Normal Stucture Property in Generalized James Spaces.

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Abstract

In an earlier paper by M.A.Khamsi a constant β_p was introduced in any Banach space with a Schauder basis, which is related to the normal structure property. We prove here an inequality for this constant in generalized James spaces and use it to investigate the normal structure property in these spaces. Then we discuss in the context of generalised James spaces a property introduced by D.Tingley.

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1 Introduction.

The class of spaces which possess the geometric property of normal structure, this property being a sufficient condition for the existence of fixed points of nonexpansive mappings ,on weakly compact convex subsets of a Banach space, has been studied widely [3,8,9,10,14]. This class is known to include not only an important subclass of reflexive spaces but also some interesting nonreflexive spaces. In this paper we investigate this property in generalised James spaces. First we prove an inequality concerning the constant β_p, introduced by one of us [9], and use it to obtain a condition for the normal structure property in a generalized James space. Then we show that a James space $J(X)$ has the normal structure property if the norm of X is uniformly monotone. This is obtained by relating uniform monotonicity to a property (*) defined and studied by D.Tingley [14].

2 Basic Definitions.

Let X be a Banach space having a Schauder basis (x_n). We assume that (x_n) is spreading and 1-unconditional. Recall that (x_n) is said to be spreading if for every (α_n) in R we have

$$||\sum_n \alpha_n x_n||_X = ||\sum_n \alpha_n x_{p_n}||_X$$

where $p_1 < p_2 < ...$ And 1-unconditional if for every (α_n) in R we have

$$||\sum_n \epsilon_n \alpha_n x_n||_X = ||\sum_n \alpha_n x_n||_X$$

where $\epsilon_n = \pm 1$.

Definition1. A family of James type spaces.
Let c_0 denotes the space of all real sequences that converge to 0.

(1) The space $J_1(X)$ consists of all sequences $(\alpha_n) \in c_0$ for which $||(\alpha_n)||_1 < \infty$ where

$$||(\alpha_n)||_1 = \sup\{||\sum_{1\leq i\leq n} (\alpha_{p_i} - \alpha_{p_{i+1}})x_i + \alpha_{p_{n+1}} x_{n+1}||_X\},$$

the supremum being taken over all finite increasing sequences of positive numbers $p_1, p_2, ..., p_{n+1}$.

(2) The space $J(X)$ consists of all sequences $(\alpha_n) \in c_0$ for which $||(\alpha_n)|| < \infty$ where

$$||(\alpha_n)|| = \sup\{|| \sum_{1 \le i \le n} (\alpha_{p_i} - \alpha_{p_{i+1}}) x_i + (\alpha_{p_{n+1}} - \alpha_{p_1}) x_{n+1} ||_X\},$$

the supremum being taken over all finite increasing sequences of positive numbers $p_1, p_2, ..., p_{n+1}$.

It is easily shown [11,13] that the family of spaces $J(X)$ and $J_1(X)$ are Banach spaces.

The original James spaces [7] is obtained by taking $X = l_2$ in the definition of $J(X)$. It is the first example of a nonreflexive Banach space with a basis that is of finite codimension in its second dual. Let us mention that this space has been used to disprove several conjectures in Banach space geometry [1,2,5,11,12].

Let K be a nonempty convex subset of X. A mapping $T : K \to K$ is said to be nonexpansive if $||Tx - Ty||_X \le ||x - y||_X$ for every $x, y \in K$. The space X is said to have the fixed point property if every nonexpasive mapping defined on a weakly-compact convex subset of X has a fixed point. One of the fundamental result [10] in this direction relates the fixed point property and a geometrical property called normal structure.

Definition 2. The Banach space X is said to have *normal structure property* if for every weakly-compact convex subset K of X there exists a point $x \in K$ such that

$$r(x, K) = \sup\{||x - y||_X ; y \in K\} < diam(K).$$

This property was introduced by Brodskii and Milman [4] who gave a simple sequential characterization of it. Indeed it is proved in [4] that X has normal structure property if and only if X does not contain a weakly-convergent diametral sequence. Recall that (z_n) is said to be diametral if

$$\lim_{n \to \infty} d(z_{n+1}, co(z_1, .., z_n)) = diam(z_n),$$

where $co(A)$ is the convex hull of the set A and

$$d(z_{n+1}, co(z_1, .., z_n)) = \inf\{||z_{n+1} - z||_X ; z \in co(z_1, .., z_n)\}.$$

3 Normal structure property in James spaces.

In [9] one of the authors introduced a constant β_p in any Banach space with Schauder basis which is related to normal structure property. Let us recall the definition of β_p. Let X be a Banach space with Schauder basis (x_n). Define $\beta_p(X)$,for $p \in [1, \infty)$, to be the infinimum of the set of numbers λ such that

$$(||u||_X^p + ||v||_X^p)^{\frac{1}{p}} \leq \lambda ||u + v||_X,$$

for every $u, v \in X$ which verify $supp(u) + 1 < supp(v)$. Recall that

$$supp(\sum_n \alpha_n x_n) = \{n; \alpha_n \neq 0\}.$$

The notation $A + k < B$, for any subsets A, B of N, means $a + k < b$ for every $a \in A$ and $b \in B$. In [9] it is proved that if $\beta_p(X) < 2^{\frac{1}{p}}$ then X has normal structure property. Using this kind of ideas Besbes [3] proved that $J_1(l_p)$ has normal structure property. Another proof of this result can be found in [6]. In the next theorem we discuss a generalization of this result.

Theorem 1. Let X ba Banach space as described above. Then for every $p \in [1, \infty)$ we have

$$\beta_p(J_1(X)) \leq \beta_p(X).$$

Proof. Let u, v be in $J_1(X)$ such that $supp(u) + 1 < supp(v)$. We can assume that v has a finite support. Therefore one can find $p_1 < p_2 < .. < p_n \leq L$ and $q_1 < q_2 < .. < q_m$ with $q_1 > L$ such that

$$||u||_{J_1(X)} = || \sum_{1 \leq i \leq n-1} (\alpha_{p_i} - \alpha_{p_{i+1}})x_i + \alpha_{p_n} x_n ||_X,$$

and

$$||v||_{J_1(X)} = || \sum_{1 \leq i \leq m-1} (\beta_{q_i} - \beta_{q_{i+1}})x_i + \beta_{q_m} x_m ||_X,$$

if $u = (\alpha_n)$ and $v = (\beta_n)$. Using the spreading behavor of the Schauder basis (x_n) and the definition of $\beta_p(X)$ we get

$$(||u||^p + ||v||^p)^{\frac{1}{p}} \leq \beta_p(X) \; || \sum_{1 \leq i \leq n+m} (\gamma_i - \gamma_{i+1})x_i + \gamma_{n+m+1} x_{n+m+1} ||_X,$$

where $\gamma_i = \alpha_{p_i}$ for every $i \in [1, n]$ and $\gamma_j = \beta_{q_{j-(n+1)}}$ for every $j \in [n+2, m+n+1]$ and $\gamma_{n+1} = 0$. Hence because of the definition of $||u+v||_{J_1(X)}$ we get

$$(||u||^p + ||v||^p)^{\frac{1}{p}} \leq \beta_p(X)\ ||u+v||_{J_1(X)}.$$

The proof of theorem 1 is therefore complete.

Using the properties of the constants β_p we get the following corollary.
Corollary.1 The following hold.

- $\beta_p(J_1(l_p)) = 1$ for every $p \in [1, \infty)$,

Using the main result of [9] one will get the following result.

Theorem 2. Let X be a Banach space as described above. Then $J_1(X)$ has normal structure property provided $\beta_p(X) < 2^{\frac{1}{p}}$.

The next natural question is under which conditions on X the James space $J(X)$ has normal structure property. This question was motivated by some partial known results. Indeed in [8] the author showed that $J(l_2)$ has the fixed point property. The method used is based on nonstandard techniques and does not depend on normal structure. As matter of fact this method was developed to show that certain Banach spaces that fail to have normal structure property, have the fixed point property. It was unkown whether $J(l_2)$ has normal structure property. This was answered by Tingley in [14]. Indeed Tingley noticed that $J(l_2)$ enjoys a property he called $(*)$ and defined as.

Definition 3. We say that a Banach space X satisfies property $(*)$ if every sequence (z_n) that converges weakly to 0 satisfies

$$(*)\quad \sup_m \{\limsup_n ||z_n - z_m||\} > \liminf_n ||z_n||.$$

It is not hard to see that if X enjoys the property $(*)$ then it does not contain a diametral sequence, and therefore X has normal structure property. The main result of [14] is to prove that $J(l_2)$ enjoys the property $(*)$. In what follows we discuss the property $(*)$ in $J(X)$.

We will say that the norm of X is uniformly monotone if for every $\alpha > 0$ there exists $\delta > 0$ such that

$$||\sum_{i\geq 2}\alpha_i x_i|| + \delta \leq ||\alpha x_1 + \sum_{i\geq 2}\alpha_i x_i||$$

for any (α_n).

Lemma 1. Let X be a Banach space described as above. Assume that the norm of X is uniformly monotone. Let (z_n) be weakly convergent to 0 in $J(X)$ for which no subsequence is convergent with respect to the norm. Then there exists a subsequence $(z_{n'})$ of (z_n) such that

$$\sup_q\{\limsup_r ||z_{r'} - z_{q'}||\} > \liminf_n ||z_{n'}||.$$

Proof. Since the conclusion is independent of how many extractions we will be making, we will denote any subsequence of (z_n) by (z_n) again. Because (z_n) is weakly convergent to 0 there exits a sequence of blocks (u_n) such that

$$\lim_{n\to\infty} ||z_n - u_n||_{J(X)} = 0.$$

One can assume that $supp(u_n) \subset [b_n, B_n]$ where $b_n < B_n < b_{n+1} - 2$ for every $n \in N$. Set $z_n = (\alpha_i^n)$ and $u_n = (\beta_i^n)$. Using the definition of $||u_n||_{J(X)}$ there exist $p_1 < p_2 < .. < p_{k+1}$, depending on n, such that

$$||u_n||_{J(X)} = ||\sum_{1\leq i\leq k}(\beta^n_{p_{i+1}} - \beta^n_{p_i})x_i + (\beta^n_{p_{k+1}} - \beta^n_{p_1})x_{k+1}||_X.$$

Since the norm is uniformly monotone one can show that $\beta^n_{p_{k+1}}\beta^n_{p_1} lleq 0$ and at least one term is not zero. Put $M = \sup_n\{||z_n||_{c_0}\} = \sup\{|\alpha^q_t|; q,t \in N\}$. From our assumption on (z_n) we deduce that $M > 0$. Let $0 < \epsilon < M$. Hence there exist t, q such that $|\alpha^q_t| > M - \epsilon$. Let $r > t$, then by definition of the norm in $J(X)$ we have

$$(*)\quad ||P_{r-1}(z_q) - u_r||_{J(X)} \geq ||\alpha^q_t x_1 - \beta^r_{p_1}x_2 - \sum_{1\leq i\leq k}(\beta^r_{p_{i+1}} - \beta^r_{p_i})x_{i+2} - (\beta^r_{p_{k+1}} + \alpha^q_t)x_{k+3}||_X$$

and

$$(**)\quad ||P_{r-1}(z_q) - u_r||_{J(X)} \geq ||-\alpha^q_t x_1 - (\beta^r_{p_1} + \alpha^q_t)x_2 - \sum_{1\leq i\leq k}(\beta^r_{p_{i+1}} - \beta^r_{p_i})x_{i+2} - \beta^r_{p_{k+1}}x_{k+3}||_X.$$

Here we have $P_s(z) = (\alpha_1, \alpha_2, .., \alpha_s, 0, 0, ..)$ if $z = (\alpha_n)$. By using the condition on $\beta^n_{p_1}$ and $\beta^n_{p_{k+1}}$ plus $(*)$ and $(**)$ one can assume that $\alpha^q_t > 0$, $\beta^r_{p_{k+1}} \geq 0 \geq \beta^r_{p_1}$. Since

$$\beta^r_{p_{k+1}} + \alpha^q_t \geq \beta^r_{p_{k+1}} + M - \epsilon$$

$$\geq \beta^r_{p_{k+1}} - \beta^r_{p_1} - \epsilon$$

and using the monotonicity of the norm we get

$$||P_{r-1}(z_q) - u_r||_{J(X)} \geq ||-\alpha^q_t x_1 - \beta^r_{p_1} x_2 - \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+2} - (\beta^r_{p_{k+1}} - \beta^r_{p_1} - \epsilon) x_{k+3}||_X.$$

Therefore

$$||P_{r-1}(z_q) - u_r||_{J(X)} \geq ||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1} - \epsilon) x_{k+2}||_X.$$

The triangle inequality implies

$$||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1} - \epsilon) x_{k+2}||_X \geq$$

$$||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1}) x_{k+2}||_X - \epsilon.$$

Hence

$$||P_{r-1}(z_q) - u_r||_{J(X)} \geq ||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1}) x_{k+2}||_X - \epsilon.$$

Let r goes to infinity, we get

$$\limsup_r ||z_q - z_r||_{J(X)} \geq \limsup_r ||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1}) x_{k+2}||_X - \epsilon.$$

Since $\alpha^q_t > M - \epsilon$ there exists $\delta > 0$ such that

$$||\alpha^q_t x_1 + \sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_{i+1} + (\beta^r_{p_{k+1}} - \beta^r_{p_1}) x_{k+2}||_X \geq$$

$$\delta + ||\sum_{1\leq i\leq k} (\beta^r_{p_{i+1}} - \beta^r_{p_i}) x_i + (\beta^r_{p_{k+1}} - \beta^r_{p_1}) x_{k+1}||_X = \delta + ||u_r||.$$

Therefore

$$\limsup_r ||z_q - z_r||_{J(X)} \geq \delta + \limsup_r ||z_r||_{J(X)} - \epsilon.$$

Hence

$$\sup_q \{\limsup_r ||z_q - z_r||_{J(X)}\} \geq \delta + \limsup_r ||z_r|| - \epsilon.$$

Let ϵ goes to 0 to get

$$\sup_q \{\limsup_r ||z_q - z_r||_{J(X)}\} > \limsup_{r\to\infty} ||z_r||_{J(X)}.$$

This clearly completes the proof of Lemma 1.

From this technical lemma we get the main result of this work.

Theorem 3. Let X be a Banach space as described above. Assume that the norm of X is uniformly monotone. Then $J(X)$ has normal structure property.

Proof. Since any diametral sequence cannot satisfy the conclusion of Lemma 1 and using the sequential characterization of normal structure property, we get the conclusion of Theorem 3.

Clearly any l_p for $p \in [1, \infty)$ satisfies the assumptions of Theorem 3. Therefore as a corollary we obtain.

Corollary 2. $J(l_p)$ has normal structure property for every $1 \leq p < \infty$.

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