On the numerical index of vector-valued function spaces

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Abstract. Let X be a Banach space and μ a positive measure. We show that $n(L_p(\mu, X)) = \lim_m n(l_p^m(X)), 1 \leq p < \infty$. Also we investigate the positivity of the numerical index of l_p-spaces.

1 Introduction.

Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $B(X)$. The *numerical range* of $T \in B(X)$ is defined by

$$V(T) = \{x^*(Tx) : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}\cdot$$

The *numerical radius* of T is then given by

$$v(T) = \sup\{|\lambda| : \ \lambda \in V(T)\}\cdot$$

Clearly, v is a semi norm on $B(X)$ and $v(T) \leq \|T\|$ for all $T \in B(X)$. The *numerical index* of X is defined by

$$n(X) = \inf\{v(T) : \ T \in S_{B(X)}\}\cdot$$

The concept of numerical index was first suggested by Lumer [**7**] in 1968. Since then a lot of attention has been paid to this constant of equivalence between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. Classical references here are [**1**], [**2**]. For recent results we refer the reader to [**3**], [**5**], [**6**], [**8**], [**10**].

In this paper we show that for any positive measure μ and Banach space X, the numerical index of $L_p(\mu, X)$, $1 \leq p < \infty$ is the limit of the sequence of numerical index of $l_p^m(X)$. This gives a partial answer to Martín's question [**9**] and generalizes the result obtained for the scalar case [**5**]. Also we study the positivity of the numerical index of l_p-space.

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Here $L_p(\mu, X)$ is the classical Banach space of p-integrable functions f from Ω into X where (Ω, Σ, μ) is a given measure space. And $l_p(X)$ is the Banach space of sequences $x = (x_n)_{n\geq 1}$, $x_n \in X$, such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. And finally $l_p^m(X)$ is the Banach space of finite sequences $x = (x_n)_{1\leq n\leq m}$, $x_n \in X$, equipped with the norm $\|x\|_p = \Big(\sum_{n=1}^{m} \|x_n\|^p\Big)^{\frac{1}{p}}$.

2 Main results.

Theorem 2.1. *Let X be a Banach space. Then, for every real number $p, 1 \leq p < \infty$, the numerical index of the Banach space $l_p(X)$ is given by*

$$n(l_p(X)) = \lim_m n(l_p^m(X)).$$

Proof. Let $m \geq 1$ and $T : l_p^m(X) \to l_p^m(X)$ $x \mapsto (T_1(x), ..., T_m(x))$. Define the linear operator $\tilde{T} : l_p(X) \to l_p(X)$ as follows for $x = (x_1, ..., x_m, x_{m+1}, ...) \in l_p(X)$, $\tilde{T}(x) = (T_1(x_1, ..., x_m), ..., T_m(x_1, ..., x_m), 0, ...)$. Clearly, $\tilde{T}$ is bounded and $\|T\| = \|\tilde{T}\|$. We have also $v(T) = v(\tilde{T})$. To prove this, let us first note that if $x = (x_1, ..., x_m, ...) \in S_{l_p(X)}$, then there exists an element, namely x_x^*, in $S_{l_q(X^*)}$, where q is the conjugate exponent to p, such that $x_x^*(x) = 1$. Explicitly $x_x^* = (\|x_1\|^{p-1}x_1^*, ..., \|x_m\|^{p-1}x_m^*, ...)$ where the x_k^*'s are taken in S_{X^*} such that $x_k^*(x_k) = \|x_k\|$. Now, let $\varepsilon > 0$. Following the expression $v(\tilde{T}) = \sup\{|x_x^*(\tilde{T}x)| : x \in S_{l_p(X)}\}$ (**[4]**, Lemma 3.2 and Proposition 1.1) there exists $x = (x_1, ..., x_m, x_{m+1}, ...) \in S_{l_p(X)}$ such that

$$\begin{aligned} v(\tilde{T}) - \varepsilon &< |x_x^*(\tilde{T}x)| \\ &= |(\|x_1\|^{p-1}x_1^*, ..., \|x_m\|^{p-1}x_m^*)(T(x_1, ..., x_m))|. \end{aligned}$$

Put $r := \big(\sum_{k=1}^{m} \|x_k\|^p\big)^{1/p} \leq 1$. Then we obtain $v(\tilde{T}) - \varepsilon < r^p v(T)$ which yields $v(\tilde{T}) \leq v(T)$. The reverse inequality is easy. Therefore

$$\{v(T) : T \in l_p^m(X), \|T\| = 1\} \subset \{v(U) : U \in l_p(X), \|U\| = 1\}$$

which yields $n(l_p(X)) \leq n(l_p^m(X))$. Consequently $n(l_p(X)) \leq \liminf_m n(l_p^m(X))$. Now we shall prove that $\limsup_m n(l_p^m(X)) \leq n(l_p(X))$. Let $T \in B(l_p(X))$. Define the sequence of operators $\{S_m\}_m$ as follows; for each $m \geq 1$, S_m is defined on $l_p^m(X)$ by

$$S_m(x) = (T_1(x_1, ..., x_m, 0, 0, ...), ..., T_m(x_1, ..., x_m, 0, 0, ...)) \qquad (x \in l_p^m(X))\cdot$$

Clearly, the S_m's are bounded and $\|S_m\| \leq \|T\|$ for all m. We claim that

$$\text{(i)} \quad \|S_m\| \to \|T\|$$

$$\text{(ii)} \quad v(S_m) \to v(T).$$

Indeed, we consider the sequence of operators $\{\tilde{S}_m\}_m$ defined on $l_p(X)$ by

$$\tilde{S}_m(x) = (T_1(x_1, ..., x_m, 0, 0, ...), ..., T_m(x_1, ..., x_m, 0, 0, ...), 0, 0, ...)$$

for all $x = (x_1, ..., x_m, x_{m+1}, ...) \in l_p(X)$. It is easy to see that $\|S_m\| = \|\tilde{S}_m\|$, and $\tilde{S}_m$ converges strongly to T. This implies that $\|T\| \leq \liminf_m \|\tilde{S}_m\|$, and it follows that $\|S_m\| \to \|T\|$. As in (i) we have also $v(S_m) = v(\tilde{S}_m)$, so it is enough to prove that $v(\tilde{S}_m) \to v(T)$. Let $\varepsilon > 0$ and fix $u \in S_X$, $u^* \in S_{X^*}$ such that $u^*(u) = 1$. There exists $x \in S_{l_p(X)}$ such that

$$|x^*_x(Tx)| > v(T) - \varepsilon \cdot \tag{1}$$

For each $n \geq 1$, consider

$$x^n = \big(x_1, ..., x_{n-1}, \lambda_n u, 0, 0, ...\big); \;\; x^*_{x^n} = \big(\|x_1\|^{p-1}x^*_{x_1}, ..., \|x_{n-1}\|^{p-1}x^*_{x_{n-1}}, \lambda_n^{p-1}u^*, 0, 0, ...\big)$$

where $\lambda_n = \big(\sum_{k=n}^{\infty} \|x_k\|^p\big)^{1/p}$. Then

$$x^*_{x^n}(x^n) = 1 = \|x^*_{x^n}\| = \|x^n\| \cdot$$

Moreover, $\|x - x^n\| \to 0$ and $\|x^*_x - x^*_{x^n}\| \to 0$ where $x^*_x = \big(\|x_1\|^{p-1}x^*_{x_1}, ..., \|x_n\|^{p-1}x^*_{x_n}, ...\big)$. It follows that $x^*_{x^n}(Tx^n) \to x^*_x(Tx)$ as n tends to infinity. Let $n_0 \geq 1$ be such that

$$|x^*_{x^n}(Tx^n)| > v(T) - \varepsilon \quad (n \geq n_0) \cdot \tag{2}$$

Since $\tilde{S}_m$ converges strongly to T, thus for fixed $n \geq n_0$, $x^*_{x^n}(\tilde{S}_m x^n)$ converges to $x^*_{x^n}(Tx^n)$ as m tends to infinity. So there is $m_0 \geq n$ such that

$$|x^*_{x^n}(\tilde{S}_m x^n)| > v(T) - \varepsilon \quad (m \geq m_0) \cdot \tag{3}$$

This yields $v(\tilde{S}_m) > v(T) - \varepsilon$ for all $m \geq m_0$ and therefore $v(\tilde{S}_m)$ converges to $v(T)$ as m tends to infinity. Now, following (i) and (ii) we have $n(l_p(X)) \geq \limsup_m n(l_p^m(X))$. Indeed, for a given $\varepsilon > 0$, we find $T \in S_{B(l_p(X))}$ such that

$$n(l_p(X)) + \varepsilon > v(T) \cdot$$

Since $v(T) = \lim_m v(\tilde{S}_m)$, there exists m_0 such that

$$n(l_p(X)) + \varepsilon > v(\tilde{S}_m) \quad (m \geq m_0) \cdot$$

But $v(\tilde{S}_m) = v(S_m) \geq n(l_p^m(X))\|S_m\|$, and $\|S_m\| \to \|T\| = 1$, so there exists $k_0 \geq m_0$ such that

$$n(l_p(X)) + \varepsilon > n(l_p^m(X))(1 - \varepsilon) \quad (m \geq k_0) \cdot$$

This implies $n(l_p(X)) \geq \limsup_m n(l_p^m(X))$ and completes the proof of Theorem 2.1. $\square$

It is well known that $n(\oplus_\lambda X_\lambda)_{l_\infty} = \inf_{\lambda \in \Lambda} n(X_\lambda)$ **[9]**. This shows that, in particular, $n(l_\infty(X)) = n(X)$ $(= \lim_m n(l_\infty^m(X)))$. So, Theorem 2.1 is also valid for $p = \infty$.

Theorem 2.2. *Let (Ω, Σ, μ) be a σ-finite measure space. Then, for every Banach space X and every real number p, $1 \leq p < \infty$,*

$$n(L_p(\mu, X)) = n(l_p(X))\cdot$$

Proof. Let us first prove that $n(L_p(\mu, X)) \leq n(l_p(X))$. For this we adapt the proof due to Javier and Martin for the scalar case (not published result). Indeed, if μ is not atomic, $L_p(\mu, X)$ is isometric to $L_p(\mu, X) \oplus_p L_p(\mu, X)$, so they have the same numerical index. Let $T = (T_1, T_2) \in B(l_p^2(X))$ and define the operator S on $L_p(\mu, X) \oplus_p L_p(\mu, X)$ by $S(f_1, f_2)(\omega) = T(f_1(\omega), f_2(\omega))$. One can check easily that $\|T\| = \|S\|$. Moreover, $v(T) = v(S)$. Indeed, let $f_1 = \sum_{i=1}^{m} x_i \frac{1_{A_i}}{\mu(A_i)^{1/p}}$, $f_2 = \sum_{i=1}^{n} y_i \frac{1_{B_i}}{\mu(B_i)^{1/p}}$ be simple functions in $L_p(\mu, X)$ with $\|(f_1, f_2)\|^p = \sum_{i=1}^{m} \|x_i\|^p + \sum_{i=1}^{n} \|y_i\|^p = 1$. For each i we can find x_i^* and y_i^* in S_{X^*} such that $x_i^*(x_i) = \|x_i\|$ and $y_i^*(y_i) = \|y_i\|$. If we set $g_1 = \sum_{i=1}^{m} \|x_i\|^{p-1} x_i^* \frac{1_{A_i}}{\mu(A_i)^{1/q}}$ and $g_2 = \sum_{i=1}^{n} \|y_i\|^{p-1} y_i^* \frac{1_{B_i}}{\mu(B_i)^{1/q}}$, we have clearly $(g_1, g_2) \in S_{L_q(\mu, X^*) \oplus_q L_q(\mu, X^*)}$ and $<(g_1, g_2), (f_1, f_2)>= 1$. Moreover,

$$\begin{aligned} |(g_1, g_2)(S(f_1, f_2))| &\leq \int_\Omega |(g_1(\omega), g_2(\omega))(T(f_1(\omega), f_2(\omega)))| d\mu(\omega) \\ &\leq v(T) \int_\Omega \|f_1(\omega)\|^p + \|f_2(\omega)\|^p d\mu(\omega) = v(T). \end{aligned}$$

Following **[4]**, we have $v(S) \leq v(T)$. For the reverse inequality, let $(x_1, x_2) \in S_{l_p^2(X)}$. Take $A \in \Sigma$ with $\mu(A) > 0$ and consider $(f_1, f_2) = \left(x_1 \frac{1_A}{\mu(A)^{\frac{1}{p}}}, x_2 \frac{1_A}{\mu(A)^{\frac{1}{p}}}\right)$. From what we have just seen $(g_1, g_2) = \left(\|x_1\|^{p-1} x_1^* \frac{1_A}{\mu(A)^{\frac{1}{q}}}, \|x_2\|^{p-1} x_2^* \frac{1_A}{\mu(A)^{\frac{1}{q}}}\right) \in S_{L_q(\mu, X^*) \oplus_q L_q(\mu, X^*)}$ and $<(g_1, g_2), (f_1, f_2)>= 1$. Moreover,

$$\big|(\|x_1\|^{p-1} x_1^*, \|x_2\|^{p-1} x_2^*)(T(x_1, x_2))\big| = \big| \int_\Omega (g_1(\omega), g_2(\omega)) S(f_1, f_2)(\omega) d\mu(\omega)\big| \leq v(S)\cdot$$

This yields $v(T) \leq v(S)$. Consequently $\{v(T) : T \in S_{l_p^2(X)}\} \subset \{v(S) : S \in S_{L_p(\mu, X) \oplus_p L_p(\mu, X)}\}$ which yields $n(L_p(\mu, X) \oplus_p L_p(\mu, X)) \leq n(l_p^2(X))$. So

$$n(L_p(\mu, X)) \leq n(l_p^2(X))\cdot$$

Now, for any integer $m \geq 1$, with the same work as above, we obtain

$$n(L_p(\mu, X)) \leq n(l_p^m(X))\cdot$$

It follows from Theorem 2.1 that

$$n(L_p(\mu, X)) \leq n(l_p(X))\cdot$$

If μ is atomic then $L_p(\mu, X)$ is isometric to $L_p(\nu, X) \oplus_p \left[\oplus_{i\in I} X\right]_{l_p}$ for a suitable set I and an atomless measure ν. With the help of Remark 2 **[9]**, we also have $n(L_p(\mu, X)) \leq n(l_p(X))$. The reverse inequality $n(L_p(\mu, X)) \geq n(l_p(X))$ follows with the same technique used in **[5]** for the scalar case. □

Corollary 2.3. *Let (Ω, Σ, μ) be a σ-finite measure space. Then, for every Banach space X and every real number $p, 1 \leq p < \infty$*

$$n(L_p(\mu, X)) = \lim_m n(l_p^m(X))\cdot$$

3 On the positivity of the numerical index of l_p-space

It was proved that the numerical index of l_p^m, $p \neq 2$, $m = 1, 2, ...$ cannot be equal to 0 this is equivalent to that the numerical radius and the operator norm are equivalent on $B(l_p^m)$, $p \neq 2$ (see Theorem 2.3 **[6]**). In this section we shall also prove that both norms are equivalent on $B(l_p, l_p^m)$.

Theorem 3.1. *For every real number $p \geq 1, p \neq 2$ and every integer m, the numerical radius is equivalent to the operator norm on $B(l_p, l_p^m)$.*
Here l_p is real and l_p^m is identified with its natural embedding in l_p.

Proof. Let $T = (t_{ik}) \in B(l_p, l_p^m)$. We first have

$$\begin{aligned} \|T\| &\leq \left\| \Big(\sum_{k=1}^{\infty} |t_{1k}|^q\Big)^{\frac{1}{q}}, ..., \Big(\sum_{k=1}^{\infty} |t_{mk}|^q\Big)^{\frac{1}{q}} \right\|_p \\ &\leq \Big(\sum_{k=1}^{\infty} |t_{1k}|^q\Big)^{\frac{1}{q}} + \cdots + \Big(\sum_{k=1}^{\infty} |t_{mk}|^q\Big)^{\frac{1}{q}}. \end{aligned}$$

Consider $\{T^j\} \in B(l_p, l_p^m)$ defined by $T^j e_k = Te_k$ for $k \neq j$ and $T^j(e_j) = 0$. Then for $x = \sum_{k=1}^{\infty} x_k e_k \in S_{l_p}$ we have

$$x_x^*(T^1 x) = \varepsilon_1 |x_1|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k \qquad (\varepsilon_j \in \{-1, 1\})\cdot$$

Take $x_1 = \varepsilon_1 2^{-1/p}$ with $\varepsilon_1 \in \{-1, 1\}$ we obtain

$$\left|x_x^*(T^1 x)\right| = \left|2^{-1/q}\Big(\sum_{k=2}^{\infty} t_{1k} x_k\Big) + \varepsilon_1 \Big\{\varepsilon_2 |x_2|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k\Big\}\right| \leq v(T^1)$$

Since ε_1 is arbitrary in $\{-1, 1\}$ then

$$2^{-1/q}\left|\sum_{k=2}^{\infty} t_{1k} x_k\right| + \left|\varepsilon_2 |x_2|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k\right| \leq v(T^1).$$

And in particular

$$2^{-1/q}\Big|\sum_{k=2}^{\infty} t_{1k}x_k\Big| \leq v(T^1)$$

for all $(x_2, ..., x_m, ...) \in l_p$ such that $\displaystyle\sum_{k=2}^{\infty}|x_k|^p = \frac{1}{2}$. That is

$$\frac{1}{2}\Big|\sum_{k=2}^{\infty} t_{1k}y_k\Big| \leq v(T^1) \qquad \forall (y_2, ..., y_m, ...) \in S_{l_p}$$

which yields

$$\frac{1}{2}\Big(\sum_{k\neq 1}|t_{1k}|^q\Big)^{\frac{1}{q}} \leq v(T^1).$$

The same work as above shows that

$$\frac{1}{2}\Big(\sum_{k\neq j}|t_{jk}|^q\Big)^{\frac{1}{q}} \leq v(T^j) \qquad (*)$$

for $j = 1, 2, ..., m$. Now let $R^j = T - T^j$ then we have

$$v(T^j) \leq v(T) + \|R^j\|.$$

And following (*) we obtain

$$\Big(\sum_{k=1}^{\infty}|t_{jk}|^q\Big)^{\frac{1}{q}} \leq 2\Big(v(T) + \|R^j\|\Big) + |t_{jj}|$$

which yields

$$\|T\| \leq 2mv(T) + 2\sum_{j=1}^{m}\|R^j\| + \sum_{j=1}^{m}|t_{jj}|.$$

Now let $\{T_n\}$ be a v-cauchy sequence in $B(l_p, l_p^m)$. Since $v(T_nP_m) = v(P_mT_nP_m) \leq v(T_n)$ where P_m is the operator projection on l_p^m (see **[5]** p 4), and using the fact that in finite dimensional space l_p^m both norms are equivalent, then each $R_n^j = T_n - T_n^j$ converges in operator norm to some R^j. Therefore $\{T_n\}$ is $\|\|$-cauchy. This completes the proof of the Theorem 3.1. □

It's still unknown if the numerical radius and the operator norm are equivalent on the Banach space $B(l_p)$, $p \neq 2$ which gives a complete answer to the question of C. Finet and D. Li.

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