepts about relational systems are the following.

Definition 2.1. Let E be a set. A *binary relation* on E is any subset r of $E \times E$. The *restriction* of r to a subset A of E is $r_{\restriction A} := r \cap (A \times A)$. The *inverse* of r is the binary relation $r^{-1} := \{(x, y) : (y, x) \in r\}$. The *diagonal* is $\Delta_E := \{(x, x) : x \in E\}$. The relation r is *symmetric* if $r = r^{-1}$; it r is *reflexive* if $\Delta_E \subseteq r$.

Let $\mathcal{E}$ be a set of binary relations on E. The pair $\mathbf{E} := (E, \mathcal{E})$ is a *binary relational system*. Set $\mathcal{E}^{-1} := \{r \subseteq E \times E : r^{-1} \in \mathcal{E}\}$.

(i) We say that $\mathbf{E} := (E, \mathcal{E})$ is *involutive* if $\mathcal{E} = \mathcal{E}^{-1}$.
(ii) We say that $\mathbf{E}$ is *reflexive* (resp. *symmetric*) if each member $r \in \mathcal{E}$ is reflexive (resp. symmetric).

For a subset A of E, the *restriction* of $\mathcal{E}$ to A is $\mathcal{E}_{\restriction A} := \{r_{\restriction A} : r \in \mathcal{E}\}$ which gives birth to the binary relational system $\mathbf{E}_{\restriction A} := (A, \mathcal{E}_{\restriction A})$. For a subset $\mathcal{E}'$ of binary relations on A, we set $\mathbf{p}_A^{-1}(\mathcal{E}') := \{r \in \mathcal{E} : r_{\restriction A} \in \mathcal{E}'\}$.

In the next definition, we discuss the maps which preserve a relational system.

Definition 2.2. Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system. Let $f : E \to E$ be a map.

(i) f is said to *preserve* $r \in \mathcal{E}$ if $(f(x), f(y)) \in r$ whenever $(x, y) \in r$. If f preserves every $r \in \mathcal{E}$, we say that f *preserves* $\mathcal{E}$.

(ii) f is said to *preserve* a subset $A \subseteq E$ if $f(A) \subseteq A$.

We denote by $End(\mathbf{E})$ the collection of self-maps which preserve $\mathcal{E}$ (since we only consider self-maps, we do not need to introduce an index set on $\mathcal{E}$).

Note that the maps which preserve $\mathcal{E}$ are in fact the relational homomorphisms (or endomorphisms) of this system. Next, we borrow some basic concepts from metric spaces.

Definition 2.3. Let $\mathbf{E} := (E, \mathcal{E})$ be a *binary relational system*. Let $r \in \mathcal{E}$ and let $x \in E$. The ball of center x and *radius* r, is the set

$$B(x, r) := \{y \in E : (x, y) \in r\}.$$

We denote by $\mathcal{B}_{\mathcal{E}}$ the set of balls whose radius belong to $\mathcal{E}$, i.e., $\mathcal{B}_{\mathcal{E}} := \{B(x, r) : x \in E, r \in \mathcal{E}\}$. We denote by $\hat{\mathcal{B}}_{\mathcal{E}}$ the set of all intersections of members of $\mathcal{B}_{\mathcal{E}}$, and we set $\hat{\mathcal{B}}^*_{\mathcal{E}} := \hat{\mathcal{B}}_{\mathcal{E}} \setminus \{\emptyset\}$.

Note that, as E is the intersection over the empty set, E belongs to $\hat{\mathcal{B}}_{\mathcal{E}}$. In the investigation of the fixed-point problem for self-maps, we will need the following concepts which can be seen as the relational analogues of the metric Chebyshev center and radius.

Definition 2.4. Let $\mathbf{E} := (E, \mathcal{E})$ be a *binary relational system*. Let A be a nonempty subset of E and $r \in \mathcal{E}$.

(i) The r-center is the set $C(A, r) := \{x \in E : A \subseteq B(x, r)\}$.
(ii) Set $\mathrm{Cov}_{\mathcal{E}}(A) := \bigcap\{B \in \mathcal{B}_{\mathcal{E}} : A \subseteq B\}$.
(iii) The *diameter* of A is the set $\delta_{\mathcal{E}}(A) := \{r \in \mathcal{E} : A \times A \subseteq r\}$.
(iv) The *radius* of A is the set $r_{\mathcal{E}}(A) := \{r \in \mathcal{E} : A \subseteq B(x, r)\ for\ some\ x \in A\}$.

Note that $\delta_{\mathcal{E}}(\emptyset) = \mathcal{E}$ and $r_{\mathcal{E}}(\emptyset) = \emptyset$. If $\mathbf{E} := (E, \mathcal{E})$, we may replace the index $\mathcal{E}$ in the previous notations by $\mathbf{E}$ which gives $\mathcal{B}_{\mathbf{E}}$, $\hat{\mathcal{B}}^*_{\mathbf{E}}$, $\mathrm{Cov}_{\mathbf{E}}(A)$, $r_{\mathbf{E}}(A)$ and $\delta_{\mathbf{E}}(A)$.

The elementary properties about center, diameter and radius are given in the following proposition:

Proposition 2.5. *Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system. Let $A \subseteq E$ and $r \in \mathcal{E}$. Then the following hold:*

(i) $A \subseteq C(A, r)$ *iff* $r \in \delta(A)$ *iff* $r^{-1} \in \delta(A)$*;*
(ii) $C(A, r) = \bigcap\{B(a, r^{-1}) : a \in A\}$*;*
(iii) If $r^{-1} \in \mathcal{E}$ *then* $C(A, r) \in \hat{\mathcal{B}}_{\mathbf{E}}$*;*
(iv) $C(A, r) = C(\mathrm{Cov}_{\mathbf{E}}(A), r)$ *whenever* $r \in \mathcal{E}$*;*
(v) $r_{\mathbf{E}}(A) \subseteq r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$ *and if* $A \neq \emptyset$*,* $\delta_{\mathbf{E}}(A) \subseteq r_{\mathbf{E}}(A)$*;*
(vi) $\delta_{\mathbf{E}}(A) = \delta_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$ *provided that* $\mathbf{E}$ *is involutive.*

Proof. (i) Immediate.
(ii) $x \in C(A, r)$ iff $A \subseteq B(x, r)$. This latter condition amounts to $x \in \bigcap\{B(a, r^{-1}) : a \in A\}$.
(iii) Follows immediately from (ii).
(iv) From the definition of the r-center, $x \in C(A, r)$ means that $A \subseteq B(x, r)$. Since $r \in \mathcal{E}$ this inclusion amounts to $\mathrm{Cov}_{\mathbf{E}}(A) \subseteq B(x, r)$. Again, from the definition of the r-center, this means $x \in C(\mathrm{Cov}_{\mathbf{E}}(A), r)$.
(v) Let $r \in r_{\mathbf{E}}(A)$ then $C(A, r) \cap A \neq \emptyset$. Since $C(A, r) = C(\mathrm{Cov}_{\mathbf{E}}(A), r)$ from (iv), we have $C(\mathrm{Cov}_{\mathbf{E}}(A), r) \cap \mathrm{Cov}_{\mathbf{E}}(A) \neq \emptyset$, hence $r \in r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$. The second assertion is obvious.
(vi) Trivially $\delta_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A)) \subseteq \delta_{\mathbf{E}}(A)$. Conversely, let $r \in \delta_{\mathbf{E}}(A)$. Then $A \subseteq B(x, r)$ for every $x \in A$, that

is $A \subseteq C(A, r)$. From (iv), this yields $A \subseteq C(\mathrm{Cov}_{\mathbf{E}}(A), r)$. Since $\mathbf{E}$ is involutive, $r^{-1} \in \mathcal{E}$, hence from (ii) we have $C(\mathrm{Cov}_{\mathbf{E}}(A), r) \in \hat{\mathcal{B}}_{\mathbf{E}}$. Since $A \subseteq C(\mathrm{Cov}_{\mathbf{E}}(A), r)$ it follows $\mathrm{Cov}_{\mathbf{E}}(A) \subseteq C(\mathrm{Cov}_{\mathbf{E}}(A), r)$ that is $r \in \delta_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$ by (i). □

Next, we introduce the notion of compact normal structure as Penot did for metric spaces [35]. Then we prove a fixed-point result.

Definition 2.6. Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system. A subset $A \subseteq E$ is *equally centered* if $r_{\mathbf{E}}(A) = \delta_{\mathbf{E}}(A)$.

As an example of an equally centered set, one may take a subset reduced to one point provided $\mathcal{E}$ is reflexive. Indeed, set $A = \{a\}$. From (v) of Proposition 2.5, we have $\delta_{\mathbf{E}}(A) \subseteq r_{\mathbf{E}}(A)$. Conversely, let $r \in r_{\mathbf{E}}(A)$. Since r is reflexive, then $(a, a) \in r$, i.e., $A \subseteq B(a, r)$. Hence $r \in \delta_{\mathbf{E}}(A)$, which completes the proof of our claim.

Definition 2.7. Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system.

(i) $\mathbf{E}$ has a *compact structure* if $\mathcal{B}_{\mathcal{E}}$ has the finite intersection property (f.i.p.), i.e., for every family $\mathcal{F}$ of members of $\mathcal{B}_{\mathcal{E}}$, the intersection of $\mathcal{F}$ is nonempty provided that the intersection of all finite subfamilies of $\mathcal{F}$ are nonempty.
(ii) $\mathbf{E}$ is said to have *normal structure* if any $A \in \hat{\mathcal{B}}^*_{\mathcal{E}}$, not reduced to one point, is not equally centered, i.e., if $|A| \neq 1$ then $r_{\mathbf{E}}(A) \neq \delta_{\mathbf{E}}(A)$.

It is easy to see that $\mathcal{B}_{\mathcal{E}}$ has the f.i.p. if and only if $\hat{\mathcal{B}}_{\mathcal{E}}$ has the f.i.p. If $\mathbf{E}$ has a compact structure, then every chain of members of $\hat{\mathcal{B}}^*_{\mathbf{E}}$ has an infimum, namely the intersection of all members of that chain. This allows us to prove the existence of minimal elements from $\mathcal{B}^*_{\mathbf{E}}$ preserved by an endomorphism of $\mathbf{E}$.

The following technical lemma will be useful throughout.

Lemma 2.8. *Let* $\mathbf{E} := (E, \mathcal{E})$ *be a binary relational system. Assume that* $\mathbf{E}$ *is involutive. Let* f *be an endomorphism of* $\mathbf{E}$. *If* $\mathbf{E}$ *has a compact structure then every member of* $\hat{\mathcal{B}}^*_{\mathbf{E}}$ *preserved by* f *contains a minimal one. If* $A \in \hat{\mathcal{B}}^*_{\mathbf{E}}$ *is a minimal member preserved by* f, *then* $\mathrm{Cov}_{\mathbf{E}}(f(A)) = A$ *and* A *is equally centered.*

Proof. First assume that $\hat{\mathcal{B}}_{\mathbf{E}}$ has the f.i.p. Let f be an endomorphism of $\mathbf{E}$. Assume there exists $C \in \hat{\mathcal{B}}^*_{\mathbf{E}}$ preserved by f, i.e., $f(C) \subseteq C$. Consider

$$\mathcal{C} = \{A \in \hat{\mathcal{B}}^*_{\mathbf{E}};\ A \subseteq C \textit{ and } f(A) \subseteq A\}.$$

Clearly $\mathcal{C}$ is not empty and partially ordered by inclusion. Since $\mathbf{E}$ has a compact structure, then Zorn's lemma ensures the existence of a minimal element. Assume $\mathbf{E}$ is involutive. Let us prove that $\mathrm{Cov}_{\mathbf{E}}(f(A)) = A$ for any minimal element A of $\mathcal{C}$. Indeed, since $f(A) \subseteq A$, we have $\mathrm{Cov}_{\mathbf{E}}(f(A)) \subseteq \mathrm{Cov}_{\mathbf{E}}(A) = A$. Hence $f(\mathrm{Cov}_{\mathbf{E}}(f(A))) \subseteq f(A) \subseteq \mathrm{Cov}_{\mathbf{E}}(f(A))$, i.e., $\mathrm{Cov}_{\mathbf{E}}(f(A))$ is preserved by f. The minimality of A implies $A = \mathrm{Cov}_{\mathbf{E}}(f(A))$. Next, let us prove that A is equally centered. We only have to prove that $r_{\mathbf{E}}(A) \subseteq \delta_{\mathbf{E}}(A)$. Let $r \in r_{\mathbf{E}}(A)$. Then there exists $x \in A$ such that $A \subseteq B_{\mathbf{E}}(x, r)$. By definition of $C(A, r)$, we have $x \in C(A, r) \cap A$. Set $A' := C(A, r) \cap A$. Since $\mathbf{E}$ is involutive, $r^{-1} \in \mathcal{E}$. Hence from (iii) of Proposition 2.5, we conclude that $A' \in \hat{\mathcal{B}}^*_{\mathbf{E}}$. Let us prove that A' is preserved by f. Note that f preserves $C(A, r)$. Since f is a relational homomorphism, we have $f(C(A, r)) \subseteq C(f(A), r)$. Indeed, if $x \in C(A, r)$, we have $a \subseteq B_{\mathbf{E}}(x, r)$. Hence $f(A) \subseteq B_{\mathbf{E}}(f(x), r)$, i.e., $f(x) \in C(f(A), r))$. Using (iv) of Proposition 2.5, we have $C(f(A), r) = C(\mathrm{Cov}_{\mathbf{E}}(f(A)), r) = C(A, r)$. Hence $C(A, r)$ is preserved by f. Clearly $A' = A \cap C(A, r)$

is therefore preserved by f. The minimality of A implies $A = A' = A \cap C(A,r)$. Hence $A \subset C(A,r)$, i.e., $r \in \delta_{\mathbf{E}}(A)$. Hence $r_{\mathbf{E}}(A) \subseteq \delta_{\mathbf{E}}(A)$. Thus, A is equally centered as claimed. □

This lemma allows us to deduce Penot's formulation [35] of Kirk's fixed point theorem [29] under our formulation.

Theorem 2.9. *Let $\mathbf{E} := (E,\mathcal{E})$ be a binary relational system. Assume $\mathbf{E}$ is involutive and has a compact normal structure. Then every endomorphism f of $\mathbf{E}$ has a fixed point.*

An easy consequence of Theorem 2.9, we have the following beautiful structural result:

Proposition 2.10. *Let $\mathbf{E} := (E,\mathcal{E})$ be a binary relational system. Assume $\mathbf{E}$ is involutive and has a compact normal structure. Let f be an endomorphism $\mathbf{E}$. Then the restriction $\mathbf{E}_{\restriction Fix(f)}$, to the set $Fix(f)$ of fixed points of f, has a compact normal structure.*

Proposition 2.10 will allow us to prove that a finite set of commuting endomorphism maps has a common fixed point and the restriction of $\mathbf{E}$ to the set of common fixed points has a compact normal structure. Obviously one would like to know whether such a conclusion still holds for infinitely many maps. In order to do this, one has to investigate carefully the structure of the fixed points of an endomorphism. This will be done in the next section.

3. One-local retracts and fixed points

The concept of retract plays a major role in investigating the fixed point problem.

Definition 3.1. Let $\mathbf{E} := (E,\mathcal{E})$ be a binary relational system. A map $g : E \to E$ is a *retraction* of $\mathbf{E}$ if g is a homomorphism of $\mathbf{E}$ such that $g \circ g = g$. For a subset A of E, we say that $\mathbf{E}_{\restriction A}$ is a *retract* of $\mathbf{E}$ if A is the image of E by some retraction of $\mathbf{E}$. We say that $\mathbf{E}_{\restriction A}$ is a *one-local retract* of $\mathbf{E}$ if for every $x \in E$, $\mathbf{E}_{\restriction A}$ is a retract of $\mathbf{E}_{\restriction A\cup\{x\}}$.

The next technical lemma will be used to prove the main result, Theorem 3.8, of this section.

Lemma 3.2. *Let $\mathbf{E} := (E,\mathcal{E})$ be a binary relational system and A be a subset of E. Assume $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}$. Then for every family of balls $\Big(B_{\mathbf{E}}(x_i,r_i)\Big)_{i\in I}$, with $x_i \in A$, $r_i \in \mathcal{E}$ for $i \in I$, such that $\bigcap_{i\in I} B_{\mathbf{E}}(x_i,r_i)$ is not empty, we have $\bigcap_{i\in I} B_{\mathbf{E}}(x_i,r_i) \cap A$ is not empty. The converse holds provided that $\mathbf{E}$ is reflexive and involutive.*

Proof. Let I be a set. Consider a family of balls $\Big(B_{\mathbf{E}}(x_i,r_i)\Big)_{i\in I}$, with $x_i \in A$, $r_i \in \mathcal{E}$ for $i \in I$, such that $B := \bigcap_{i\in I} B_{\mathbf{E}}(x_i,r_i)$ is not empty. Let $a \in B$ and let h be a retraction from $\mathbf{E}_{\restriction A\cup\{a\}}$ onto $\mathbf{E}_{\restriction A}$. The map h fixes A and preserves the relations induced by $\mathcal{E}$ on $A \cup \{a\}$. Fix $i \in I$. Since $a \in B_{\mathbf{E}}(x_i,r_i)$, we have $(x_i,a) \in r_i$. Since h preserves the relations induced by $\mathcal{E}$ on $A \cup \{a\}$, $(h(x_i),h(a)) \in r_i$. Using the fact $h(x_i) = x_i$, we get $(x_i,h(a)) \in r_i$, hence $h(a) \in B_{\mathbf{E}}(x_i,r_i)$. Since i is arbitrarily picked in I, we get $h(a) \in \bigcap_{i\in I} B_{\mathbf{E}}(x_i,r_i) \cap A$. Conversely, assume that $\mathbf{E}$ is reflexive and involutive. Let us prove $\mathbf{E}_{\restriction A}$ is a one-local retract provided the intersection property of balls is satisfied. Let $a \in E \setminus A$. Let

$$\mathcal{B} := \{B_{\mathbf{E}}(u,r) :\ u \in A,\ a \in B(u,r) \text{ and } r \in \mathcal{E}\}.$$

Set $B := \bigcap \mathcal{B}$. If $\mathcal{B} = \emptyset$, then $B = E$. Otherwise we have $a \in B$ which implies $B \neq \emptyset$. According to the ball's property, $B \cap A \neq \emptyset$. Let $a' \in B \cap A$. We claim that the map $h : A \cup \{a\} \to A$ which is the identity on A and satisfies $h(a) = a'$ is a retraction of $\mathbf{E}_{\restriction A \cup \{a\}}$. Since h is the identity on A, it suffices to check that for every $r \in \mathcal{E}$ and $u \in A$:

(i) $(u, a) \in r$ implies $(u, a') \in r$;
(ii) $(a, u) \in r$ implies $(a', u) \in r$;
(iii) $(a, a) \in r$ implies $(a', a') \in r$.

The first item holds by our choice of a'. The second item is equivalent to the first because $\mathbf{E}$ is involutive and the third item holds because $\mathbf{E}$ is reflexive. □

The following lemma gives some basic properties of one-local retracts.

Lemma 3.3. *Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system and $A \subseteq B \subseteq E$.*

(i) If $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}$ then it is a one-local retract of $\mathbf{E}_{\restriction B}$.
(ii) If $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}_{\restriction B}$ and $\mathbf{E}_{\restriction B}$ is a one-local retract of $\mathbf{E}$, then $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}$.

Proof. The proof relies on the basic fact that $(\mathbf{E}_{\restriction D})_{\restriction C} = \mathbf{E}_{\restriction C}$ whenever $C \subseteq D$. Hence the statement (i) is immediate.
(ii) Let $x \in E \setminus A$. If $x \in B$ then $\mathbf{E}_{\restriction A}$ since it is a one-local retract of $\mathbf{E}_{\restriction B}$. Otherwise assume $x \notin B$. Since $\mathbf{E}_{\restriction B}$ is a one-local retract of $\mathbf{E}$, it is a retract of $\mathbf{E}_{\restriction B \cup \{x\}}$ by some map g. Let $y := g(x)$. If $y \in A$, then $g_{\restriction A \cup \{x\}}$ is a retraction of $\mathbf{E}_{\restriction A \cup \{x\}}$ onto $\mathbf{E}_{\restriction A}$. Otherwise if $y \notin A$, then $\mathbf{E}_{\restriction A}$ is a retract of $\mathbf{E}_{\restriction A \cup \{x\}}$ by some map h since it is a one-local retract of $\mathbf{E}_{\restriction B}$. The map $h \circ g$ is a retraction of $\mathbf{E}_{\restriction A \cup \{x\}}$ onto $\mathbf{E}_{\restriction A}$. □

The next result is the most important one as it shows that a one-local retract enjoys the same properties as the larger set.

Lemma 3.4. *Let $\mathbf{E} := (E, \mathcal{E})$ be a binary relational system and $X \subseteq E$ a nonempty subset. Assume $\mathbf{E}_{\restriction X}$ is a one-local retract of $\mathbf{E}$. If $\mathbf{E}$ has a compact structure, then $\mathbf{E}_{\restriction X}$ also has a compact structure. Moreover if $\mathbf{E}$ is involutive and has a normal structure, then $\mathbf{E}_{\restriction X}$ has a normal structure as well.*

Proof. Let $\mathbf{E}' := \mathbf{E}_{\restriction X}$ and $\mathcal{E}' := \mathcal{E}_{\restriction X} := \{r \cap X \times X : r \in \mathcal{E}\}$. We prove the first assertion. Let $\mathcal{B}' := \{B_{\mathbf{E}'}(x'_i, r'_i) : i \in I, r'_i \in \mathcal{E}'\}$ be a family of balls of $\mathbf{E}'$ whose finite intersections are nonempty. For each $i \in I$, $r'_i = r_i \cap (X \times X)$ for some $r_i \in \mathcal{E}$. The family $\mathcal{B} := \{B_{\mathbf{E}}(x'_i, r_i) : i \in I\}$ of balls of $\mathbf{E}$ satisfies the f.i.p. hence has a nonempty intersection. Let $x \in \bigcap \mathcal{B}$. A retraction g from $\mathbf{E}_{\restriction X \cup \{x\}}$ onto $\mathbf{E}_{\restriction X} = \mathbf{E}'$ will send x into $\bigcap \mathcal{B}'$, proving that this set is nonempty. Next, we prove the second assertion. Let $A \in \hat{\mathcal{B}}^*_{\mathbf{E}'}$. We claim that

$$\delta_{\mathbf{E}}(A) = \delta_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A)), \tag{3.1}$$

and

$$r_{\mathbf{E}}(A) = r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A)). \tag{3.2}$$

Indeed, equality (3.1) is item (vi) of Proposition 2.5. Concerning equality (3.2), note that the inclusion $r_{\mathbf{E}}(A) \subseteq r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$ is item (v) of Proposition 2.5. For the converse, let $r \in r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A))$. Then, there is some $x \in \mathrm{Cov}_{\mathbf{E}}(A)$ such that $\mathrm{Cov}_{\mathbf{E}}(A) \subseteq B_{\mathbf{E}}(x, r)$. Since $\mathbf{E}'$ is a one-local retract of $\mathbf{E}$, there is a retraction g of $\mathbf{E}_{\restriction X \cup \{x\}}$ onto $\mathbf{E}' := \mathbf{E}_{\restriction X}$ which fixes X. Let $a := g(x)$. Since $A \subseteq \mathrm{Cov}_{\mathbf{E}}(A) \subseteq B_{\mathbf{E}}(x, r)$, we have $A \subseteq B_{\mathbf{E}}(x, r)$. Because g fixes A, $A \subseteq B_{\mathbf{E}}(a, r)$. We claim that $a \in A$. Indeed, $A = \bigcap\{B_{\mathbf{E}'}(x'_i, r'_i) : i \in I\}$ with $x'_i \in X, r'_i \in \mathcal{E}_{\restriction X}$. For each $i \in I$, choose $r_i \in \mathcal{E}$ such that $r'_i = r_i \cap X \times X$. Then $\mathrm{Cov}_{\mathbf{E}}(A) \subseteq A_1 := \bigcap\{B_{\mathbf{E}}(x'_i, r_i) : i \in I\}$. Since $x \in \mathrm{Cov}_{\mathbf{E}}(A)$, $x \in A_1$. Since g fixes each x'_i, we get $a \in A_1$. Since $a \in X$, $a \in A$, proving our claim. Hence $r \in r_{\mathbf{E}}(A)$. From (3.1) and (3.2), we obtain:

$$\delta_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A)) = \mathbf{p}_X^{-1}(\delta_{\mathbf{E}'}(A)), \tag{3.3}$$

and

$$r_{\mathbf{E}}(\mathrm{Cov}_{\mathbf{E}}(A)) = \mathbf{p}_X^{-1}(r_{\mathbf{E}'}(A)). \tag{3.4}$$

Let us prove first (3.3). By definition $\mathbf{p}_X^{-1}(\delta_{\mathbf{E}'}(A)) = \{r \in \mathcal{E} : r_{\restriction X} \in \delta_{\mathbf{E}'}(A)\} = \{r \in \mathcal{E} : A \times A \subseteq r\} = \delta_{\mathbf{E}}(A)$ since $A \subseteq X$. Equality (3.3) then follows from equality (3.1).

Next, we prove (3.4). By definition, we have

$$\begin{aligned} \mathbf{p}_X^{-1}(r_{\mathbf{E}'}(A)) &= \{r \in \mathcal{E} : r_{\restriction X} \in r_{\mathbf{E}'}(A)\} \\ &= \{r \in \mathcal{E} : A \subseteq B_{\mathbf{E}'}(a', r_{\restriction X}) \text{ for some } a' \in A\} \\ &= \{r \in \mathcal{E} : A \subseteq B_{\mathbf{E}}(a', r) \text{ for some } a' \in A\} = r_{\mathbf{E}}(A). \end{aligned}$$

Then equality (3.4) follows from equality (3.2). Finally, suppose that A is equally centered in $\mathbf{E}'$, that is $\delta_{\mathbf{E}'}(A) = r_{\mathbf{E}'}(A)$. From the equations above, we deduce that $\delta_{\mathbf{E}}(\mathrm{Cov}(A)) = r_{\mathbf{E}}(\mathrm{Cov}(A))$. Hence $\mathrm{Cov}_{\mathbf{E}}(A)$ is equally centered. If $\mathbf{E}$ has a normal structure, $\mathrm{Cov}_{\mathbf{E}}(A)$ is reduced to one point. Since $A \subseteq \mathrm{Cov}_{\mathbf{E}}(A)$ and $A \neq \emptyset$, we conclude that A is reduced to one point. Hence $\mathbf{E}' = \mathbf{E}_{\restriction X}$ has a normal structure. □

Recall that Proposition 2.10 gives information about the fixed point set of an endomorphism. In the next result, we show that in fact we have a better property satisfied by these sets which implies the conclusion of Proposition 2.10 as well.

Proposition 3.5. *Let $\mathbf{E} := (E, \mathcal{E})$ be a reflexive and involutive binary relational system. Assume $\mathbf{E}$ has a compact normal structure. Then for every homomorphism f of $\mathbf{E}$, the set of fixed points $Fix(f)$ of f is a nonempty one-local retract of $\mathbf{E}$. Thus $\mathbf{E}_{\restriction Fix(f)}$ has a compact normal structure.*

Proof. Let I be a set. Consider a family of balls $\Big(B_{\mathbf{E}}(x_i, r_i)\Big)_{i \in I}$, with $x_i \in Fix(f)$ and $r_i \in \mathcal{E}$ for $i \in I$, such that $A := \bigcap_{i \in I} B_{\mathbf{E}}(x_i, r_i)$ is not empty. Since each x_i belongs to $Fix(f)$, f preserves A. According to Lemma 2.8, since A is an intersection of balls, A contains an intersection of balls A' which is minimal, preserved by f, and equally centered. From the normality of $\mathbf{E}$, A' is reduced to a single element, i.e., A' is reduced to a fix-point of f. Consequently, $A \cap Fix(f) \neq \emptyset$. According to Lemma 3.2, $Fix(f)$ is a one-local retract. □

In order to prove the existence of a common fixed point for a family of nonexpansive mappings in the context of hyperconvex metric spaces, Baillon [5] discovered an intersection property satisfied by this class of metric spaces. In order to prove an analogue to Baillon's conclusion under our setting, we will need the following lemma.

Lemma 3.6. *Let $\mathbf{E} := (E, \mathcal{E})$ be a reflexive and involutive binary relational system with a compact normal structure. Let κ be an infinite cardinal. For every ordinal α, $\alpha < \kappa$, let B_α and E_α be subsets of E such that:*

(1) $B_\alpha \supseteq B_{\alpha+1}$ *and* $E_\alpha \supseteq E_{\alpha+1}$ *for every* $\alpha < \kappa$*;*
(2) $\bigcap\limits_{\gamma<\alpha} B_\gamma = B_\alpha$ *and* $\bigcap\limits_{\gamma<\alpha} E_\gamma = E_\alpha$ *for each limit ordinal* $\alpha < \kappa$*;*
(3) $\mathbf{E}_\alpha := \mathbf{E}_{\restriction E_\alpha}$ *is a one-local retract of* $\mathbf{E}$ *and* B_α *is a nonempty intersection of balls of* $\mathbf{E}_\alpha$*.*

Then $B_\kappa := \bigcap\limits_{\alpha<\kappa} B_\alpha \neq \emptyset$*.*

Proof. Let $\mathfrak{A}$ be the collection of all descending sequences $\mathcal{A} := (A_\alpha)_{\alpha<\kappa}$ such that each A_α is a nonempty intersection of balls of $\mathbf{E}_{\restriction E_\alpha}$ contained into B_α. Set $\mathbf{E}_\alpha := \mathbf{E}_{\restriction E_\alpha}$ and $\mathfrak{B} := \Pi_{\alpha<\kappa}\hat{B}^*_{\mathbf{E}_\alpha}$. The sequence $\mathcal{B} := (B_\alpha)_{\alpha<\kappa}$ belongs to $\mathfrak{A}$ and $\mathfrak{A}$ is contained into $\mathfrak{B}$. The set $\mathfrak{B}$ is ordered as follows:

$$(A'_\alpha)_{\alpha<\kappa} \preceq (A''_\alpha)_{\alpha<\kappa} \text{ if } A'_\alpha \subseteq A''_\alpha \text{ for every } \alpha < \kappa. \tag{3.5}$$

Since $\mathbf{E}_\alpha$ is a one-local retract of $\mathbf{E}$, $\mathbf{E}_\alpha$ has a normal and compact structure (Lemma 3.4). Since it has a compact structure, every descending sequence in $\hat{B}^*_{\mathbf{E}_\alpha}$ has an infimum (Lemma 2.8). Thus, there is a minimal sequence $\mathcal{A} := (A_\alpha)_{\alpha<\kappa}$ with $\mathcal{A} \preceq \mathcal{B}$. Let $\alpha < \kappa$ and $A_\alpha \subseteq B_{\mathbf{E}}(x, r)$, with $r \in \mathcal{E}$ and $x \in E_\alpha$. Then $A_\beta \subseteq B_{\mathbf{E}}(x, r)$ for each $\beta < \kappa$. Indeed, set $B := B_{\mathbf{E}}(x, r)$. For $\xi < \kappa$ set $A'_\xi := A_\xi \cap B$ if $\xi \leq \alpha$ and $A'_\xi = A_\xi$ otherwise. The family $\mathcal{A}' := (A'_\xi)_{\xi<\kappa}$ belongs to $\mathfrak{A}$ and satisfies $\mathcal{A}' \preceq \mathcal{A}$. Since $\mathcal{A}$ is minimal, we get $\mathcal{A}' = \mathcal{A}$. Thus $A_\xi = A_\xi \cap B$ for $\xi \leq \alpha$ that is $A_\xi \subseteq B$. Since $A_\xi \subseteq A_\alpha \subseteq B$ for $\xi \geq \alpha$, it follows that $A_\xi \subseteq B$. Let $\alpha < \kappa$. From the hypotheses of the lemma, there is a family $\mathcal{B}' := (B_{\mathbf{E}_{\restriction E_\alpha}}(x'_i, r'_i))_{i\in I}$, with $x'_i \in E_\alpha$, $r'_i \in \mathcal{E}_{\restriction E_\alpha}$ such that $A_\alpha = \bigcap \mathcal{B}'$. For each $i \in I$, let r_i such that $r_{i\restriction E_\alpha} = r'_i$. Let $\mathcal{B} := (B_{\mathbf{E}_{\restriction E_\alpha}}(x'_i, r'_i))_{i\in I}$, $B = \bigcap \mathcal{B}$. Then $A_\alpha = B \cap E_\alpha \subseteq B_{\mathbf{E}}(x, r)$, for any $\alpha < \kappa$. Next, for any $\alpha < \kappa$, we have

(a) $A_\beta \subseteq B \cap E_\beta$ for every $\beta < \alpha$;
(b) $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$ if α is a limit ordinal;
(c) $r_{\mathbf{E}}(A_\alpha) \subseteq r_{\mathbf{E}}(A_\beta)$ for every $\beta < \alpha$;
(d) $r_{\mathbf{E}}(A_\beta) \subseteq r_{\mathbf{E}}(A_\alpha)$ for every $\beta < \alpha$.

Indeed, we have $A_\alpha \subseteq B_{\mathbf{E}}(x'_i, r_i)$ which implies $A_\beta \subseteq B_{\mathbf{E}}(x, r)$. This yields $A_\beta \subseteq \bigcap \mathcal{B} = B$ which gives (a). From $\bigcap\limits_{\gamma<\alpha} E_\gamma = E_\alpha$ and (a), we get

$$A_\alpha = B \cap E_\alpha = \bigcap_{\beta<\alpha} B \cap E_\beta \supseteq \bigcap_{\beta<\alpha} A_\beta.$$

This implies $A_\alpha \supseteq \bigcap\limits_{\beta<\alpha} A_\beta$. Since $\mathcal{A}$ is decreasing, we have $A_\alpha \subseteq \bigcap\limits_{\beta<\alpha} A_\beta$. Hence, $A_\alpha = \bigcap\limits_{\beta<\alpha} A_\beta$. As for (c), let $r \in r_{\mathbf{E}}(A_\alpha)$. Then $A_\alpha \subseteq B_{\mathbf{E}}(x, r)$, for some $x \in A_\alpha$ which implies $A_\beta \subseteq B_{\mathbf{E}}(x, r)$. Since $A_\alpha \subseteq A_\beta$, we get $x \in A_\beta$ which implies $r \in r_{\mathbf{E}}(A_\beta)$. As for (d), let $r \in r_{\mathbf{E}}(A_\beta)$ and $x \in A_\beta$ such that $A_\beta \subseteq B_{\mathbf{E}}(x, r)$. From (a), we have $A_\beta \subseteq B$. Thus $x \in B \cap \bigcap\limits_{u\in A_\alpha} B_{\mathbf{E}}(u, r^{-1})$ holds. Since $\mathbf{E}_{\restriction E_\alpha}$ is a one-local retract, there is some $y \in B \cap \bigcap\limits_{u\in A_\alpha} B_{\mathbf{E}}(u, r^{-1}) \cap E_\alpha$. This means $A_\alpha \subseteq B_{\mathbf{E}}(y, r)$ which in turns implies $r \in r_{\mathbf{E}}(A_\alpha)$. Hence if we combine (c) and (d), we get $r_{\mathbf{E}}(A_\beta) = r_{\mathbf{E}}(A_\alpha)$, for any $\alpha, \beta < \kappa$. Next, we claim that $\delta_{\mathbf{E}}(A_\alpha) = r_{\mathbf{E}}(A_\alpha)$ for every $\alpha < \kappa$. Indeed, let $\mathbf{r}$ be the common value of all $r_{\mathbf{E}}(A_\alpha)$, for $\alpha < \kappa$. Let $r \in \mathbf{r}$. Set $C_r(A_\alpha) := \{x \in E_\alpha : A_\alpha \subseteq B_{\mathbf{E}}(x, r)\}$, $A^r_\alpha := A_\alpha \cap C_r(A_\alpha)$ and $\mathcal{A}^r := (A^r_\alpha)_{\alpha<\kappa}$. We claim

(1) A^r_α is a nonempty intersection of balls of $\mathbf{E}_{\restriction E_\alpha}$;
(2) $A^r_\alpha \subseteq A_\alpha$;
(3) $A^r_\beta \supseteq A^r_\alpha$ for $\beta < \alpha$.

(1). Since $r \in r_{\mathbf{E}}(A_\alpha)$, $A_\alpha \subseteq B_{\mathbf{E}}(x, r)$ for some $x \in A_\alpha$, hence $x \in C_r(A_\alpha)$ proving that A^r_α is nonempty. Since $\mathbf{E}$ is involutive, $r^{-1} \in \mathcal{E}$. Thus from (iii) of Proposition 2.5, $C_r(A_\alpha)$ is an intersection of balls of $\mathbf{E}_{\restriction E_\alpha}$ with centers in A_α. Hence, A^r_α is a nonempty intersection of balls of $\mathbf{E}_{\restriction E_\alpha}$ which proves (1). Clearly (2) is obvious. As for (3), let $\beta < \alpha$. By construction of $\mathcal{A}$, we have $A_\beta \supseteq A_\alpha$. Let $x \in A^r_\alpha$. By definition, we have $A_\alpha \subseteq B_{\mathbf{E}}(x, r)$. We have $A_\beta \subseteq B_{\mathbf{E}}(x, r)$. It follows that $x \in C_r(A_\beta)$. Since $x \in A_\beta$, $x \in A^r_\beta$ which proves that (3) holds. On the other hand, if we use the minimality of $\mathcal{A}$, we obtain $\mathcal{A}^r = \mathcal{A}$. From this it follows that $A_\alpha \subseteq C_r(A_\alpha)$. Since this inclusion holds for every $r \in r_{\mathbf{E}}(A_\alpha)$, we get $\delta_{\mathbf{E}}(A_\alpha) = r_{\mathbf{E}}(A_\alpha)$, for any $\alpha < \kappa$, as claimed. Since $\mathbf{E}$ has a normal structure, we deduce that each A_α is a singleton. Since $\mathcal{A}$ is decreasing, $A_\kappa := \bigcap_{\alpha<\kappa} A_\alpha$ is a singleton too. Hence, $B_\kappa \neq \emptyset$ which completes the proof of Lemma 3.6. □

Next, let $\mathbf{E} := (E, \mathcal{E})$ be involutive and reflexive. Assume $\mathbf{E}$ has a compact normal structure. Let P be the set, ordered by inclusion, of nonempty subsets A of E such that $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}$. As any ordered set, every down-directed subset of P has an infimum iff every totally ordered subset of P has an infimum (see [15] Proposition 5.9 p 33). We claim that P is closed under intersection of every chain of its members. Indeed, we argue by induction on the size of totally ordered families of one-local retracts of $\mathbf{E}$. First we may suppose that E has more than one element. Next, we may suppose that these families are dually well ordered by induction. Thus, given an infinite cardinal κ, let $(\mathbf{E}_{\restriction E_\alpha})_{\alpha<\kappa}$ be a descending sequence of one-local retracts of $\mathbf{E}$. From the induction hypothesis, we may suppose that the restriction of $\mathbf{E}$ to $E'_\alpha := \bigcap_{\gamma<\alpha} E_\gamma$ is a one-local retract of $\mathbf{E}$ for each limit ordinal $\alpha < \kappa$. Hence, we may suppose that $E_\alpha = \bigcap_{\gamma<\alpha} E_\gamma$ for each limit ordinal $\alpha < \kappa$. Since $\mathbf{E}_\alpha$ is a one-local retract of $\mathbf{E}$ and $\mathbf{E}$ has a normal structure, $\mathbf{E}_\alpha$ has a normal structure (Lemma 3.4). Hence, either E_α is a singleton, say x_α, or $r_{\mathbf{E}_\alpha}(E_\alpha) \setminus \delta_{\mathbf{E}_\alpha}(E_\alpha) \neq \emptyset$. In both cases, E_α is a ball of $\mathbf{E}_\alpha$ (since $\mathbf{E}$ is reflexive, $(x_\alpha, x_\alpha) \in r$ for any $r \in \mathcal{E}$). Hence the first case, $E_\alpha = B_{\mathbf{E}_\alpha}(x_\alpha, r_{\restriction E_\alpha})$, whereas in second case, $E_\alpha \subseteq B_{\mathbf{E}_\alpha}(x, r)$ for some $x \in E_\alpha, r \in r_{\mathbf{E}_\alpha}(E_\alpha) \setminus \delta_{\mathbf{E}_\alpha}(E_\alpha)$. Hence, Lemma 3.6 applies with $B_\alpha = E_\alpha$ and gives that E_κ is nonempty. Let us prove that $\mathbf{E}_\kappa := \mathbf{E}_{\restriction E_\kappa}$ is a one-local retract of $\mathbf{E}$. We apply Lemma 3.2. Let $(B_{\mathbf{E}}(x_i, r_i))_{i\in I}, x_i \in E_\kappa$, $r_i \in \mathcal{E}$ be a family of balls such that the intersection is nonempty. Since $\mathbf{E}_\alpha$ is a one-local retract of $\mathbf{E}$, the intersection $B_\alpha := E_\alpha \bigcap \bigcap_{i\in I} B_{\mathbf{E}}(x_i, r_i)$ is nonempty for every $\alpha < \kappa$. Now, Lemma 3.6 applied to the sequence $(E_\alpha, B_\alpha)_{\alpha<\kappa}$ tells us that $B_\kappa := E_\kappa \bigcap \bigcap_{i\in I} B_{\mathbf{E}}(x_i, r_i)$ is nonempty. According to Lemma 3.2, $\mathbf{E}_{\restriction B_\kappa}$ is a one-local retract of $\mathbf{E}$.

The above conclusion on P translates into the following result:

Theorem 3.7. *Let $\mathbf{E} := (E, \mathcal{E})$ be a reflexive and involutive binary relational system. Assume $\mathbf{E}$ has a compact normal structure. Then the intersection of every down-directed family of one-local retracts is a nonempty one-local retract.*

As Baillon [5] did, we next give the main result about the existence of common fixed points of a commutative family of endomorphisms.

Theorem 3.8. *Let $\mathbf{E} := (E, \mathcal{E})$ be a reflexive and involutive binary relational system. Assume $\mathbf{E}$ has a compact normal structure. Then any commuting family $\mathcal{F}$ of homomorphisms of $\mathbf{E}$ has a common fixed point. Furthermore, the restriction of $\mathbf{E}$ to the set $Fix(\mathcal{F})$ of common fixed points of $\mathcal{F}$ is a one-local retract of $\mathbf{E}$.*

Proof. For a subset $\mathcal{F}'$ of $\mathcal{F}$, let $Fix(\mathcal{F}')$ be the set of fixed points of $\mathcal{F}'$. Using Proposition 3.5, we conclude that $\mathbf{E}_{\restriction Fix(\mathcal{F}')}$ is a nonempty one-local retract of $\mathbf{E}$ for every finite subset $\mathcal{F}'$ of $\mathcal{F}$. Indeed, we show this by induction on the number n of elements of $\mathcal{F}'$. If $n = 1$, this is the conclusion of Proposition 3.5. Let $n \geq 1$. Suppose that the property holds for every subset $\mathcal{F}''$ of $\mathcal{F}'$ such that $|\mathcal{F}''| < n$. Let $f \in \mathcal{F}'$ and

$\mathcal{F}'' := \mathcal{F}' \setminus \{f\}$. From our inductive hypothesis, $\mathbf{E}_{\restriction Fix(\mathcal{F}'')}$ is a one-local retract of $\mathbf{E}$. Thus, according to Lemma 3.4, $\mathbf{E}_{\restriction Fix(\mathcal{F}'')}$ has a compact normal structure. Now since f commutes with every member g of $\mathcal{F}''$, f preserves $Fix(\mathcal{F}'')$ setwise. Indeed, if $u \in Fix(\mathcal{F}'')$, we have $g(f(u)) = f(g(u)) = f(u)$, that is $f(u) \in Fix(\mathcal{F}'')$. Thus f induces an endomorphism f'' of $\mathbf{E}_{\restriction Fix(\mathcal{F}'')}$. According to Proposition 3.5, the restriction of $\mathbf{E}_{\restriction Fix(\mathcal{F}'')}$ to $Fix(f'')$, that is $\mathbf{E}_{\restriction Fix(\mathcal{F}')}$, is a nonempty one-local retract of $\mathbf{E}_{\restriction Fix(\mathcal{F}'')}$. Since the notion of one-local retract is transitive it follows that $\mathbf{E}_{\restriction Fix(\mathcal{F}')}$ is a nonempty one-local retract of $\mathbf{E}$. Let $\mathcal{P} := \{Fix(\mathcal{F}'') : |\mathcal{F}''| < \aleph_0\}$ and $P := \bigcap \mathcal{P}$. According to Theorem 3.7, $\mathbf{E}_{\restriction P}$ is a one-local retract of $\mathbf{E}$. Since $P = Fix(\mathcal{F})$ the conclusion follows. □

4. Illustrations

Inspired by the work of Baillon, our results apply naturally to metric spaces and also to various kinds of generalizations of metric spaces, including ordered sets, graphs and transition systems.

In this section, we consider first generalized metric spaces and particularly those for which the set of distance values is a *dual involutive integral quantale* (previously called Heyting algebra). Then, we go on from ordinary metric spaces to ordered sets and graphs.

4.1. Towards generalized metric spaces

We present first a natural association of generalized metric spaces and binary relational systems.

We recall that if ρ and τ are two binary relations on the same set E, then their composition $\rho \circ \tau$ is the binary relation made of pairs (x, y) such that $(x, z) \in \tau$ and $(z, y) \in \rho$ for some $z \in E$. It is customary to denote it $\tau \cdot \rho$.

The set $\mathrm{Inv}_2(\mathcal{F})$ of binary relations on E preserved by all f belonging to a set $\mathcal{F}$ of self maps on E has some very simple properties which we state below (the proofs are left to the reader). For a construction of many more properties by means of primitive positive formulas (see [45]).

Lemma 4.1. *Let $\mathcal{F}$ be a set of unary operations on a set E. Then the set $\mathcal{R} := \mathrm{Inv}_2(\mathcal{F})$ of binary relations on E preserved by all $f \in \mathcal{F}$ satisfies the following properties:*

(a) $\Delta_E \in \mathcal{R}$;
(b) *$\mathcal{R}$ is closed under arbitrary intersections; in particular $E \times E \in \mathcal{R}$;*
(c) *$\mathcal{R}$ is closed under arbitrary unions;*
(d) *If $\rho, \tau \in \mathcal{R}$ then $\rho \circ \tau \in \mathcal{R}$;*
(e) *If $\rho \in \mathcal{R}$ then $\rho^{-1} \in \mathcal{R}$.*

Let $\mathcal{R}$ be a set of binary relations on a set E satisfying items (a), (b), (d) and (e) of the above lemma (we do not require (c)). To make things more transparent, denote by 0 the set Δ_E, set $\rho \oplus \tau := \rho \cdot \tau$. Then $\mathcal{R}$ becomes a monoid. Set $\overline{\rho} := \rho^{-1}$, this defines an involution on $\mathcal{R}$ which reverses the monoid operation. With this involution $\mathcal{R}$ is an *involutive monoid.* With the inclusion order, that we denote $\leq$, this involutive monoid is an *involutive complete ordered monoid.*

With these definitions, we have immediately:

Lemma 4.2. *Let $\mathcal{R}$ be an involutive complete ordered monoid of the set of binary relations on E and let d be the map from $E \times E$ into $\mathcal{R}$ defined by*

$$d(x, y) := \bigcap \{\rho \in \mathcal{R} : (x, y) \in \rho\}.$$

Then, the following properties hold:

(i) $d(x, y) \leq 0$ *iff* $x = y$;
(ii) $d(x, y) \leq d(x, z) \oplus d(z, y)$;
(iii) $\overline{d(y, x)} = d(x, y)$.

The general setting to deal with this situation is the following. Let V be an ordered monoid equipped with an involution. We denote by $\oplus$ the monoid operation, by 0 its neutral element and by $^-$ the involution, so that $\overline{p \oplus q} = \overline{q} \oplus \overline{p}$ for all $p, q \in V$.

Following [38], we say that a set E equipped with a map d from $E \times E$ into V and which satisfies properties $(i), (ii), (iii)$ stated in Lemma 4.2 is a *V-distance*, and the pair (E, d) a *V-metric space*. Lemma 4.2 justifies writing $d(x, y) \leq \rho$ for the fact that a pair (x, y) belongs to a binary relation ρ on the set E. We then use notions borrowed from the theory of metric spaces.

Remark 4.3. From now on, we suppose that the neutral element of the monoid V is the least element of V for the ordering. In [17] (cf. p.82) the corresponding V-metric spaces are called *generalized distance spaces* and the maps d are called *generalized metrics*.

If (E, d) is a V-metric space and A a subset of E, the restriction of d to $A \times A$, denoted by $d_{\restriction A}$ is a V-distance and $(A, d_{\restriction A})$ is a *restriction* of (E, d). As in the case of ordinary metric spaces, if (E, d) and (E', d') are two V-metric spaces, a map $f : E \longrightarrow E'$ is a *non-expansive map* (or a *contraction*) from (E, d) to (E', d') provided that $d'(f(x), f(y)) \leq d(x, y)$ holds for all $x, y \in E$ (and the map f is an *isometry* if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in E$). The space (E, d) is a *retract* of (E', d'), if there are two non-expansive maps $f : E \longrightarrow E'$ and $g : E' \longrightarrow E$ such that $g \circ f = id_E$ (where id_E is the identity map on E). In this case, f is a *coretraction* and g a *retraction*. If E is a subspace of E', then clearly E is a retract of E' if there is a non-expansive map from E' to E such $g(x) = x$ for all $x \in E$. We can easily see that every coretraction is an isometry. We say that $(A, d_{\restriction A})$ is a *one-local retract* if it is a retract of $(A \cup \{x\}, d_{\restriction A \cup \{x\}})$ (via the identity map on A) for every $x \in E$.

Let (E, d) be a V-metric space; for $x \in E$ and $v \in V$, the set $B(x, v) := \{y \in E : d(x, y) \leq v\}$ is a *ball*. One can define diameter and radius as in ordinary metric spaces, but in order to avoid a problem with the existence of joins and meets, we suppose that V is a complete lattice. The *diameter* $\delta(A)$ of a subset A of E is $\bigvee\{d(x, y) : x, y \in A\}$, while the *radius* $r(A)$ is $\bigwedge\{v \in V : A \subseteq B(x, v)$ for some $x \in A\}$. A subset A of E is *equally centered* if $\delta(A) = r(A)$. Following Penot, who defined the notions for ordinary metric spaces, a V-metric space (E, d) has a *compact structure* if the collection of balls has the finite intersection property and it has a *normal structure* if for every intersection of balls A, either $\delta(A) = 0$ or $r(A) < \delta(A)$. This condition amounts to the fact that the only equally centered intersections of balls are singletons.

The correspondence between the notions defined for metric spaces and for binary relational systems is given in the lemma below.

Lemma 4.4. *Let (E, d) be a V-metric space. For $v \in V$, set $\delta_v := \{(x, y) : d(x, y) \leq v\}$ and $\mathbf{E} := (E, \{\delta_v : v \in V\})$. Then $\mathbf{E}$ is reflexive and involutive. Furthermore:*

(a) *A self map f on E is non-expansive iff it is an endomorphism of $\mathbf{E}$.*
(b) *(E, d) has a compact structure iff $\mathbf{E}$ has a compact structure.*
(c) *For every subset A of E, $(A, d_{\restriction A})$ is a one-local retract of (E, d) iff $\mathbf{E}_{\restriction A}$ is a one-local retract of $\mathbf{E}$.*
(d) *For every subset A of E, $\delta(A)$ is the least element of the set of $v \in V$ such that $A \subseteq \delta_v$; equivalently $\delta_{\mathbf{E}}(A) = \{\delta_v : \delta(A) \leq v\}$. Also, $r(A) = \bigwedge\{v \in V : \delta_v \in r_{\mathbf{E}}(A)\}$.*
(e) *A subset A of E is equally centered w.r.t. the space (E, d) iff it is equally centered w.r.t. the binary relational system $\mathbf{E}$.*

Proof. The first three items are obvious.

Item (d). Let $r := \delta(A)$. By definition, $r = \bigvee\{d(x,y) : (x,y) \in A^2\}$. In particular, $A \subseteq \delta_r$. Let v such that $A \subseteq \delta_v$; this means $d(x,y) \leq v$ for every $(x,y) \in A^2$, hence $r \leq v$. This proves that $\delta(A) = \text{Min}\{v \in V : \delta_v \in \delta_{\mathbf{E}}(A)\}$. The verification of the other assertions is immediate. Item (e). By Item (d), $r(A) := \bigwedge\{v \in V : \delta_v \in r_{\mathbf{E}}(A)\}$ and $\delta(A) := \text{Min}\{v \in V : \delta_v \in \delta_{\mathbf{E}}(A)\}$. If $r_{\mathbf{E}}(A) = \delta_{\mathbf{E}}(A)$, this immediately implies $r(A) = \delta(A)$. Conversely, suppose that $r(A) = \delta(A)$. In this case $A \neq \emptyset$, hence $\delta_{\mathbf{E}}(A) \subseteq r_{\mathbf{E}}(A)$. If $\delta_{\mathbf{E}}(A) \subsetneq r_{\mathbf{E}}(A)$ then since $\delta(A) = \text{Min}\{v \in V : \delta_v \in \delta_{\mathbf{E}}(A)\}$ and $r(A) = \bigwedge\{v \in V : \delta_v \in r_{\mathbf{E}}(A)\}$ it follows that $r(A) < \delta(A)$, a contradiction. □

With this lemma in hand, Theorem 3.8 becomes:

Theorem 4.5. *If a generalized metric space (E,d) has a compact normal structure then every commuting family $\mathcal{F}$ of non-expansive self maps has a common fixed point. Furthermore, the restriction of (E,d) to the set $Fix(\mathcal{F})$ of common fixed points of $\mathcal{F}$ is a one-local retract of (E,d).*

The fact that a space has a compact structure is an infinistic property (any finite metric space enjoys it). A description of generalized metric spaces with a compact normal structure eludes us. In the next subsection we describe a large class of generalized metric spaces with a compact normal structure.

4.2. Hyperconvexity

We say that a generalized metric space (E,d) is *hyperconvex* if for every family of balls $B(x_i, r_i)$, $i \in I$, with $x_i \in E, r_i \in V$, the intersection $\bigcap_{i \in I} B(x_i, r_i)$ is nonempty provided that $d(x_i, x_j) \leq r_i \oplus \overline{r}_j$ for all $i, j \in I$. This property amounts to the fact that the collection of balls of (E,d) has the 2-*Helly property* (that is an intersection of balls is nonempty provided that these balls intersect pairwise) and the following *convexity property*:

$$\text{Any two balls } B(x,r), B(y,s) \text{ intersect if and only if } d(x,y) \leq r \oplus \overline{s}. \tag{4.1}$$

An element $v \in V$ is *self-dual* if $\overline{v} = v$, it is *accessible* if there is some $r \in V$ with $v \not\leq r$ and $v \leq r \oplus \overline{r}$ and *inaccessible* otherwise. Clearly, 0 is inaccessible; every inaccessible element v is self-dual (otherwise, $\overline{v}$ is incomparable to v and we may choose $r := \overline{v}$). We say that a space (E,d) is *bounded* if 0 is the only inaccessible element below $\delta(E)$.

Lemma 4.6. *Let A be an intersection of balls of (E,d). If $\delta(A)$ is inaccessible then A is equally centered; the converse holds if (E,d) is hyperconvex.*

Proof. Suppose that $v := \delta(A)$ is inaccessible. According to (d) of Lemma 4.4, $r(A) = \bigwedge r_{\mathbf{E}}(A)$. Let $r \in r_{\mathbf{E}}(A)$. Then there is some $x \in A$ such that $A \subseteq B(x,r)$. This yields $d(a,b) \leq d(a,x) \oplus d(x,b) \leq \overline{r} \oplus r$ for every $a, b \in A$. Thus $v \leq \overline{r} \oplus r$. Since v is inaccessible, $v \leq r$, hence $v \leq r(A)$. Thus $v = r(A)$. Suppose that A is equally centered. Let r be such that $v \leq r \oplus \overline{r}$. The balls $B(x,r)$ $(x \in A)$ intersect pairwise and intersect each of the balls containing A; since (E,d) is hyperconvex, these balls have a nonempty intersection. Any member a of this intersection is in A and satisfies $A \subseteq B(a, \overline{r})$, hence $\overline{r} \in r_{\mathbf{E}}(A)$. Since A is equally centered $r(A) = v$. Hence, $v \leq \overline{r}$. Since v is self-dual, $v \leq r$. Thus v is inaccessible. □

This lemma, with the fact that the 2-Helly property implies that the collection of balls has the finite intersection property, yields:

Corollary 4.7. *If a generalized metric space (E,d) is bounded and hyperconvex then it has a compact normal structure.*

From Theorem 4.5, we obtain:

Theorem 4.8. *If a generalized metric space (E, d) is bounded and hyperconvex then every commuting family of non-expansive self maps has a common fixed point.*

4.3. Metric spaces over an involutive Heyting algebra, alias a dual integral involutive quantale

Hyperconvex spaces have a simple characterization provided that the set V of values of the distance is a complete lattice and satisfies the following distributivity condition:

$$\bigwedge_{\alpha\in A,\beta\in B} u_\alpha \oplus v_\beta = \bigwedge_{\alpha\in A} u_\alpha \oplus \bigwedge_{\beta\in B} v_\beta \tag{4.2}$$

for all $u_\alpha \in V$ $(\alpha \in A)$ and $v_\beta \in V$ $(\beta \in B)$.

In this case, we say that V is an *involutive Heyting algebra*; according to Kaarli and Radeleczki [23], we could call it a *dual integral involutive quantale* (see [41,20] about quantales).

On an involutive Heyting algebra V, we may define a V-distance. This fact relies on the classical notion of residuation (see [13] for the context of residuation). Let $v \in V$. Given $\beta \in V$, the sets $\{r \in V : v \leq r \oplus \beta\}$ and $\{r \in V : v \leq \beta \oplus r\}$ have least elements, that we denote by $\lceil v \oplus -\beta\rceil$ and $\lceil -\beta \oplus v\rceil$ respectively and call the *right* and *left quotient* of v by β (note that $\overline{\lceil -\beta \oplus v\rceil} = \lceil \bar{v} \oplus -\bar{\beta}\rceil$). It follows that for all $p, q \in V$, the set

$$D(p, q) := \{r \in V : p \leq q \oplus \bar{r} \text{ and } q \leq p \oplus r\} \tag{4.3}$$

has a least element. This least element is $\lceil \bar{p} \oplus -\bar{q}\rceil \vee \lceil -p \oplus q\rceil$, we denote it by $d_V(p, q)$.

As shown in [22], the map $(p, q) \longrightarrow d_V(p, q)$ is a $V-$distance.

Let $((E_i, d_i))_{i\in I}$ be a family of V-metric spaces. The *direct product* $\prod_{i\in I}(E_i, d_i)$, is the metric space (E, d) where E is the cartesian product $\prod_{i\in I} E_i$ and d is the "sup" (or ℓ^∞) distance defined by $d\left((x_i)_{i\in I}, (y_i)_{i\in I}\right) = \bigvee_{i\in I} d_i(x_i, y_i)$. We recall the following result of [22].

Theorem 4.9. (V, d_V) *is a hyperconvex V-metric space and every V-metric space embeds isometrically into a power of* (V, d_V).

This result is due to the fact that for every V-metric space (E, d) and for all $x, y \in E$ the following equality holds:

$$d(x, y) = \bigvee_{z\in E} d_V(d(z, x), d(z, y)). \tag{4.4}$$

A generalized metric space E is an *absolute retract* if it is a retract of every isometric extension. The space E is *injective* if for all V-metric space E' and E'', for each non-expansive map $f : E' \longrightarrow E$ and for every isometry $g : E' \longrightarrow E''$ there is a non-expansive map $h : E'' \longrightarrow E$ such that $h \circ g = f$.

With this result follows the characterization given in [22].

Theorem 4.10. *For metric spaces over an involutive Heyting algebra V, the notions of absolute retract, injective, hyperconvex and retract of a power of* (V, d_V) *coincide.*

Note that if v is accessible in V and V is an involutive Heyting algebra, then v is accessible in the initial segment $\downarrow v$ of V (indeed, if $v \leq r \oplus s$ then since by distributivity $(r \wedge v) \oplus (\overline{r} \wedge v) = (r \oplus \overline{r}) \wedge (r \oplus v) \wedge (v \oplus \overline{r}) \wedge (v \oplus v)$, we have $v \leq (r \wedge v) \oplus (\overline{r} \wedge v)$.

4.4. One-local retracts and hole-preserving maps

In this subsection, we relate the notions of one-local retracts and the notion of hole-preserving maps. A large part is borrowed from subsection II-4 of [22].

Let E and E' be two V-metric spaces. If f is a non-expansive map from E into E', and h is a map from E into V, the *image* of h is the map h_f from E' into V defined by $h_f(y) : \bigwedge\{h(x) : f(x) = y\}$ (in particular $h_f(y) = 1$ where 1 is the largest element of V for every y not in the range of f). A *hole* of E is any map $h : E \to V$ such that the intersection of balls $B(x, h(x))$ of E $(x \in E)$ is empty. If h is a hole of E, the map f *preserves* h provided that h_f is a hole of E'. The map f is *hole-preserving* if the image of every hole is a hole.

Let $\mathcal{B} := (B(x_i, r_i))_{i\in I}$ be a family of balls of E. For every $x \in E$, set $V_{\mathcal{B}}(x) = \{r \in V : B(x_i, r_i) \subseteq B(x, r)$ for some $i \in I\}$ and $h_{\mathcal{B}}(x) := \bigwedge V_{\mathcal{B}}(x)$. We note that:

$$\bigcap \mathcal{B} = \bigcap_{x\in E} B(x, h_{\mathcal{B}}(x)). \tag{4.5}$$

We omit the routine proof.

A hole h of E is *finite* if $\bigcap_{x\in F} B(x, h(x)) = \emptyset$ for some finite subset F of E, otherwise it is infinite.

A poset is *well-founded* if every nonempty subset contains some minimal element. We recall that if a lattice is well-founded, every element x which is the infimum of some subset X is the infimum of some finite subset. In general, the order on a Heyting algebra is not well-founded, still there are interesting examples (see Subsection 4.6 and 4.8).

The following lemma relates holes and compactness of the collection of balls (it contains a correction of Proposition II-4.9. of [22]):

Lemma 4.11. *If a generalized space E has a compact structure then every hole is finite; the converse holds if V is well-founded.*

Proof. Let h be a hole. Then, by definition, $\bigcap_{x\in E} B(x, h(x)) = \emptyset$. Since E has a compact structure, $\bigcap_{x\in F} B(x, h(x)) = \emptyset$ for some finite subset, hence h is finite. Conversely, let $\mathcal{B} := (B(x_i, r_i))_{i\in I}$ be a family of balls of E such that $\bigcap \mathcal{B} = \emptyset$. There are two ways of associating a finite hole to $\mathcal{B}$. We may define $h : E \to V$ by setting $h(x) := \bigwedge\{r_i : x_i = x\}$. We may also associate $h_{\mathcal{B}}$. By Formula (4.5), this is a hole. These holes are finite. We conclude by using $h_{\mathcal{B}}$. Let F be some finite subset of E such that $\bigcap_{x\in F} B(x, h_{\mathcal{B}}(x)) = \emptyset$. Since V is well-founded, for each $x \in E$, there is some finite subset V_x of $V_{\mathcal{B}}(x) = \{r \in V : B(x_i, r_i) \subseteq B(x, r)$ for some $i \in I\}$ such that $\bigwedge V_{\mathcal{B}}(x) = \bigwedge V_x$. For each $x \in F$, there is a finite subset I_x such that for each $r \in V_x$ there is some $i \in I_x$ such that $B(x_i, r_i) \subseteq B(x, r)$. Setting $I_F := \bigcup_{x\in F} I_x$ we get $\bigcap_{i\in I_F} B(x_i, r_i) = \emptyset$ proving that the intersection of the finite subfamily $(B(x_i, r_i))_{i\in I_F}$ of $\mathcal{B}$ is nonempty. □

Lemma 4.12. *A non-expansive map f from a V-metric space E into a V-metric space E' is hole-preserving iff f is an isometry of E onto its image and this image is a* 1*-local retract of E'.*

The routine proof is based on Lemma 3.2. We omit it.

Replacing isometries by hole-preserving maps in the definition of absolute retracts and injectives, we have the notions of absolute retracts and injectives w.r.t. hole preserving maps.

We recall the following result of [22].

Theorem 4.13. *On an involutive Heyting algebra V, the absolute retracts and the injectives w.r.t. hole-preserving maps coincide. The class $\mathcal{H}$ of these objects is closed under products and retractions. Moreover, every metric space embeds into some member of $\mathcal{H}$ by some hole-preserving map.*

The proof relies on the notion of the replete space $H(E)$ of a metric space E. The space E is an absolute retract w.r.t. hole-preserving maps or not depending whether E is a retract of $H(E)$ or not. Furthermore, with the introduction of this space, one can prove the *transferability* of hole-preserving maps (Lemma II-4.6 of [22]), that is the fact that for every non-expansive map $f : E \to F$, and every hole-preserving map $g : E \to G$ there is a hole-preserving map $g' : F \to E'$ and a non-expansive map $f' : G \to E'$ such that $g' \circ f = f' \circ g$. Indeed, one may choose $E' = H(E)$. For details, see [22], notably Lemma II-4.4 and Lemma II-4.5.

Problems 1. Let E be a generalized metric space with a compact normal structure.

(a) When is a one-local retract of E a retract?
(b) When is the set $Fix(f)$ of fixed points of a non-expansive self map a retract?

Note that if (a) has a positive answer then spaces with a compact normal structure are absolute retracts w.r.t. hole-preserving maps. For these problems, it could be fruitful to consider the case of posets; there is a vast literature on fixed-point and this type of question (see [43,4,34]).

4.5. The case of ordinary metric spaces

Let $\mathbb{R}^+$ be the set of non negative reals with the addition and natural order, the involution being the identity. Let $V := \mathbb{R}^+ \cup \{+\infty\}$. Extend the addition and order to V in a natural way. Then, metric spaces over V are direct sums of ordinary metric spaces (the distance between elements in different components being $+\infty$). The set V is an involutive Heyting algebra, the distance d_V, when restricted to $\mathbb{R}^+$, is derived from the absolute value. The inaccessible elements are 0 and $+\infty$ hence, if one deals with ordinary metric spaces, unbounded spaces in the above sense are those which are unbounded in the ordinary sense. If one deals with ordinary metric spaces, infinite products can yield spaces for which $+\infty$ is attained. One may replace powers with ℓ^∞-spaces (if I is any set, $\ell^\infty_{\mathbb{R}}(I)$ is the set of bounded families $(x_i)_{i\in I}$ of reals numbers, endowed with the sup-distance). Doing so, the notions of absolute retract, injective, hyperconvex and retract of some $\ell^\infty_{\mathbb{R}}(I)$ space coincide.

According to Corollary 4.7, a hyperconvex metric space has a normal structure iff its diameter is bounded. In fact, if a subset A of a hyperconvex space is an intersection of balls, its radius is half of its diameter. No description of metric spaces with a compact normal structure seems to be known.

The existence of a fixed point for a non-expansive map on a bounded hyperconvex space is the famous result of Sine and Soardi. Theorem 3.8 applied to a bounded hyperconvex metric space is Baillon's fixed point theorem. Applied to a metric space with a compact normal structure, this is the result obtained by the first author [28].

4.6. The case of ordered sets

In this subsection, we consider posets as binary relational systems as well as metric spaces over an involutive Heyting algebra.

Let $P := (E, \leq)$ be an ordered set. Let $\mathcal{E} := \{\leq, \leq^{-1}\}$ and $\mathbf{E} := (E, \mathcal{E})$. By definition, $\mathbf{E}$ is reflexive and involutive. For $x \in E$, set $\uparrow x := \{y \in E : x \leq y\}$ and $\downarrow x := \{y \in E : y \leq x\}$; these sets are called the *principal* final, resp. *initial*, segment generated by x. With our terminology of balls of $\mathbf{E}$, these sets are the balls $B(x, \leq)$ and $B(x, \leq^{-1})$.

Let V be the following structure. The domain is the set $\{0, +, -, 1\}$. The order is $0 \leq +, - \leq 1$ with $+$ incomparable to $-$; the involution exchanges $+$ and $-$ and fixes 0 and 1; the operation $\oplus$ is defined by $p \oplus q := p \vee q$ for every $p, q \in V$. As it is easy to check, V is an involutive Heyting algebra.

If (E,d) is a V-metric space, then $P_d := (E, \delta_+)$, where $\delta_+ := \{(x,y) : d(x,y) \leq +\}$, is an ordered set. Conversely, if $P := (E, \leq)$ is an ordered set, then the map $d : E \times E \to V$ defined by $d(x,y) := 0$ if $x = y$, $d(x,y) := +$ if $x < y$, $d(x,y) := -$ if $y < x$ and $d(x,y) := 1$ if x and y are incomparable. Clearly, if (E,d) and (E',d') are two V-metric spaces, a map $f : E \to E'$ is non-expansive from (E,d) into (E',d') iff it is order-preserving from P_d into $P_{d'}$. Depending on the value of $v \in V$, a V-metric space has four types of balls: singletons, corresponding to $v = 0$, the full space, corresponding to $v = 1$, the principal final segments, $\uparrow x := \{y \in E : x \leq y\}$, corresponding to balls $B(x,+)$, and principal initial segments, $\downarrow x := \{y \in E : y \leq x\}$, corresponding to balls $B(x,-)$. The set V can be equipped with the distance d_V given by means of the formula (4.3). The corresponding poset is the four element lattice $\{-,0,1,+\}$ with $- < 0, 1 < +$. The retracts of powers of this lattice are all complete lattices. This is confirmed by the following fact.

Proposition 4.14. *A metric space (E,d) over V is hyperconvex iff the corresponding poset is a complete lattice.*

Proof. Suppose that (E,d) is hyperconvex. Let $\leq := \delta_+$ and $P_d := (E, \delta_+)$. We prove that every subset A has a supremum in P_d. This amounts to prove that $A^+ := \{y \in E : x \leq y$ for all $x \in A\}$ has a least element. Since (E,d) satisfies the convexity property, and $+ \vee \overline{+} = + \vee - = 1$, $B(x',+) \cap B(x'',+) \neq \emptyset$ for every $x', x'' \in E$; since (E,d) satisfies the 2-Helly property, $A^\Delta = \bigcap_{x \in A} B(x,+) \neq \emptyset$. Applying again the convexity and the 2-Helly property, we get that the intersection of balls $B(x,+)$ for $x \in A$ and $B(y,-)$, for $b \in A^\Delta$ is nonempty. This intersection contains just one element, this is the supremum of A. A similar argument yields the existence of the infimum of A, hence P_d is a complete lattice. Conversely, let $B(x_i, r_i)$, $(i \in I)$, be a family of balls such that $d(x_i, x_j) \leq r_i \vee \overline{r}_j$. We prove that $C := \bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$. If there is some $i \in I$ such that $r_i = 0$, then $x_i \in C$. If not, let $A := \{i \in I : r_i = +\}$, $B := \{j \in I : r_j = -\}$. Then $x_i \leq x_j$ for all $x_i \in A$, $x_j \in B$. Set $c := \bigvee A$ and observe that $c \in C$. □

Since 0 is the only inaccessible element of V, Theorem 4.8 applies: *Every commuting family of order-preserving maps on a complete lattice has a common fixed point.* This is Tarski's theorem (in full).

Posets coming from V-metric spaces with a compact normal structure are a bit more general than complete lattices, hence Theorem 3.8 on compact normal structure could say a bit more than Tarski's fixed point theorem. As we will see, in the case of one order-preserving map this is no more than Abian-Brown's fixed-point theorem.

We observe first that the f.i.p. property of the collection of balls $\mathcal{B}_E := \{\downarrow x : x \in E\} \cup \{\uparrow y : y \in E\}$ is an infinistic condition: it holds for every finite poset. In order to describe it we introduce the following notions.

A pair of subsets (A,B) of E is called a *gap* of P if every element of A is dominated by every element of B but there is no element of E which dominates every element of A and is dominated by every element of B (cf. [19]). In other words: $(\bigcap_{x \in A} B(x, \leq)) \cap (\bigcap_{y \in B} B(y, \geq)) = \emptyset$ while $B(x, \leq) \cap B(y, \geq) \neq \emptyset$ for every $x \in A, y \in B$. A *subgap* of (A,B) is any pair (A',B') with $A' \subseteq A$, $B' \subseteq B$, which is a gap. The gap (A,B) is *finite* if A and B are finite, otherwise it is *infinite*. Say that an ordered set Q *preserves* a gap (A,B) of P if there is an order-preserving map g of P to Q such that $(g(A), g(B))$ is a gap of Q. On the preservation of gaps, see [34].

Lemma 4.15. *Let $P := (E, \mathcal{E})$ be a poset. Then:*

- *(a) P is a complete lattice iff P contains no gaps;*
- *(b) An order-preserving map $f : P \to Q$ preserves all gaps of P iff it preserves all holes of P with values in $V \setminus \{0\}$ iff $f(P)$ is a one-local retract of Q;*

(c) $\mathcal{B}_{\mathbf{E}}$ satisfies the f.i.p. iff every gap of P contains a finite subgap iff every hole is finite.

The routine proof is omitted. We may note the similarity of (b) and Lemma 4.12.

From item (c) of Lemma 4.15 it follows that every nonempty chain in a poset P for which the collection of balls has the *f.i.p*, has a supremum and an infimum. Such a poset is called *chain-complete*.

Abian-Brown's theorem [1] asserts that *in a chain-complete poset with a least or largest element, every order-preserving map has a fixed point.* The fact that the collection of intersection of balls of P has a normal structure means that every nonempty intersection of balls of P has either a least or largest element. Being the intersection of the empty family of balls, P has either a least element or a largest element. Consequently, if P has a compact normal structure, we may suppose without loss of generality that it has a least element. Since every nonempty chain has a supremum, it follows from Abian-Brown's theorem that every order preserving map has a fixed point.

On the other hand, a description of posets with a compact normal structure is still open. We only mention some examples. Let $\bigvee$ be the 3-element poset consisting of $0, +, -$ with $0 < +, -$ and $+$ incomparable to $-$. We denote by $\bigwedge$ its dual. Then the reader will observe that retracts of powers of $\bigvee$ have a compact normal structure.

Theorem 3.8 above yields a fixed point theorem for a commuting family of order-preserving maps on any retract of a power of $\bigvee$ or of a power of $\bigwedge$. But this result says nothing about retracts of products of $\bigvee$ and $\bigwedge$. These two posets fit in the category of fences. A *fence* is a poset whose comparability graph is a path. For example, a two-element chain is a fence. Each larger fence has two orientations, for example on the three vertices path, these orientations yield the $\bigvee$ and the $\bigwedge$.

From Theorem 4.23, proved in Subsection 4.7.1, follows:

Theorem 4.16. *If a poset Q is a retract of a product P of finite fences of bounded length, every commuting set of order-preserving maps on Q has a fixed point.*

Since every complete lattice is a retract of a power of the two-element chain, this result contains Tarski's fixed point theorem.

4.7. The case of directed graphs

A *directed graph* G is a pair $(E, \mathcal{E})$ where $\mathcal{E}$ is a binary relation on E. We say that G is *reflexive* if $\mathcal{E}$ is reflexive and that G is *oriented* if $\mathcal{E}$ is antisymmetric (that is (x, y) and (y, x) cannot be in $\mathcal{E}$ simultaneously except if $x = y$). All graphs we consider will be reflexive. If $\mathcal{E}$ is symmetric, we identify it with a subset of pairs of E and we say that the graph is *undirected*.

If $G := (E, \mathcal{E})$ and $G' := (E', \mathcal{E}')$ are two directed graphs, a *homomorphism from G to G'* is a map $h : E \to E'$ such that $(h(x), h(y)) \in \mathcal{E}'$ whenever $(x, y) \in \mathcal{E}$ for every $(x, y) \in E \times E$.

Let us recall that a finite *path* is an undirected graph $L := (E, \mathcal{E})$ such that one can enumerate the vertices into a non-repeating sequence $v_0, \ldots, v_n$ such that edges are the pairs $\{v_i, v_{i+1}\}$ for $i < n$. A *reflexive zigzag* is a reflexive graph such that the symmetric hull is a path. If L is a reflexive oriented zigzag, we may enumerate the vertices in a non-repeating sequence $v_0 := x, \ldots, v_n := y$ and to this enumeration we may associate the finite sequence $ev(L) := \alpha_0 \cdots \alpha_i \cdots \alpha_{n-1}$ of $+$ and $-$, where $\alpha_i := +$ if (v_i, v_{i+1}) is an edge and $\alpha_i := -$ if (v_{i+1}, v_i) is an edge. We call such a sequence a *word* over the *alphabet* $\Lambda := \{+, -\}$. If the path has just one vertex, the corresponding word is the empty word, that we denote by $\Box$. Conversely, to a finite word $u := \alpha_0 \cdots \alpha_i \cdots \alpha_{n-1}$ over Λ we may associate the reflexive oriented zigzag $L_u := (\{0, \ldots n\}, \mathcal{L}_u)$ with end-points 0 and n (where n is the length $\ell(u)$ of u) such that $\mathcal{L}_u = \{(i, i+1) : \alpha_i = +\} \cup \{(i+1, i) : \alpha_i = -\} \cup \Delta_{\{0,\ldots,n\}}$.

4.7.1. The zigzag distance

Let $G := (E, \mathcal{E})$ be a reflexive directed graph. For each pair $(x, y) \in E \times E$, the *zigzag distance* from x to y is the set $d_G(x, y)$ of words u such that there is a non-expansive map h from L_u into G which sends 0 on x and $n := \ell(u)$ (the length of u) on y.

This notion is due to Quilliot [39,40] (Quilliot considered reflexive directed graphs, not necessarily oriented, and in defining the distance, considered only oriented paths). A general study is presented in [22]; some developments appear in [42] and [25].

Because of the reflexivity of G, every word obtained from a word belonging to $d_G(x, y)$ by inserting letters will also be in $d_G(x, y)$. This leads to the following framework.

Let Λ^* be collection of words over the alphabet $\Lambda := \{+, -\}$. Extend the involution on Λ to Λ^* by setting $\overline{\square} := \square$ and $\overline{u_0 \cdots u_{n-1}} := \overline{u_{n-1}} \cdots \overline{u_0}$ for every word in Λ^*. Order Λ^* by the *subword ordering*, denoted by $\leq$. If $u := \alpha_1\alpha_2 \dots \alpha_m, v := \beta_1\beta_2 \dots \beta_n \in \Lambda^*$ set

$$u \leq v \text{ if and only if } \ \alpha_j = \beta_{i_j} \text{ for all } j = 1, \dots m \text{ with some } 1 \leq j_1 < \dots j_m \leq n.$$

Let $\mathbf{F}(\Lambda^*)$ be the set of final segments of Λ^*, that is subsets F of Λ^* such that $u \in F$ and $u \leq v$ imply $v \in F$. Setting $\overline{X} := \{\overline{u} : u \in X\}$ for a set X of words, we observe that $\overline{X}$ belongs to $\mathbf{F}(\Lambda^*)$. Order $\mathbf{F}(\Lambda^*)$ by reverse of the inclusion, denote by 0 its least element (that is Λ^*), set $X \oplus Y$ the concatenation $X \cdot Y := \{uv : u \in X, v \in Y\}$. Then, one sees that $\mathcal{H}_\Lambda := (\mathbf{F}(\Lambda^*), \oplus, \supseteq, 0, -)$ is an involutive Heyting algebra. This leads us to consider distances and metric spaces over $\mathcal{H}_\Lambda$. There are two simple and crucial facts about the zigzag distance (see [22]).

Lemma 4.17. *A map from a reflexive directed graph G into an other is a graph-homomorphism iff it is non-expansive.*

Lemma 4.18. *The distance d of a metric space (E, d) over $\mathcal{H}_\Lambda$ is the zigzag distance of a reflexive directed graph $G := (E, \mathcal{E})$ iff it satisfies the following property for all $x, y, z \in E$, $u, v \in \mathbf{F}(\Lambda^*)$: $uv \in d(x, y)$ implies $u \in d(x, z)$ and $v \in d(z, y)$ for some $z \in E$. When this condition holds, $(x, y) \in \mathcal{E}$ iff $+ \in d(x, y)$.*

Due to Lemma 4.18, the various metric spaces mentioned above (injective, absolute retracts, etc.) are graphs equipped with the zigzag distance; in particular, the distance $d_{\mathcal{H}_\Lambda}$ defined on $\mathcal{H}_\lambda$ is the zigzag distance of some graph. A fairly precise description of absolute retracts in the category of reflexive directed graphs is given in [25].

4.8. The case of oriented graphs

The situation of oriented graphs is different. These graphs cannot be modeled over a Heyting algebra (Theorem IV-3.1 of [22] is erroneous), but the absolute retracts in this category can be (this was proved by Bandelt, Saïdane and the second author and included in [42]). The appropriate Heyting algebra is the *MacNeille completion* of Λ^*.

The MacNeille completion of Λ^* is in some sense the least complete lattice extending Λ^*. The definition goes as follows. If X is a subset of Λ^* ordered by the subword ordering then

$$X^\Delta := \bigcap_{x \in X} \uparrow x$$

is the *upper cone* generated by X, and

$$X^\nabla := \bigcap_{x \in X} \downarrow x$$

is the *lower cone* generated by X. The pair (Δ, ∇) of mappings on the complete lattice of subsets of Λ^* constitutes a Galois connection. Thus, a set Y is an upper cone if and only if $Y = Y^{\nabla\Delta}$, while a set W is an lower cone if and only if $W = W^{\Delta\nabla}$. This Galois connection (Δ, ∇) yields the *MacNeille completion* of Λ^*. This completion is realized as the complete lattice $\{W^{\nabla} : W \subseteq \Lambda^*\}$ ordered by inclusion or alternatively $\{Y^{\Delta} : Y \subseteq \Lambda^*\}$ ordered by reverse inclusion. We choose as completion the set $\{Y^{\Delta} : Y \subseteq \Lambda^*\}$ ordered by reverse inclusion that we denote by $\mathbf{N}(\Lambda^*)$. This complete lattice is studied in detail in [7].

We recall the following characterization of members of the MacNeille completion of Λ^*.

Proposition 4.19. *[7] Corollary 4.5. A member Z of $\mathbf{F}(\Lambda^*)$ belongs to $\mathbf{N}(\Lambda^*)$ if and only if it satisfies the following cancellation rule: if $u + v \in Z$ and $u - v \in Z$ then $uv \in Z$.*

The concatenation, order and involution defined on $\mathbf{F}(\Lambda^*)$ induce an involutive Heyting algebra $\mathcal{N}_\Lambda$ on $\mathbf{N}(\Lambda^*)$ (see Proposition 2.2 of [7]). Being an involutive Heyting algebra, $\mathcal{N}_\Lambda$ supports a distance $d_{\mathcal{N}_\Lambda}$ and this distance is the zigzag distance of a graph $G_{\mathcal{N}_\Lambda}$. But it is not true that every oriented graph embeds isometrically into a power of that graph. For example, an oriented cycle cannot be embedded. The following result characterizes graphs which can be isometrically embedded, via the zigzag distance, into products of reflexive and oriented zigzags. It is stated in part in Subsection IV-4 of [22], cf. Proposition IV-4.1.

Theorem 4.20. *For a directed graph $G := (E, \mathcal{E})$ equipped with the zigzag distance, the following properties are equivalent:*

(i) G is isometrically embeddable into a product of reflexive and oriented zigzags;
(ii) G is isometrically embeddable into a power of $G_{\mathcal{N}_\Lambda}$;
(iii) The values of the zigzag distance between vertices of E belong to $\mathcal{N}_\Lambda$.

The proof follows the same lines as the proof of Proposition IV-5.1 p.212 of [22].

We may note that the product can be infinite even if the graph G is finite. Indeed, if G consists of two vertices x and y with no value on the pair $\{x, y\}$ (that is the underlying graph is disconnected) then we need infinitely many zigzags of arbitrarily long length.

Theorem 4.21. *An oriented graph $G := (V, \mathcal{E})$ is an absolute retract in the category of oriented graphs if and only if it is a retract of a product of oriented zigzags.*

We just give a sketch. For details, see Chapter V of [42] and the forthcoming paper [8]. The proof has three steps. Let G be an absolute retract. First, one proves that G has no 3-element cycle. Second, one proves that the zigzag distance between two vertices of G satisfies the cancelation rule. From Proposition 4.19, it belongs to $\mathbf{N}(\Lambda^*)$; from Theorem 4.20, G isometrically embeds into a product of oriented zigzags. Since G is an absolute retract, it is a retract of that product.

As illustrated by the results of Tarski and Sine and Soardi, absolute retracts are appropriate candidates for the fixed point property. Reflexive graphs with the fixed point property must be antisymmetric, i.e., oriented. Having described absolute retracts among oriented graphs, we derive from Theorem 4.8 that the bounded ones have the fixed point property.

We start with a characterization of accessible elements of $\mathcal{N}_\Lambda$. The proof is omitted.

Lemma 4.22. *Every element v of $\mathcal{N}_\Lambda \setminus \{\Lambda^*, \emptyset\}$ is accessible.*

Theorem 4.23. *If a graph G, finite or not, is a retract of a product of reflexive and directed zigzags of bounded length then every commuting set of endomorphisms has a common fixed point.*

Proof. We may suppose that G has more than one vertex. The diameter of G equipped with the zigzag distance belongs to $\mathcal{N}_\Lambda \setminus \{\Lambda^*, \emptyset\}$. According to Lemma 4.22, it is accessible, hence as a metric space, G is bounded. Being a retract of a product of hyperconvex metric spaces it is hyperconvex. Theorem 4.8 applies. □

4.9. Bibliographical comments

Generalizations of the notion of a metric space are as old as the notion of ordinary metric space and arise from geometry and logic, as well as probability. The generalization we consider, originating in [22], is one among several; the paper [22] contains 71 references, e.g. Blumenthal and Menger [10], [11] [12], as well as Lawvere [30], to mention just a few. It was motivated by the work of Quilliot on graphs and posets [39,40]. The characterization of hyperconvex spaces due to Aronszjan-Panitchpakdi [3] and the existence of an injective envelope, obtained for ordinary metric spaces by Isbell [21], and developed by Dress [18], was extended to metric spaces over some ordered monoid (what we called here an involutive Heyting algebra). A study of hole-preserving maps and a characterization of absolute retracts w.r.t. these maps by means of the replete space were also obtained. For more recent developments, see [7,24,27]. It was pointed out recently to the second author that the study of these Heyting algebras goes back to the late 1930's and the work of Ward and Dilworth (1939) [48]. The term "quantale" was introduced in 1984 by Mulvey [33] as a combination of "quantum logic" and "locale". We find it convenient to use the terminology of [23]. A huge literature has developed about quantales (see Rosenthal [41] and the recent book of Eklund et al. [20]) and spaces over these objects. Inevitably, there is some overlap with the work mentioned above (e.g. compare the main result of [9] with the existence of injective envelopes obtained previously in [22]; see also [2]).

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