LAMBDA-HYPERCONVEXITY IN METRIC SPACES

M.A. Khamsi, H. Knaust, N.T. Nguyen and M.D. O'Neill

Abstract. We introduce the concept of λ-hyperconvexity in metric spaces, generalizing the classical notion of a hyperconvex metric space. We show that a bounded metric space which is λ-hyperconvex has the fixed point property for nonexpansive mappings provided $\lambda < 2$. Uniformly convex Banach spaces are examples of such λ-hyperconvex spaces for some $\lambda < 2$. We furthermore investigate the relationship between Penot's Intersection Property and 2-hyperconvexity.

§1. Introduction

Aronszajn and Panitchpakdi [AP] define a metric space to be *hyperconvex* if it satisfies the following condition: Whenever a collection of balls intersects pairwise, the intersection of all balls in the collection is not empty. We generalize this concept by calling a metric space λ-*hyperconvex* if *every pairwise intersecting collection of balls with centers in a given admissible set has non-empty intersection in the admissible set if the radii of the balls are increased by the factor* λ (see Definition 2 below for a precise definition). While the Hilbert space ℓ_2 fails to be hyperconvex, it naturally satisfies the condition of λ-hyperconvexity for $\lambda = \sqrt{2}$.

The recent interest into hyperconvex spaces goes back to results of Sine [Si2] and Soardi [Soa] who proved independently that the fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many interesting results [Ba, KLS, KR, LS, Si1] have been shown to hold in hyperconvex spaces. We show in Section 2 that for $\lambda < 2$ the fixed point property holds also for nonexpansive mappings in bounded λ-hyperconvex spaces.

In Section 3 we show that uniformly convex Banach spaces are λ-hyperconvex for some $\lambda < 2$, while in Section 4 we investigate the relationship between Penot's Intersection Property and 2-hyperconvexity.

1991 *Mathematics Subject Classification.* Primary 46B20, 47E10.

Key words and phrases. Hyperconvexity, fixed point.

Definition 1. A metric space (M,d) is *hyperconvex* if and only if for any family of points $(x_\alpha)_{\alpha\in\Lambda}$ and any family of positive numbers $(r_\alpha)_{\alpha\in\Lambda}$ such that $d(x_\alpha,x_\beta)\le r_\alpha+r_\beta$ for all $\alpha,\beta\in\Lambda$ we must have

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha,r_\alpha)\neq\emptyset,$$

where $B(x,r)$ denotes the closed ball centered at $x\in M$ with radius r.

$\ell_\infty(I)$ for any index set I is the classical example of a hyperconvex metric space, while the Hilbert space ℓ_2 fails to be hyperconvex.

We may refine this definition by considering the cardinality of the index set Λ. We will say that (M,d) (or M for short) is *finitely hyperconvex* (resp. *sequentially hyperconvex*) if the property above is satisfied for any finite set Λ (resp. countable set Λ).

Recall that a subset A of M is called *admissible* if and only if A is an intersection of closed balls. The family of admissible subsets of M will be denoted by $\mathcal{A}(M)$.

This article is centered around the following notion:

Definition 2. Let M be a metric space and let $\lambda\ge 1$. We say that the metric space M is λ-*hyperconvex* if for every non-empty admissible set A of M, for any family of closed balls $\{B(x_\alpha,r_\alpha)\}_{\alpha\in\Lambda}$, each of radius r_α, centered at $x_\alpha\in A$ for $\alpha\in\Lambda$, the condition $d(x_\alpha,x_\beta)\le r_\alpha+r_\beta$ for every $\alpha,\beta\in\Lambda$, implies

$$A\cap\Big(\bigcap_{\alpha\in\Lambda} B(x_\alpha,\lambda r_\alpha)\Big)\neq\emptyset.$$

We let $\Lambda(M)$ be the infimum of all constants λ such that M is λ-hyperconvex, and say that $\Lambda(M)$ is *exact* if M is $\Lambda(M)$-hyperconvex.

Grünbaum [Gr1,Gr2,Gr3] and other authors have studied a similar property not involving the underlying admissible set A. For our purposes it is essential to include the restriction that the intersection of the expanded balls also intersects with the admissible set containing the centers of the original balls. Otherwise we will not be able to connect this concept to the normal structure property (see the proof of Theorem 2 for more details).

Let us recall Grünbaum's definition: For a metric space M, let the *expansion constant* $E(M)$ be the infimum of all constants μ such that the following holds: Whenever a collection $\{B(x_\alpha,r_\alpha):\ \alpha\in\Lambda\}$ intersects pairwise, then

$$\bigcap_{\alpha\in\Lambda} B(x_\alpha,\mu\cdot r_\alpha)\neq\emptyset.$$

We say $E(M)$ is *exact*, if the condition is even satisfied for $\mu=E(M)$.

Trivially, $E(M)\le\Lambda(M)$ holds in metrically convex spaces. On the other hand, if M is a two element metric space, then $E(M)=1$, while $\Lambda(M)=2$, so both concepts do not coincide in general.

Let us first summarize some basic properties of λ-hyperconvex metric spaces, some of which are trivial, while the others can be easily derived from corresponding results about expansion constants:

Theorem 1. (1) *A metric space is hyperconvex if and only if it is* 1*-hyperconvex.*

(2) *For every complete metric space* M*,* $\Lambda(M) \leq 2$ [Gr2]*. Every* λ*-hyperconvex metric space is complete.*

(3) *If* $\mathcal{A}(M)$ *has the Intersection Property, then* $\Lambda(M)$ *is exact* [Gr2, Theorem 3]*. Here we say,* $\mathcal{A}(M)$ *has the Intersection Property if and only if any family of elements* $\{A_\alpha\}_{\alpha\in\Lambda}$ *in* $\mathcal{A}(M)$ *has a nonempty intersection provided*

$$\bigcap_{\alpha\in\Lambda_f} A_\alpha \neq \emptyset$$

for any finite subset Λ_f *of* Λ [Pe]*.*

(4) *In particular, reflexive Banach spaces and dual Banach spaces are* 2*-hyperconvex.*

(5) *There is a subspace* X *of* ℓ_1*, which fails to be* 2*-hyperconvex* [Gr2]*.*

(6) *Hilbert space is* $\sqrt{2}$*-hyperconvex* [Ju, Gr1]*.*

Remark. The counterexample in (5) is the following example, first considered by Klee [Kl] in a different context:

$$X = \left\{ (x_n)_{n\in\mathbb{N}} \in \ell_1 : \sum_{n=1}^{\infty} \frac{n}{n+1} x_n = 0 \right\}.$$

§2. λ-Hyperconvexity and the Fixed Point Property

We establish the following generalization of the Fixed Point Theorem by Sine [Si2] and Soardi [Soa]:

Theorem 2. *Let* M *be a bounded* λ*-hyperconvex space. If* $\lambda < 2$*, then any nonexpansive map* $f : M \to M$ *has a fixed point.*

A map $f : M \to M$ is *nonexpansive* if

$$d(f(x), f(y)) \leq d(x, y) \text{ for every } x, y \in M.$$

The proof of Theorem 2 makes use of the following notation: Let A be a bounded subset of a metric space M. We set

$$\begin{aligned} R(x, A) &:= \sup\{d(x, y) : y \in A\} \text{ for } x \in M, \\ \delta(A) &:= \sup\{R(x, A) : x \in A\} \\ \text{and} \quad R(A) &:= \inf\{R(x, A) : x \in A\} \end{aligned}$$

We also need the following definitions: Let $\mathcal{F}$ be a family of subsets in a metric space M. We say that $\mathcal{F}$ defines a *convexity structure on* M if it contains the balls and is stable by intersection.

Assume that M is a metric space with a convexity structure $\mathcal{F}$. We say that $\mathcal{F}$ is a *uniform normal structure on* M if there exists $c < 1$ such that

$$R(A) \leq c\,\delta(A) \text{ for every } A \in \mathcal{F}.$$

The proof of Theorem 2 is based on the following fact established in [Kh].

Theorem. *Let M be a bounded complete metric space. If M has a uniform normal structure, then M has the fixed point property for nonexpansive maps.*

Proof of Theorem 2. Let M be a bounded λ-hyperconvex space, with $\lambda < 2$. By a previous remark, M is complete. By Khamsi's result it suffices to prove that M has a uniform normal structure.

The family of admissible sets $\mathcal{A}(M)$ defines a convexity structure on M.

We shall show that $\mathcal{A}(M)$ is a uniform normal structure on M. Let $A \in \mathcal{A}(M)$ with $\delta(A) > 0$. For every $x \in A$, let $B(x, r_x)$ denote the ball centered at x with constant radius $r_x = \frac{1}{2}\delta(A)$. Then we have

$$d(x,y) \leq \delta(A) = \frac{1}{2}\delta(A) + \frac{1}{2}\delta(A) = r_x + r_y \text{ for every } x, y \in A.$$

Since M is λ-hyperconvex we can find

$$x_0 \in A \cap \bigcap_{\alpha\in\Lambda} B(x_\alpha, \lambda r_\alpha).$$

Thus we have

$$d(x, x_0) \leq \frac{1}{2}\lambda\,\delta(A) \text{ for every } x \in A.$$

It follows that

$$R(A) \leq \frac{1}{2}\lambda\,\delta(A).$$

Since $\frac{1}{2}\lambda < 1$, $\mathcal{A}(M)$ is a uniform normal structure on M and the theorem is proved. $\square$

Remark. Alspach [Al] (see also Lim [Lim]) constructed an example of a weakly compact convex set in $L^1[0,1]$ which fails the fixed point property for an isometry. From Theorem 1 we can conclude that this example is 2-hyperconvex, while Theorem 2 shows that this example is not λ-hyperconvex for any $\lambda < 2$. In particular, Theorem 2 fails for $\lambda = 2$.

§3. Uniformly Convex Spaces

In this section, we prove the following result:

Theorem 3. *A uniformly convex Banach space is λ-hyperconvex for some $\lambda < 2$.*

Proof. Let X be a uniformly convex Banach space. Since X is reflexive, it is enough to show that for any admissible subset A, any choice of points $x_1, \dots, x_n \in A$ and positive numbers $r_1, \dots, r_n$ such that

$$d(x_i, x_j) \leq r_i + r_j \text{ for all } i, j = 1, \dots, n,$$

we have

$$A \cap \Big(\bigcap_{1\leq i\leq n} B(x_i, \lambda r_i)\Big) \neq \emptyset$$

for some $\lambda < 2$ independent of the set A and the points $\{x_i\}$ as well as the radii $\{r_i\}$. Without loss of generality, we may assume that $r_1 \leq r_2 \leq \ldots \leq r_n$. We know from the proof of Part (3) of Theorem 1 that

$$x_1 \in \bigcap_{1 \leq i \leq n} B(x_i, 2\, r_i)\ .$$

We may assume that there exists $i \in \{1, \ldots, n\}$ such that $d(x_1, x_i) > r_1$. Otherwise, we have

$$x_1 \in \bigcap_{1 \leq i \leq n} B(x_i, r_1) \subset \bigcap_{1 \leq i \leq n} B(x_i, r_i) \subset \bigcap_{1 \leq i \leq n} B(x_i, \lambda r_i)$$

for any $\lambda \geq 1$. Set $\Lambda = \{i :\ d(x_1, x_i) > r_1\}$. Let i_0 be the smallest element of Λ. Then we know that

$$x_{i_0} \in \bigcap_{j \in \Lambda} B(x_j, 2\, r_j)$$

Since $d(x_1, x_{i_0}) > r_1$, then there exists $x_* \in \text{Segment}[x_1, x_{i_0}]$ such that $d(x_1, x_*) = r_1$. For any $i \notin \Lambda$, then we have

$$d(x_*, x_i) \leq d(x_*, x_1) + d(x_1, x_i) \leq 2r_1 \leq 2r_i\ .$$

This clearly implies

$$x_* \in \Big(\bigcap_{1 \leq i \leq n} B(x_i, 2\, r_i) \Big) \cap \text{conv}(x_i)$$

where $\text{conv}(x_i)$ is the convex hull of the points $x_1, \ldots, x_n$. Let $\varepsilon > 0$. Let $i \in \{1, \ldots, n\}$. We consider two cases:

<u>Case 1:</u> $r_1 < \varepsilon\, r_i$.

Then we have

$$d\left(x_i, \frac{x_1 + x_*}{2}\right) \leq d(x_i, x_1) + d\left(x_1, \frac{x_1 + x_*}{2}\right) \leq r_i + r_1 + \frac{r_1}{2} = \frac{2r_i + 3r_1}{2}$$

Using the above condition on r_1 and r_i, we get

$$d\left(x_i, \frac{x_1 + x_*}{2}\right) \leq \frac{2 + 3\varepsilon}{2}\ r_i$$

<u>Case 2:</u> $\varepsilon\, r_i \leq r_1$.

Since

$$d(x_1, x_*) = r_1 \geq 2\, r_i \frac{\varepsilon}{2}$$

and

$$d(x_i, x_1) \leq 2r_i \ \text{ as well as } \ d(x_i, x_*) \leq 2r_i\ ,$$

we get from the definition of the modulus of uniform convexity

$$d\left(x_i, \frac{x_1 + x_*}{2}\right) \le 2\, r_i\Big(1 - \delta\left(\varepsilon/2\right)\Big)\,.$$

Set

$$\lambda = 2 \min_{\varepsilon > 0}\left\{\max\left\{\frac{2+3\varepsilon}{4}, 1 - \delta\left(\varepsilon/2\right)\right\}\right\}$$

Then we have

$$\frac{x_1 + x_*}{2} \in \Big(\bigcap_{1 \le i \le n} B(x_i, \lambda\, r_i)\Big) \cap \mathrm{conv}(x_i) \subset A \cap \Big(\bigcap_{1 \le i \le n} B(x_i, \lambda\, r_i)\Big),$$

because $\mathrm{conv}(x_i) \subset A$. Let us show that $\lambda < 2$. Indeed, for $\varepsilon < 2/3$, we have

$$\max\left\{\frac{2+3\varepsilon}{4}, 1 - \delta\left(\varepsilon/2\right)\right\} < 1$$

which clearly implies $\lambda < 2$. The proof of our claim is therefore complete. □

Remarks. A look at the proof reveals that one does not need the assumption that X is uniformly convex. It is enough to assume that the *characteristic of uniform convexity of* X is less than $1/3$, that is

$$\varepsilon_0(X) := \sup\{\varepsilon \ : \ \delta(\varepsilon) = 0\} \ < \frac{1}{3}\ .$$

We do not know whether the above result still holds if we only assume that $\varepsilon_0(X) < 1$.

§4. Hyperconvexity and the Intersection Property

Clearly if M is a hyperconvex space, then $\mathcal{A}(M)$ has the Intersection Property. Kirk [Ki] asks whether this is still true in the case of 2-hyperconvexity. We will show that c_0 provides a counterexample to this question.

Theorem 4. (1) c_0 *fails the Intersection Property.*
(2) c_0 *fails to be hyperconvex.*
(3) c_0 *is finitely hyperconvex.*
(4) c_0 *is* 2*-hyperconvex.*

Proof. Note that (1) is a direct consequence of (2) and (3). It is easy to find an example which shows that c_0 fails to be hyperconvex, establishing (2).

To prove (3), let $x_1, \dots, x_K$ be elements in c_0 and $r_1, \dots r_K$ be positive numbers such that

$$\|x_i - x_j\| \le r_i + r_j$$

for $i, j = 1, 2, \ldots, K$. We may assume that $0 < r_1 \leq r_2 \leq \cdots \leq r_K$. If we write $x_i = (x_i(n))$, then there exists $n_0 \geq 1$ such that $|x_i(n)| \leq r_1$ for all $i = 1, \ldots, K$, and all $n \geq n_0$. Let $x \in c_0$ defined by

$$x(n) \in \Big[\max_{1\leq i\leq n_0} (x_i - r_i) \ , \ \min_{1\leq i\leq n_0} (x_i + r_i) \Big] \text{ for } 1 \leq n \leq n_0,$$

and

$$x(n) = 0 \text{ for } n > n_0.$$

Clearly, each of the intervals used in the definition is non-empty, and

$$x \in \bigcap_{1\leq i\leq K} B(x_i, r_i) \ ,$$

establishing (3).

It remains to prove that c_0 is 2-hyperconvex. Let A be an admissible set in c_0, and let A_∞ denote the intersection of the same balls in ℓ_∞, *i.e.,* $A = A_\infty \cap c_0$. Let a collection of balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$ be given, each centered at some $x_\alpha \in A$ with positive radius r_α. We assume that the collection satisfies the conditions

$$\|x_\alpha - x_\beta\| \leq r_\alpha + r_\beta \text{ for all } \alpha, \beta \in \Lambda.$$

Since ℓ_∞ is hyperconvex, we can find $y = (y(n)) \in A_\infty$ such that

$$\|y - x_\alpha\| \leq r_\alpha \text{ for all } \alpha \in \Lambda.$$

If we write $a = (a(n))$ for an element in A_∞, then

$$\{a(n) \ : \ a \in A_\infty\}$$

is a closed interval on the real line for all $n \in \mathbb{N}$. Thus we may assume that

$$\inf_{\alpha\in\Lambda} x_\alpha(n) \leq y(n) \leq \sup_{\alpha\in\Lambda} x_\alpha(n) \text{ for all } n \in \mathbb{N}.$$

Set $r = \inf_{\alpha\in\Lambda} r_\alpha$. Note that for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|y(n)| \leq r + \varepsilon.$$

Indeed, pick $\alpha_0 \in \Lambda$ such that $r_{\alpha_0} < r + \varepsilon$ and observe that

$$\lim_{n\to\infty} x_{\alpha_0}(n) = 0.$$

Next we define $x = (x(n)) \in \ell_\infty$ as follows:

$$x(n) := \begin{cases} \max\{y(n) - r, \inf_{\alpha\in\Lambda} x_\alpha(n)\} & \text{if } y(n) \geq r & (1) \\ \max\{0, \inf_{\alpha\in\Lambda} x_\alpha(n)\} & \text{if } 0 \leq y(n) < r & (2) \\ \min\{0, \sup_{\alpha\in\Lambda} x_\alpha(n)\} & \text{if } -r < y(n) < 0 & (3) \\ \min\{y(n) + r, \sup_{\alpha\in\Lambda} x_\alpha(n)\} & \text{if } y(n) \leq -r & (4) \end{cases}$$

If we perform this change (in the space ℓ_∞) from y to x one coordinate at a time, it is not hard to see that our construction ensures that $x \in A_\infty$.

Since $\|x - y\| \leq r$, it follows that

$$\|x - x_\alpha\| \leq r_\alpha + r \leq 2\, r_\alpha \text{ for all } \alpha \in \Lambda.$$

Thus the proof of (4) will be complete once we show that $x \in c_0$.

Suppose $x \notin c_0$. Then there is an $\varepsilon > 0$ and a subsequence $(n_k) \in \mathbb{N}$ such that $|x(n_k)| > \varepsilon$ for all $k \in \mathbb{N}$. Let

$$N_1 := \{k \in \mathbb{N} : \text{ Condition (1) in the definition of } x(n_k) \text{ applies}\}.$$

N_2, N_3 and N_4 are defined analogously.

Suppose $k \in N_1$. Then

$$y(n_k) > r + \varepsilon \text{ or } \inf_{\alpha\in\Lambda} x_\alpha(n_k) > \varepsilon.$$

By our previous observation about y, and the fact that the centers x_α lie in the space c_0, this can happen only finitely often. Thus N_1 is finite.

The proofs that N_2, N_3 and N_4 are also finite are similar. This yields a contradiction to our assumption; consequently $x \in c_0$. $\square$

Remark. It follows from Sobczyk's Theorem [Sob] that $E(c_0) = 2$ and is exact. Since the proofs of Sobczyk's Theorem known to us do not reveal whether the points of intersection still lie in the admissible set containing the centers of the balls, we have presented our own construction.

The example of c_0 above can be modified to obtain an example of a Banach space which is sequentially hyperconvex, but fails to be hyperconvex.

Definition 3. Let $\ell_\infty^c([0,1])$ denote the closed subspace of $\ell_\infty([0,1])$, defined by

$$x = (x_i) \in \ell_\infty^c([0,1]) \text{ if and only if } \{i \in [0,1] : \ x_i \neq 0\} \text{ is countable.}$$

Theorem 5. (1) $\mathcal{A}(\ell^c_\infty([0,1]))$ *fails the Intersection Property.*
(2) $\ell^c_\infty([0,1])$ *fails to be hyperconvex.*
(3) $\ell^c_\infty([0,1])$ *is sequentially hyperconvex.*
(4) $\ell^c_\infty([0,1])$ *is* 2*-hyperconvex.*

Proof. (1) follows directly from (2) and (3). Let us first prove that the space $\ell^c_\infty([0,1])$ is not hyperconvex. Indeed, for any $j \in [0,1]$, set $e_j = (\delta^j_i)$ where $\delta^j_i = 0$ whenever $i \neq j$ and $\delta^j_j = 1$. It is easy to check that $\|e_i - e_j\| = 1$ for any $i \neq j$. Whenever a point $x = (x(i)) \in l_\infty([0,1])$ satisfies $\|x - e_j\| \leq 1/2$ for all $j \in [0,1]$, then x will also satisfy $x(i) \geq 1/2$ for all $i \in [0,1]$. Thus the support of x will not be countable. Hence

$$\ell^c_\infty([0,1]) \cap \Big(\bigcap_{j\in[0,1]} B(e_j, 1/2) \Big) = \emptyset,$$

which means $\ell^c_\infty([0,1])$ is not hyperconvex.

Let us show next that $\ell^c_\infty([0,1])$ is sequentially hyperconvex. Indeed, let $(x_n)_{n\geq 1}$ be in $\ell^c_\infty([0,1])$ such that

$$\|x_n - x_m\| \leq r_n + r_m \ \text{ for any } n, m = 1, 2, \dots$$

for some positive numbers (r_n). Write $x_n = (x_n(i))_{i\in[0,1]}$. Set

$$I = \bigcup_{n\geq 1} \{i :\ x_n(i) \neq 0\} \ .$$

By definition of the space $\ell^c_\infty([0,1])$, we deduce that I is countable. Since $l_\infty(I)$ is hyperconvex, there exists $x = (x_i) \in l_\infty(I)$ such that

$$\sup_{i\in I} \big| |x_i - x_n(i)| \big| \leq r_n$$

If we set $\hat{x} = (\hat{x}_i)$ defined by

$$\hat{x}_i = \begin{cases} x_i & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$$

then we have

$$\hat{x} \in \ell^c_\infty([0,1]) \cap \Big(\bigcap_{n\geq 1} B(x_n, r_n) \Big)$$

which completes the proof of our claim.

It remains to show that $\ell^c_\infty([0,1])$ is 2-hyperconvex. Let A be a nonempty admissible subset of $\ell^c_\infty([0,1])$. Let $x_\alpha \in A$, and r_α positive numbers for $\alpha \in \Lambda$, such that

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \text{ for every } \alpha, \beta \in \Lambda \ .$$

Set $r = \inf\limits_{\alpha\in\Lambda} r_\alpha$. Hence there exists a sequence (r_{α_n}) such that $\lim_{n\to+\infty} r_{\alpha_n} = r$. Set

$$I = \bigcup_{n\geq 1} \{t \in [0,1] : x_{\alpha_n}(t) \neq 0\}$$

By the definition of the space $\ell^c_\infty([0,1])$, the set I is countable. Clearly, we have $x_{\alpha_n} \in l_\infty(I)$ for any $n \geq 1$. Consider the family of admissible sets $\mathcal{A}(l_\infty(I))$. For any $n \geq 1$, set

$$A_n := \mathrm{co}(x_{\alpha_m})_{m\geq n} = \bigcap \{A \in \mathcal{A}(l_\infty(I)) :\ x_{\alpha_m} \in A \text{ for } m \geq n\} \ .$$

Since $l_\infty(I)$ is hyperconvex, then the family $\mathcal{A}(l_\infty(I))$ satisfies the Intersection Property. Therefore, the set $\bigcap\limits_{n\geq 1} A_n$ is not empty. Let

$$\omega \in \bigcap_{n\geq 1} A_n \subset l_\infty(I) \ .$$

Set

$$\omega_e = (\omega_e(t))_{t\in[0,1]} = \begin{cases} \omega(t) & \text{if } t \in I \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have $\omega_e \in \ell^c_\infty([0,1])$. We claim that

$$\omega_e \in A \cap \Big(\bigcap_{\alpha\in\Lambda} B(x_\alpha, 2r_\alpha) \Big)$$

which will complete the proof of the that $\ell^c_\infty([0,1])$ is 2-hyperconvex. First note that if $x \in l_\infty(I)$ and $x_n \in B(x,r)$ for any $n \geq K$, for some positive number r and $K \geq 1$, then we must have $\omega \in B(x,r)$. This is the case because $\omega \in A_K \subset B(x,r)$. On the other hand, if $x \in \ell^c_\infty([0,1])$ such that $x_{\alpha_n} \in B(x,r)$ for any $n \geq K$, for some $K \geq 1$. Then we have

$$\omega_e \in B(x,r)$$

Indeed, let

$$x_r = (x_r(t))_{t\in I} = (x(t))_{t\in I} \in l_\infty(I) \ .$$

Hence the condition $x_n \in B(x,r)$ will imply

$$\Big| x_{\alpha_n}(t) - x(t) \Big| \leq r \quad \text{for } t \in I$$

$$\Big| x(t) \Big| \leq r \quad \text{for } t \notin I$$

In particular, we have $x_{\alpha_n} \in B(x_r, r)$, for $n \geq K$, which implies $\omega \in B(x_r, r)$ in view of the result above. Therefore, we have

$$\Big|\omega(t) - x(t)\Big| \leq r \quad \text{for } t \in I$$

$$\Big|x(t)\Big| \leq r \quad \text{for } t \notin I$$

This clearly translates into $\omega_e \in B(x, r)$. From this, we conclude that

$$\omega_e \in A\,.$$

Let us now show that $\omega_e \in B(x_\alpha, 2r_\alpha)$ for any $\alpha \in \Lambda$. Let $\varepsilon > 0$. There exists $K \geq 1$ such that $r_{\alpha_n} \leq r + \varepsilon$, for any $n \geq K$. Hence, for any $\alpha \in \Lambda$, we have

$$d\Big(x_{\alpha_n}, x_\alpha\Big) \leq r_{\alpha_n} + r_\alpha \leq r + \varepsilon + r_\alpha$$

for any $n \geq K$. The above results imply

$$d\Big(\omega_e, x_\alpha\Big) \leq r + \varepsilon + r_\alpha$$

Since ε was arbitrary, then we must have

$$d\Big(\omega_e, x_\alpha\Big) \leq r + r_\alpha \leq 2\,r_\alpha$$

for any $\alpha \in \Lambda$. This completes the proof that

$$\omega_e \in A \cap \Big(\bigcap_{\alpha \in \Lambda} B(x_\alpha, 2r_\alpha)\Big)$$

which in turn completes our proof. $\square$

References

[Al] D.E. Alspach, *A Fixed Point Free Nonexpansive Map*, Proc. Amer. Math. Soc. **82** (1981), 423–424.

[AP] N. Aronszajn, P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439.

[Ba] J.B. Baillon, *Nonexpansive mappings and hyperconvex spaces*, Contemp. Math. **72** (1988), 11–19.

[Gr1] B. Grünbaum, *On some covering and intersection properties in Minkowski spaces*, Pacific J. Math. **9** (1959), 487-494.

[Gr2] ———, *Some applications of expansion constants*, Pacific J. Math. **10** (1960), 193–201.

[Gr3] ———, *Projection constants*, Trans. Amer. Math. Soc. **95** (1960), 451–465.

[Ju] H.W.E. Jung, *Über die kleinste Kugel, die eine räumliche Figur einschließt*, J. Reine Angew. Math. **123** (1901), 241–257.

[Kh] M.A. Khamsi, *On Metric Spaces With Uniform Normal Structure*, Proc. Amer. Math. Soc. **106** (1989), 723–726.

[KLS] M.A. Khamsi, M. Lin, R.C. Sine, *On the fixed points of commuting nonexpansive maps in hyperconvex spaces*, J. Math. Anal. Appl. **168** (1992), 372–380.

[KR] M.A. Khamsi, S. Reich, *Nonexpansive mappings and semigroups in hyperconvex spaces*, Math. Japon. **35** (1990), 467–471.

[Ki] W. Kirk, Private communication.

[Kl] V.L. Klee Jr., *The support property of a convex set in a linear normed space*, Duke Math. J. **15** (1948), 767–772.

[Lim] T.C. Lim, *Asymptotic Centers and Nonexpansive Mappings in Conjugate Banach Spaces*, Pacific J. Math. **90** (1980), 135–143.

[LS] M. Lin, R.C. Sine, *Retractions on the fixed point set of semigroups of nonexpansive maps in hyperconvex spaces* (to appear).

[Pe] J.P. Penot, *Fixed point theorems without convexity*, Bull. Soc. Math. France **60** (1979), 129–152.

[Si1] R.C. Sine, *Hyperconvexity and approximate fixed points*, Nonlinear Anal. **13** (1989), 863–869.

[Si2] ———, *On linear contraction semigroups in sup norm spaces*, Nonlinear Anal. **3** (1979), 885–890.

[Soa] P. Soardi, *Existence of fixed points of nonexpansive mappings in certain Banach lattices*, Proc. Amer. Math. Soc. **73** (1979), 25–29.

[Sob] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–947.

Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, USA

E-mail address: mohamed@math.utep.edu

Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, USA

E-mail address: helmut@math.utep.edu

Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, USA

E-mail address: nguyen@math.utep.edu

Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968-0514, USA

E-mail address: michael@math.utep.edu