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Inequalities in metric spaces with applications

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ABSTRACT

Analogues of the parallelogram identity and the (CN) inequality of Bruhat and Tits in uniformly convex metric spaces are established. As an application of the new inequalities, we prove two fixed point results for single-valued and multi-valued Lipschitzian mappings.

1. Introduction

The Hilbert parallelogram identity is the following:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for any x, y in a Hilbert space H, and it plays a major role in proving many basic results. This identity implies

$$\|\lambda x + (1-\lambda)y\|^2 + \lambda(1-\lambda)\|x-y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2$$

for any x, y in H and for any $\lambda \in [0,1]$. In [1], Xu gave a nice extension of these identities to uniformly convex Banach spaces. His work has been used as an important tool in proving many interesting results. In order to extend Xu's ideas to metric spaces, Beg [2] had to change the definition of uniform convexity in metric spaces. One of the difficulties in carrying out such extensions lies in the heavy use of the linear structure of the Banach spaces.

In this paper, we use the classical definition of uniform convexity in metric spaces and obtain an analogue of the parallelogram inequality and the (CN) inequality of Bruhat and Tits [3] in these spaces. Then we give several applications of our paper as in [1]. To the best of our knowledge this is the first attempt that successfully carries out such an extension on a nonlinear domain.

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2. Uniform convexity in metric spaces

Throughout this paper, (M, d) will stand for a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in M are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = \beta d(x, y), \quad \text{and} \quad d(z, y) = (1-\beta)d(x, y).$$

Such metric spaces are usually called *convex metric spaces* [4]. Moreover, if we have

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \le \frac{1}{2}d(x, y),$$

for all p, x, y in M, then M is said to be a *hyperbolic metric space* (see [5]).

Obviously, normed linear spaces are hyperbolic spaces. One can consider, as nonlinear examples, the Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [7], and the CAT(0) spaces [8–10] (see Example 2.1). We will say that a subset C of a hyperbolic metric space M is convex if $[x, y] \subset C$ whenever x, y are in C.

Let τ be another topology on M that is weaker than the metric topology. We will assume that τ is lower semi-continuous, that is,

$$d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n)$$

for every $\{x_n\}$ and $\{y_n\}$ in M which are τ-convergent to x and y, respectively.

Definition 2.1. Let (M, d) be a hyperbolic metric space. We say that M is uniformly convex (for short, UC) if for any $a \in M$, for every $r > 0$, and for each $\epsilon > 0$,

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\, d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x, a) \le r, d(y, a) \le r, d(x, y) \ge r\varepsilon\right\} > 0.$$

The definition of uniform convexity finds its origin in Banach spaces [11]. To the best of our knowledge, the first attempt to generalize this concept to metric spaces was made in [12]. The reader may also consult [7,5].

From now on we assume that M is a hyperbolic metric space and if (M, d) is uniformly convex, then for every $s \ge 0$, $\epsilon > 0$, there exists $\eta(s, \epsilon) > 0$ depending on s and ϵ such that

$$\delta(r, \varepsilon) > \eta(s, \epsilon) > 0 \quad \text{for any } r > s.$$

Remark 2.1. (i) Let us observe that $\delta(r, 0) = 0$, and $\delta(r, \varepsilon)$ is an increasing function of ε for every fixed r.

(ii) For $r_1 \le r_2$ it holds that

$$1 - \frac{r_2}{r_1}\left(1 - \delta\left(r_2, \varepsilon\frac{r_1}{r_2}\right)\right) \le \delta(r_1, \varepsilon).$$

(iii) If (M, d) is uniformly convex, then (M, d) is strictly convex, i.e., whenever

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) = d(x, a) = d(y, a)$$

for any $x, y, a \in M$, then we must have $x = y$.

Lemma 2.1. *Assume that (M, d) is uniformly convex. Let $\{C_n\} \subset M$ be a sequence of nonempty, nonincreasing, convex, bounded and closed sets. Let $x \in M$ be such that*

$$0 < d = \lim_{n\to\infty} d(x, C_n) < \infty.$$

Let $x_n \in C_n$ be such that $d(x, x_n) \to d$. Then $\{x_n\}$ is a Cauchy sequence.

Proof. Assume to the contrary that this is not the case. Passing to a subsequence if necessary, we can assume that there exists $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \le d(x_k, x_p), \quad k \ne p.$$

Set $d_k = d(x, x_k)$, for any $k \ge 1$. By (UC), we have

$$d\left(x, \frac{1}{2}x_k \oplus \frac{1}{2}x_p\right) \le \left(1 - \delta\left(d(k, p), \frac{\varepsilon_0}{d(k, p)}\right)\right) d(k, p),$$

where $d(k,p) = \max\{d_k, d_p\}$. Without loss of generality, we may assume that $d(k,p) \le 2d$ for each $k, p \ge N$, where N is fixed. Hence

$$d\left(x, \frac{1}{2}x_k \oplus \frac{1}{2}x_p\right) \le \left(1 - \delta\left(d(k,p), \frac{\varepsilon_0}{2d}\right)\right) d(k,p).$$

By (UC), there exists $\eta = \eta\left(\frac{d}{3}, \frac{\varepsilon_0}{2d}\right) > 0$ such that

$$\delta\left(d(k,p), \frac{\varepsilon_0}{2d}\right) > \eta\left(\frac{d}{3}, \frac{\varepsilon_0}{2d}\right) > 0,$$

which implies, in view of the fact that the sets $C_n,\ n \ge 1$, are convex and nonincreasing, that

$$\min\{d(x, C_k), d(x, C_p)\} \le d\left(x, \frac{1}{2}x_k \oplus \frac{1}{2}x_p\right) \le (1-\eta)d(k,p).$$

Hence

$$\min\{d(x, C_k), d(x, C_p)\} \le (1-\eta)d(k,p).$$

Letting k and p go to infinity, we get that $0 < d \le (1-\eta)d$, where $\eta > 0$. This is contradiction. $\square$

Recall that a hyperbolic metric space (M, d) is said to have the property (R) if any nonincreasing sequence of nonempty, convex, bounded and closed sets has a nonempty intersection.

Our next result deals with the existence and the uniqueness of the best approximants of convex, closed and bounded sets in a uniformly convex metric space. This result is of interest in itself, as uniform convexity implies the property (R), which reduces to reflexivity in the linear case.

Theorem 2.1. *Assume that (M, d) is complete and uniformly convex. Let $C \subset M$ be nonempty, convex and closed. Let $x \in M$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C, i.e., there exists a unique $x_0 \in C$ such that*

$$d(x, x_0) = d(x, C).$$

Proof. Set $d_0 = d(x, C)$. Let $\{x_n\} \subset C$ be such that

$$d_0 = d(x, C) = \lim_{n\to\infty} d(x_n, x).$$

If $d_0 = 0$, then $x \in C$ since C is closed. Then we must have $x_0 = x$. So we can assume then that $d_0 > 0$. Hence, from Lemma 2.1 (applied to $C_n = C$), the sequence $\{x_n\}$ is Cauchy. Since M is complete and C is closed, there exists then $x_0 \in C$ such that

$$\lim_{n\to\infty} d(x_n, x_0) = 0.$$

We claim that x_0 is the best approximant that we are seeking. Indeed, we have

$$d(x, C) \le d(x, x_0) = \lim_{n\to\infty} d(x, x_n) = d(x, C).$$

Hence $d(x, x_0) = d(x, C)$, i.e. x_0 is an approximant. Assume that there exists another approximant $y \in C$, i.e. $d(x, y) = d(x, C)$. Since C is convex, we get

$$d(x, C) \le d\left(x, \frac{1}{2}x_0 \oplus \frac{1}{2}y\right) \le \frac{d(x, x_0) + d(x, y)}{2} = d(x, C).$$

Hence

$$d\left(x, \frac{1}{2}x_0 \oplus \frac{1}{2}y\right) = d(x, x_0) = d(x, y),$$

which implies that $x_0 = y$ since (M, d) is strictly convex. $\square$

The following result is the analogue of the well-known theorem that states that any uniformly convex Banach space is reflexive. For a reference, we refer to Theorem 2.1 in [7].

Theorem 2.2. *If (M, d) is complete and uniformly convex, then (M, d) has the property (R).*

Proof. Let $\{C_n\} \subset M$ be a sequence of nonempty, nonincreasing, convex, bounded and closed sets. We need to prove that this sequence of sets has nonempty intersection. Let $x \in M$. Since sets in $\{C_n\}$ are bounded and nonincreasing, the sequence $\{d(x, C_n)\}$ is increasing and bounded. Hence $\lim_{n\to\infty} d(x, C_n) = d_1$ exists. Let $x_n \in C_n$ be an approximant of x, i.e. $d(x, x_n) = d(x, C_n)$, for any $n \geq 1$. Lemma 2.1 implies that $\{x_n\}$ is Cauchy. Hence there exists $y \in M$ such that $\lim_{n\to\infty} x_n = y$. Since sets in $\{C_n\}$ are nonincreasing and closed, therefore $y \in C_n$, for any $n \geq 1$. Hence $\bigcap_{n\geq 1} C_n$ is not empty. □

Remark 2.2. Note that any hyperbolic metric space M which satisfies the property (R) is complete. Indeed, let $\{x_n\}$ be a Cauchy sequence in M. Set

$$\varepsilon_n = \sup\{d(x_m, x_s);\ m, s \geq n\}, \quad n = 1, \ldots.$$

Our assumption implies that $\lim_{n\to\infty} \varepsilon_n = 0$. In hyperbolic metric spaces, closed balls are convex. Therefore the property (R) implies that $\bigcap_{n\geq 1} B(x_n, \varepsilon_n) \neq \emptyset$. It is easy to check that this intersection is reduced to one point which is the limit of $\{x_n\}$.

The following lemma is needed to establish a metric version of the main results of [1] proved in the setting of the Banach space.

Lemma 2.2. *Let (M, d) be uniformly convex. Assume that there exists $R \in [0, +\infty)$ such that*

$$\limsup_{n\to\infty} d(x_n, a) \leq R, \qquad \limsup_{n\to\infty} d(y_n, a) \leq R, \quad \text{and} \quad \lim_{n\to\infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = R.$$

Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Proof. Without loss of generality, we may assume that $R > 0$. Assume that the conclusion is not true. Let $\gamma > 0$ be arbitrarily chosen. For n sufficiently large, passing to subsequences if necessary, we may assume that there exists $\varepsilon > 0$ such that $d(x_n, a) \leq R + \gamma$, $d(y_n, a) \leq R + \gamma$ and $d(x_n, y_n) \geq R\varepsilon,\ n \geq 1$. Since (M, d) is uniformly convex, we have

$$0 < \eta(R, \varepsilon) < \delta(R + \gamma, \varepsilon) \leq 1 - \frac{1}{R + \gamma} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) \to \frac{\gamma}{R + \gamma}.$$

Letting $\gamma \to 0$, we get a contradiction. □

A metric version of the parallelogram identity goes as follows (see [5,1]).

Theorem 2.3. *Let (M, d) be uniformly convex. Fix $a \in M$. For each $0 < r$ and for each $\varepsilon > 0$ set*

$$\Psi(r, \varepsilon) = \inf\left\{\frac{1}{2}d^2(a, x) + \frac{1}{2}d^2(a, y) - d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)\right\},$$

where the infimum is taken over all $x, y \in M$ such that $d(a, x) \leq r$, $d(a, y) \leq r$, and $d(x, y) \geq r\varepsilon$. Then $\Psi(r, \varepsilon) > 0$ for any $0 < r$ and for each $\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have:

(i) $\Psi(r, 0) = 0$;
(ii) $\Psi(r, \varepsilon)$ *is a nondecreasing function of* ε;
(iii) *if* $\lim_{n\to\infty} \Psi(r, t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

Proof. Assume on the contrary that there exist $0 < r < \infty$ and $\varepsilon > 0$ such that $\Psi(r, \varepsilon) = 0$. Then there exist $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} \left[\frac{1}{2}d^2(a, x_n) + \frac{1}{2}d^2(a, y_n) - d^2\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right)\right] = 0$$

where $d(a, x_n) \leq r$, $d(a, y_n) \leq r$, and $d(x_n, y_n) \geq r\varepsilon$. Using the inequality $2ab \leq a^2 + b^2$, for any $a, b \in \mathbb{R}$, we get

$$d^2\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) \leq \left(\frac{d(a, x_n) + d(a, y_n)}{2}\right)^2 \leq \frac{d^2(a, x_n) + d^2(a, y_n)}{2}.$$

Hence

$$\left(\frac{d(a, x_n) - d(a, y_n)}{2}\right)^2 \leq \frac{1}{2}d^2(a, x_n) + \frac{1}{2}d^2(a, y_n) - d^2\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right),$$

which implies that $\lim_{n\to\infty} |d(a,x_n) - d(a,y_n)| = 0$. Since the sequence $(d(a,x_n))$ is bounded, passing to a subsequence if necessary, we can assume that $\lim_{n\to\infty} d(a,x_n) = R \le r$ exists. Our assumptions will then imply

$$\lim_{n\to\infty} d(a,y_n) = \lim_{n\to\infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = R.$$

Now from Lemma 2.2, we get that $\lim_{n\to\infty} d(x_n,y_n) = 0$ which contradicts the fact that $d(x_n,y_n) \ge r\varepsilon > 0$. The proofs of (i)–(iii) are immediate. □

The concept of p-uniform convexity was used extensively by Xu [1] (see also [13, p. 310]); its nonlinear version for $p = 2$ is given below.

Definition 2.2. We will say that (M,d) is 2-uniformly convex if

$$c_M = \inf\left\{\frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2}; r > 0, \varepsilon > 0\right\} > 0.$$

Note that (M,d) is 2-uniformly convex if and only if

$$\inf\left\{\frac{\delta(r,\varepsilon)}{\varepsilon^2}; r > 0, \varepsilon > 0\right\} > 0.$$

Example 2.1. Let (X,d) be a metric space. A *geodesic* from x to y in X is a mapping c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d\left(c(t), c\left(t'\right)\right) = \left|t - t'\right|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x,y]$, and called the segment joining x to y.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1,x_2,x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}\left(\bar{x}_i,\bar{x}_j\right) = d\left(x_i,x_j\right)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [14]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x,y) \le d(\bar{x},\bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [9]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$, which will be denoted by $\frac{y_1\oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \le \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. As for the Hilbert space, the (CN) inequality implies that CAT(0) spaces are uniformly convex with

$$\delta(r,\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

One may also find the modulus of uniform convexity via similar triangles. The (CN) inequality also implies that

$$\Psi(r,\varepsilon) = \frac{r^2\varepsilon^2}{4}.$$

This clearly implies that any CAT(0) space is 2-uniformly convex with $c_M = \frac{1}{4}$.

Recall that $\tau : M \to \mathbb{R}_+$ is called a type if there exists $\{x_n\}$ in M such that

$$\tau(x) = \limsup_{n\to\infty} d(x,x_n).$$

Theorem 2.4. *Assume that (M,d) is complete and uniformly convex. Let C be any nonempty, closed, convex and bounded subset of M. Let τ be a type defined on C. Then any minimizing sequence of τ is convergent. Its limit is independent of the minimizing sequence.*

Proof. Let $\{x_n\}$ be a sequence in C such that $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$. Set $\tau_0 = \inf\{\tau(x);\ x \in C\}$. Let $\{y_k\}$ be a minimizing sequence of τ. Since C is bounded, there exists $R > 0$ such that $d(x, y) \le R$ for any $x, y \in C$. Since (M, d) is uniformly convex, from Theorem 2.3, we get

$$d^2\left(\frac{1}{2}y_m \oplus \frac{1}{2}y_k, x_n\right) \le \frac{1}{2}d^2(y_m, x_n) + \frac{1}{2}d^2(y_k, x_n) - \Psi\left(R, \frac{1}{R}d(y_m, y_k)\right).$$

When n goes to infinity, we get

$$\tau^2\left(\frac{1}{2}y_m \oplus \frac{1}{2}y_k\right) \le \frac{1}{2}\tau^2(y_k) + \frac{1}{2}\tau^2(y_m) - \Psi\left(R, \frac{1}{R}d(y_m, y_k)\right).$$

Hence

$$\tau_0^2 \le \frac{1}{2}\tau^2(y_k) + \frac{1}{2}\tau^2(y_m) - \Psi\left(R, \frac{1}{R}d(y_m, y_k)\right),$$

for any $k, m \ge 1$. Therefore

$$\Psi\left(R, \frac{1}{R}d(y_m, y_k)\right) \le \frac{1}{2}\tau^2(y_k) + \frac{1}{2}\tau^2(y_m) - \tau_0^2.$$

Consequently, $\lim_{k,m\to\infty} \Psi\left(R, \frac{1}{R}d(y_m, y_k)\right) = 0$. The properties satisfied by Ψ imply that $\{y_k\}$ is Cauchy. Since M is complete and C is closed, the sequence $\{y_k\}$ is convergent to a point $z \in C$. Now we will prove that any other minimizing sequence also converges to z. Indeed let $\{u_n\} \in C$ be any other minimizing sequence of τ. Using the above argument, we have

$$\tau_0^2 \le \tau^2\left(\frac{1}{2}y_n \oplus \frac{1}{2}u_n\right) \le \frac{1}{2}\tau^2(y_n) + \frac{1}{2}\tau^2(u_n) - \Psi\left(R, \frac{1}{R}d(y_n, u_n)\right),$$

which implies

$$\Psi\left(R, \frac{1}{R}d(y_n, u_n)\right) \le \frac{1}{2}\tau^2(y_n) + \frac{1}{2}\tau^2(u_n) - \tau_0^2,$$

for any $n \ge 1$. As before, we get $\lim_{n\to\infty} d(y_n, u_n) = 0$. □

Remark 2.3. Assume in the proof of Theorem 2.4 that $\tau_0 = 0$ and let

$$C_\tau = \bigcap_{n\ge 1} \overline{\mathrm{conv}}\Big(\Big\{x_k;\ k \ge n\Big\}\Big),$$

which is nonempty in view of property (R). Let $x_\infty \in C_\tau$. Let $y \in C$, and $\varepsilon > 0$. By the definition of τ, there exists $n_0 \ge 1$ such that for every $n \ge n_0$

$$d(x_n, y) \le \tau(y) + \varepsilon.$$

Therefore, x_n belongs to the closed ball $B(y, \tau(y) + \varepsilon)$, which is convex, for any $n \ge n_0$. Hence

$$C_\tau \subset \overline{\mathrm{conv}}(\{x_n;\ n \ge n_0\}) \subset B(y, \tau(y) + \varepsilon).$$

Since this is true for every $\varepsilon > 0$, then $C_\tau \subset B(y, \tau(y))$ holds for any $y \in C$. In particular, we have $x_\infty \in B(y, \tau(y))$ and

$$d(x_\infty, y) \le \tau(y),$$

for any $y \in C$. If $\{y_n\}$ is any minimizing sequence of τ, then we have

$$d(x_\infty, y_n) \le \tau(y_n) \to 0 \quad \text{as } n \to \infty,$$

which means that $\{y_n\}$ converges to x_∞. In particular, C_τ is reduced to one point.

3. Applications

In this section we give several applications of our results. In particular, we discuss the existence of fixed points of uniformly Lipschitzian mappings.

Definition 3.1. A mapping $T : C \to C$ (a subset of M) is said to be *Lipschitzian* if there exists a non-negative number k such that $d(Tx, Ty) \le kd(x, y)$ for all x and y in C. The smallest such k is called a *Lipschitz constant* and will be denoted by $\mathrm{Lip}(T)$. The mapping T is called *uniformly Lipschitzian* if $\sup_{n\ge 1} \mathrm{Lip}(T^n) < \infty$.

It is well-known that if a mapping is uniformly Lipschitzian, then one may find an equivalent distance for which the mapping is nonexpansive (see [7, pages 34–38]). Indeed, let $T : C \to C$ be uniformly Lipschitzian. Put

$$\rho(x, y) = \sup\{d(T^n x, T^n y) : n = 0, 1, 2\ldots\}$$

for all $x, y \in C$; one can obtain a metric ρ on C which is equivalent to the metric d and relative to which T is nonexpansive. In this context, it is natural to ask the question: if a set C has the fixed point property (fpp) for nonexpansive mappings with respect to the metric d, then does C also have (fpp) for mappings which are nonexpansive relative to an equivalent metric? This is known as the stability of (fpp). The first result in this direction is due to Goebel and Kirk [15]. Motivated by such questions, we investigate the fixed point property of uniformly Lipschitzian mappings in uniformly convex hyperbolic metric spaces.

Recall that the normal structure coefficient $N(M)$ of the hyperbolic metric space M is defined (see [16]) by

$$N(M) = \inf\left\{\frac{\mathrm{diam}(C)}{R(C)};\, C \text{ bounded convex subset of } M \text{ with diam } (C) > 0\right\},$$

where $\mathrm{diam}(C) = \sup\{d(x, y);\; x, y \in C\}$ is the diameter of C, and

$$R(C) = \inf\{\sup_{y\in C} d(x, y);\; x \in C\}$$

is the Chebyshev radius of C.

For further development, we will need the following technical lemmas.

Lemma 3.1. *Let (M, d) be hyperbolic metric space and let C be a nonempty, closed and convex subset of M. Assume that M is 2-uniformly convex. Let $\{x_n\}$ be a bounded sequence in C. Then there exists a unique point $z \in C$ such that*

$$\limsup_{n\to\infty} d^2(x_n, z) + 2c_M d^2(z, x) \le \limsup_{n\to\infty} d^2(x_n, x)$$

for any $x \in C$.

Proof. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $d(x_n, x_m) \le R$, for any n, m. From Theorem 2.4, we know that there exists a unique $z \in C$ such that

$$\limsup_{n\to\infty} d(x_n, z) = \inf\left\{\limsup_{n\to\infty} d(x_n, x);\, x \in C\right\}.$$

Let $x \in C$. Since M is 2-uniformly convex, then we have

$$d^2\left(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\right) \le \frac{1}{2}d^2(x, x_n) + \frac{1}{2}d^2(z, x_n) - \Psi\left(R, \frac{1}{R}d(x, z)\right),$$

for any n, which implies

$$\limsup_{n\to\infty} d^2\left(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\right) \le \frac{1}{2}\limsup_{n\to\infty} d^2(x, x_n) + \frac{1}{2}\limsup_{n\to\infty} d^2(z, x_n) - \Psi\left(R, \frac{1}{R}d(x, z)\right).$$

The definition of z implies that

$$\limsup_{n\to\infty} d^2(z, x_n) \le \frac{1}{2}\limsup_{n\to\infty} d^2(x, x_n) + \frac{1}{2}\limsup_{n\to\infty} d^2(z, x_n) - \Psi\left(R, \frac{1}{R}d(x, z)\right),$$

or

$$\frac{1}{2}\limsup_{n\to\infty} d^2(z, x_n) \le \frac{1}{2}\limsup_{n\to\infty} d^2(x, x_n) - \Psi\left(R, \frac{1}{R}d(x, z)\right).$$

Using the 2-uniform convexity, we get

$$\frac{1}{2}\limsup_{n\to\infty} d^2(z, x_n) \le \frac{1}{2}\limsup_{n\to\infty} d^2(x, x_n) - c_M d^2(x, z),$$

which implies the desired inequality. $\square$

The following result is similar to Theorem 3 of [1].

Theorem 3.1. *Let (M, d) be a hyperbolic metric space which is 2-uniformly convex. Let C be a nonempty, closed, convex and bounded subset of M. Let $T : C \to C$ be uniformly Lipschitzian with*

$$\lambda(T) = \sup_{n\geq 1} \mathrm{Lip}(T^n) < \left(\frac{1+\sqrt{1+8c_M N(M)^2}}{2}\right)^{1/2}.$$

Then T has a fixed point in C.

Proof. Fix $x_0 \in C$. Using Lemma 3.1, one can construct inductively a sequence $\{x_m\}$ in C such that x_{m+1} is the point z found in Lemma 3.1 associated with the sequence $\{T^n(x_m)\}$, for any $m \geq 0$. For any $m \geq 0$, set

$$r_m = \limsup_{n\to\infty} d(x_{m+1}, T^n(x_m)) \quad \text{and} \quad R_m = \sup_{n\geq 1} d(x_m, T^n(x_m)).$$

Set $C^* = \overline{\mathrm{conv}}\{T^n(x_m);\ n \geq 1\}$. Then the property (R) which is satisfied by M implies the existence of a point $z \in C^*$ such that

$$\sup_{n\geq n_0} d(z, T^n(x_m)) \leq \frac{1}{N(M)} \mathrm{diam}(C^*) = \frac{1}{N(M)} \mathrm{diam}(\{T^n(x_m); n \geq 1\}).$$

Since $r_m \leq \limsup_{n\to\infty} d(z, T^n(x_m))$ and

$$\mathrm{diam}(\{T^n(x_m); n \geq 1\}) \leq \lambda(T) \sup_{n\geq 1} d(x_m, T^n(x_m)),$$

we get

$$r_m \leq \frac{\lambda(T)}{N(M)} R_m, \quad m = 1, \ldots.$$

This result is similar to Theorem 1 in [17]. Using Lemma 3.1, we get

$$r_m^2 + c_M d^2(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^2 + \frac{1}{2} \limsup_{n\to\infty} d^2(T^s(x_{m+1}), T^n(x_m)),$$

which implies that

$$r_m^2 + c_M d^2(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^2 + \frac{\lambda(T)^2}{2} \limsup_{n\to\infty} d^2(x_{m+1}, T^{n-s}(x_m)),$$

or

$$r_m^2 + c_M d^2(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^2 + \frac{\lambda(T)^2}{2} r_m^2.$$

Hence

$$c_M R_{m+1}^2 = c_M \sup_{s\geq 1} d^2(x_{m+1}, T^s(x_{m+1})) \leq \frac{\lambda(T)^2 - 1}{2} r_m^2 \leq \frac{(\lambda(T)^2 - 1)}{2} \frac{\lambda(T)^2}{N(M)^2} R_m^2,$$

which implies that $R_{m+1} \leq A R_m,\ m = 1, \ldots,$ where

$$A = \left(\frac{(\lambda(T)^2 - 1)\lambda(T)^2}{2c_M N(M)^2}\right)^{1/2}.$$

Our assumption on $\lambda(T)$ leads to $A < 1$. Since $R_m \leq A^{m-1} R_1$, for any $m \geq 1$, we conclude that $\sum_{m\geq 1} R_m$ is convergent. Since $d(x_m, x_{m+1}) \leq r_m + R_m \leq 2R_m$, for any $m \geq 1$, the series $\sum d(x_m, x_{m+1})$ is also convergent, and therefore $\{x_m\}$ is Cauchy. Since M satisfies the property (R), it is complete. Let $z \in C$ be the limit of $\{x_m\}$. Hence

$$d(z, T(z)) \leq d(z, x_m) + d(x_m, T(x_m)) + d(T(x_m), T(z)) \leq (1 + \mathrm{Lip}(T)) d(z, x_m) + R_m$$

for any $m \geq 1$. If we let $m \to \infty$, then we get $d(z, T(z)) = 0$ and therefore $T(z) = z$. □

Next we extend the above theorem to the case of multi-valued mappings. As in [18], we start by giving the definition of uniformly Lipschitzian multi-valued mappings via the generalized orbits.

Definition 3.2. Let (M, d) be a metric space. Let $T : M \to N(M)$, where $N(M)$ is the set of all nonempty subsets of M, be a multi-valued mapping. For any $x \in M$, the sequence $\{x_n\}$ is called a generalized orbit of x if $x_1 = x$ and $x_{n+1} \in T(x_n)$, for any $n \geq 1$.

It is clear that for a given $x \in M$, the mapping T may have many different orbits generated by x.

Definition 3.3. A multi-valued mapping $T : M \to N(M)$ is called a uniformly k-Lipschitzian mapping (with $k > 0$) if for any $x, y \in M$, and for any generalized orbit $\{x_n\}$ of x, there exists a generalized orbit $\{y_n\}$ of y such that

$$d(x_{n+h}, y_{n+s}) \leq kd(x_h, y), \quad \text{and} \quad d(x_{n+h}, x_{n+s}) \leq kd(x_h, x_s),$$

for any $h, n, s = 1, 2, \ldots$. The smallest such k will be denoted by $\lambda(T)$.

Note that when T is single-valued, then the above definition coincides with the traditional definition since any x will have one orbit generated by iterating T.

Next we prove a multi-valued version of Theorem 3.1.

Theorem 3.2. *Let (M, d) be hyperbolic metric space which is 2-uniformly convex. Let C be a nonempty closed, bounded and convex subset of M. Let us have $T : C \to \mathcal{C}(C)$, i.e. $T(x)$ is a nonempty closed subset of C, for any $x \in C$. If T is uniformly Lipschitzian with*

$$\lambda(T) < \left(\frac{1 + \sqrt{1 + 8c_M N(M)^2}}{2}\right)^{1/2},$$

then T has a fixed point, i.e. there exists $x \in C$ such that $x \in T(x)$.

Proof. Let $x \in C$ and let $\{x_n\}$ be a generalized orbit of x. Consider the type function generated by $\{x_n\}$, i.e. $\tau(z) = \limsup_{n\to} d(x_n, z)$, for $z \in C$. Using Lemma 3.1, there exists a unique $\omega \in C$ such that $\tau(\omega) = \inf\{\tau(z); z \in C\}$. Set $\sigma(x) = \omega$. Note that $\sigma^2(x)$ is the minimizer of the type function generated by a generalized orbit of $\sigma(x)$. Since T is uniformly Lipschitzian, there exists a generalized orbit $\{\sigma(x)_n\}$ such that

$$d(x_{n+h}, \sigma(x)_{n+m}) \leq \lambda(T)d(x_h, \sigma(x)_m),$$

for any $n, h, m \geq 1$. By induction, one will construct a sequence $\{\sigma^n(x)\}$ and generalized orbit $\{\sigma^n(x)_m\}_{m\geq 1}$ of $\sigma^n(x)$ for any $n \geq 1$, such that $\sigma^{n+1}(x)$ is the unique minimum point of the type generated by the generalized orbit $\{\sigma^n(x)_m\}_{m\geq 1}$. Set

$$r_m = \limsup_{n\to\infty} d(\sigma^m(x), \sigma^m(x)_n), \quad \text{and} \quad R_m = \sup_{n\geq 1} d(\sigma^m(x), \sigma^m(x)_n),$$

for any $m \geq 1$. As in the proof of Theorem 3.1, one can show that

$$r_m \leq \frac{\lambda(T)}{N(M)} R_m,$$

for any $m \geq 1$. Using Lemma 3.1, we get

$$r_m^2 + c_M d^2(\sigma^{m+1}(x), \sigma^{m+1}(x)_s) \leq \frac{1}{2} r_m^2 + \frac{1}{2} \limsup_{n\to\infty} d^2(\sigma^{m+1}(x)_s, \sigma^m(x)_n),$$

for any $s \geq 1$, which implies

$$r_m^2 + c_M d^2(\sigma^{m+1}(x), \sigma^{m+1}(x)_s) \leq \frac{1}{2} r_m^2 + \frac{\lambda(T)^2}{2} \limsup_{n\to\infty} d^2(\sigma^{m+1}(x), \sigma^m(x)_{n-s}),$$

for any $s \geq 1$, or

$$r_m^2 + c_M d^2(\sigma^{m+1}(x), \sigma^{m+1}(x)_s) \leq \frac{1}{2} r_m^2 + \frac{\lambda(T)^2}{2} r_m^2,$$

for any $s \geq 1$. Hence

$$d^2(\sigma^{m+1}(x), \sigma^{m+1}(x)_s) \leq \frac{(\lambda(T)^2 - 1)}{2c_M} r_m^2,$$

for any $s \geq 1$, which implies

$$R_{m+1}^2 = \sup_{s\geq 1} d^2(\sigma^{m+1}(x), \sigma^{m+1}(x)_s) \leq \frac{(\lambda(T)^2 - 1)}{2c_M} r_m^2.$$

Hence $R_{m+1} \leq AR_m,\ m = 1, \ldots,$ where

$$A = \left(\frac{(\lambda(T)^2 - 1)\lambda(T)^2}{2c_M N(M)^2}\right)^{1/2}.$$

Our assumption on $\lambda(T)$ leads to $A < 1$. Hence the series $\sum R_m$ is convergent. Since

$$d(\sigma^m(x), \sigma^{m+1}(x)) \leq r_m + R_m \leq 2R_m,$$

for any $m \geq 1$, the series $\sum d(\sigma^m(x), \sigma^{m+1}(x))$ is convergent. Hence $\{\sigma^m(x)\}$ is Cauchy. Let $z \in C$ be its limit. Next we prove that z is a fixed point of T, i.e. $z \in T(z)$. Indeed, we have

$$d(\sigma^m(x), \sigma^m(x)_1) \leq d(\sigma^m(x), \sigma^m(x)_n) + d(\sigma^m(x)_n, \sigma^m(x)_1)$$

which implies

$$d(\sigma^m(x), \sigma^m(x)_1) \leq d(\sigma^m(x), \sigma^m(x)_n) + \lambda(T)d(\sigma^m(x)_{n-1}, \sigma^m(x)),$$

and if we let $n \to \infty$, in the above inequality, we get

$$d(\sigma^m(x), \sigma^m(x)_1) \leq (1 + \lambda(T))r_m,$$

for any $m \geq 1$. Hence $\{\sigma^m(x)_1\}$ also converges to z. Using the uniform Lipschitzian behavior of T, for any $m \geq 1$, there exists a generalized orbit $\{z_n^m\}$ of z such that

$$d(\sigma^m(x)_n, z_n^m) \leq \lambda(T)d(\sigma^m(x), z)$$

for any $n \geq 1$. In particular, we have $d(\sigma^m(x)_1, z_1^m) \leq \lambda(T)d(\sigma^m(x), z)$. Hence $\{z_1^m\}$ also converges to z. But $z_1^m \in T(z)$ for any $m \geq 1$ and $T(z)$ is closed. This gives that $z \in T(z)$ as required. $\square$

Remark 3.1. For some related fixed point and best approximant results in Banach spaces and metric spaces and their applications, we refer to [19].

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