One-local retract and common fixed point for commuting mappings in metric spaces.

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Abstract

In this paper, we introduce and study one-local retract of metric spaces. In particular we prove that any commutative family of non-expansive mappings defined on a metric space with a compact and normal convexity structure has a common fixed point. This conclusion was known in Banach spaces with no similar result in metric spaces.

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1 Introduction

A mapping T defined in metric space (M, d) is said to be nonexpansive if $d(T(x), T(y)) \leq d(x, y)$ for every x, y in M. For such mapping we will denote by $Fix(T)$ the set of its fixed points, i.e. $Fix(T) = \{x \in M;\ T(x) = x\}$. An extensive fixed point theory exists for such mappings, most of which is couched within Banach space framework. Within that framework, many results involve geometrical structure properties and compactness of the domain in some topology which is weaker than the norm topology, usually the weak topology itself (for other topologies see [7]). From the beginning, it was natural to ask whether Kirk's conclusion holds for any commutative family of nonexpansive mappings. The first positive attempt was based on the introduction of complete normal structure property [3]. Later on Lim [11] noticed that in Banach spaces, normal structure property and complete normal structure property are equivalent.

The search for similar fixed results in metric spaces was extensive and exiting. Penot [12] was the first to give a correct formulation of Kirk's result in metric spaces. Under this formulation, it was natural to ask whether Kirk's conclusion holds for any commutative family of nonexpansive mappings. Since Lim's ideas are purely linear, this problem remained open for many years.

Inspired by Baillon's [2] result on hyperconvex metric spaces, we introduce 1-local retract of metric spaces. The investigation of this new concept led to the proof of the stated open question, i.e. any commutative family of nonexpansive mappings defined on a metric space with a compact and normal convexity structure has a common fixed point.

This work forms a part of the author's Ph.D dissertation [6] which was never published. I wish to thank Professor M. Pouzet with whom we had wonderful discussions regarding this work. It is worth to mention that the notion of 1-local retract is due to him.

2 Basic definitions and results

We begin by describing Penot's framework. Let (M, d) be a metric space. $B(x, r)$ will stand for the closed ball centered at $x \in M$ with radius $r \geq 0$. For a bounded subset $A \subset M$, we set

$$r_x(A) = \sup\{d(x,a);\ a \in A\}, \quad x \in M,$$

$$r(A) = \inf\{r_a(A);\ a \in A\},$$

$$\delta(A) = diam(A) = \sup\{r_a(A);\ a \in A\} = \sup\{d(x,y);\ x,\ y \in A\},$$

$$\mathcal{C}(A) = \{a \in A;\ r_a(A) = r(A)\}.$$

For a bounded subset A of M set

$$cov(A) = \bigcap\{B(x,r);\ x \in M \ \ A \subset B(x,r)\}.$$

We will say that A is an *admissible* set if and only if $A = cov(A)$, i.e. A is an intersection of closed balls. The family of all admissible subsets of M will be denoted by $\mathcal{A}(M)$.

A family $\mathcal{F} \subset 2^M$ is called a *convexity structure* [12] if

(i) $\emptyset,\ M \in \mathcal{F}$,

(ii) $\{x\} \in \mathcal{F}$ for every $x \in M$,

(iii) $\mathcal{F}$ is closed under arbitrary intersections.

Following [12], we will say that a convexity structure $\mathcal{F}$ of M, is *compact* (resp. *countably compact*) if each descending chain (resp. sequence) of nonempty sets in $\mathcal{F}$ has nonempty intersection. In this work, we will always assume that closed balls are in any convexity structure. Note that in this case, the smallest convexity structure is the family $\mathcal{A}(M)$ of admissible subsets of M. It is easy to see that if $\mathcal{A}(M)$ is countably compact then (M,d) is complete.

The fundamental existence theorem [9] requires one additional assumption: a convexity structure $\mathcal{F}$ is said to be *normal* if for each $A \in \mathcal{F}$ we have either $\delta(A) = 0$ or $r(A) < \delta(A)$.

Theorem 1.(Kirk [9], Penot [12]). *Let (M,d) be a nonempty bounded metric space that possesses a convexity structure $\mathcal{F}$ which is compact and normal.*

Then every nonexpansive mapping $T : M \to M$ has a fixed point.

The original proof requires the use of Zorn's lemma which explains why the compactness of $\mathcal{F}$ is assumed. Other constructive proofs [10] are known which only require $\mathcal{F}$ to be countably compact. In this work, we will not discuss these ideas. Interested readers can refer to [4]. For more on fixed point property see [1,5].

For the rest of this work, we will need the following technical result.

Proposition 2. *Let (M, d) be a nonempty metric space and A be a nonempty bounded subset of M. Then we have*

(1) $cov(A) = \bigcap\{B(x, r_x(A));\ x \in M\}$

(2) $r_x(A) = r_x(cov(A))$ *for every* $x \in M$

(3) $r(cov(A)) \leq r(A)$

(4) $\delta(cov(A)) = \delta(A)$.

Proof. (1) Since $B(x, r_x(A))$ is the smallest ball centered at x which contains A, we get the conclusion of (1).

(2) Let x be in M. Then since $A \subset cov(A)$, we get $r_x(A) \leq r_x(cov(A))$. On the other hand from (1) we have $cov(A) \subset B(x, r_x(A))$. Hence we have $r_x(cov(A)) \leq r_x(A)$. The proof of (2) is therefore complete.

(3) Obvious from the definition of r and (2).

(4) Because $A \subset cov(A)$ it is enough to prove that $\delta(cov(A)) \leq \delta(A)$. Let $z \in cov(A)$. Using (1), we get $d(x, z) \leq r_x(A)$ for every $x \in M$. Hence $d(a, z) \leq r_a(A) \leq \delta(A)$ for every $a \in A$. Therefore, we have $A \subset B(z, \delta(A))$ which implies $cov(A) \subset B(z, \delta(A))$. Hence for every $w \in cov(A)$ we have $d(w, z) \leq \delta(A)$. This clearly implies $\delta(cov(A)) \leq \delta(A)$.

The proof of Proposition 2 is therefore complete.

Note that in general, the set $\mathcal{C}(A)$, for $A \in \mathcal{F}$ not reduced to one point, may be empty. Assume that $\mathcal{F}$ is compact, then $\mathcal{C}(A)$ is not empty and belongs to $\mathcal{F}$. Indeed, using the definition of $r(A)$, for every $n > 1$, there exists $x \in A$ such that $r_x(A) \leq r(A) + 1/n$. This clearly implies that $A_n = \bigcap_{a \in A} B\left(a, r(A) + \frac{1}{n}\right)$ is not empty ($x \in A_n$). It is obvious that $A_n \in \mathcal{F}$ for every n and the sequence (A_n) is decreasing. Compactness of $\mathcal{F}$ implies that

$$\mathcal{C}(A) = \bigcap_{n \geq 1} \bigcap_{a \in A} B\left(a, r(A) + \frac{1}{n}\right) = \bigcap_{n \geq 1} A_n$$

is not empty and belongs to $\mathcal{F}$.

3 One-local retract

From now on, we only consider the convexity structure of admissible subsets $\mathcal{A}(M)$ for any metric space. With minor modification one can easily extend the results to arbitrary convexity structures. A subset A of M is said to be a *1-local retract* of M if for every family $\{B_i; i \in I\}$ of closed balls centered in A with nonempty intersection, it is the case that $A \cap (\cap B_i) \neq \emptyset$. It is immediate that each nonexpansive retract of M is a 1-local retract (but not conversely). Recall that $A \subset M$ is a nonexpansive retract of M if there exists a nonexpansive map $r : M \rightarrow A$ such that $r(a) = a$ for every $a \in A$.

The following theorem will throw some light on this notion.

Theorem 3. *Let (M, d) be a nonempty metric space and N be a nonempty subset of M. The following are equivalent.*

(i) N is a 1-local retract of M.

(ii) N is a nonexpansive retract of $N \cup \{x\}$, for every $x \in M$.

(iii) Let H be a metric space and $T : N \rightarrow H$ be a Lipschitzian map, i.e. there exists $\alpha \geq 0$ such that $d_H(T(a), T(b)) \leq \alpha\, d_M(a, b)$, for every $a, b \in N$, where d_H and d_M stand respectively for the distances on H and M. Then, for every $x \in M$, there exists an extension $T^* : N \cup \{x\} \rightarrow H$, such that $d_H(T^*(a), T^*(b)) \leq \alpha\, d_M(a, b)$, for every $a, b \in N \cup \{x\}$ and $T^*(a) = T(a)$ for every $a \in N$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in M$. We may assume that $x \notin N$. In order to construct a nonexpansive retract $r : N \cup \{x\} \to N$, we only need to find $r(x) \in N$ such that $d(a, r(x)) \leq d(a, x)$, for every $a \in N$. Since $x \in \bigcap_{a \in N} B(a, d(a,x))$ in M, then $\bigcap_{a \in N} B(a, d(a,x)) \neq \emptyset$. Therefore we have $N \bigcap \bigcap_{a \in N} B(a, d(a,x)) \neq \emptyset$ because N is a 1-local retract of M. Then any point in $N \bigcap \bigcap_{a \in N} B(a, d(a,x))$ will work as $r(x)$.

$(ii) \Rightarrow (iii)$ Let $x \in M$, H be a metric space and $T : N \to H$ be a Lipschitzian map such that $d_H(T(a), T(b)) \leq \alpha\ d_M(a,b)$, for every $a, b \in N$, for some $\alpha \geq 0$. Define $T^* : N \cup \{x\} \to H$ by $T^*(a) = T(r(a))$ for every $a \in N \cup \{x\}$, where r is a nonexpansive retract from $N \cup \{x\}$ into N. T^* satisfies the conclusion of (ii).

$(iii) \Rightarrow (ii)$ Take $H = N$ and $T = id_N$ the identity map of N.

$(ii) \Rightarrow (i)$ In order to prove that N is a 1-local retract, let $(B_i)_{i \in I}$ be a family of closed balls of M centered in N such that $\bigcap_{i \in I} B_i \neq \emptyset$. We need to prove that $N \bigcap \bigcap_{i \in I} B_i \neq \emptyset$. Let $x \in \bigcap_{i \in I} B_i$. If $x \in N$, we have nothing to prove. Assume that $x \notin N$. Let r be a nonexpansive retract from $N \cup \{x\}$ into N. Then it is easy to see that $r(x) \in N \bigcap \bigcap_{i \in I} B_i$.

The proof of Theorem 3 is therefore complete.

We have the following technical lemma.

Lemma 4. *Let (M, d) be a nonempty metric space and N be a nonempty 1-local retract of M. Then we have*

$$r(cov(A)) = r(A)$$

for every $A \in \mathcal{A}(N)$.

Proof. Assume that A is not reduced to one point. According to Proposition 2, we have $r(cov(A)) \leq r(A)$. Let us prove that $r(A) \leq r(cov(A))$.

Since $A \in \mathcal{A}(N)$, then $A = N \bigcap \bigcap_{i\in I} B_i$, where $(B_i)_{i\in I}$ is a family of closed balls of M centered in N. Let $z \in cov(A)$. Set $m = r_z(cov(A))$. Then z belongs to $S = \bigcap_{a\in A} B(a,m) \bigcap \bigcap_{i\in I} B_i$. Since N is a 1-local retract of M, then $N \cap S$ is not empty. Let $a \in N \cap S$. Clearly we have $A \subset B(a,m)$. Then $r_a(A) \leq m$. Therefore $r(A) \leq m = r_z(cov(A))$. Since z was arbitrary in $cov(A)$, we get $r(A) \leq r(cov(A))$. The proof of Lemma 4 is complete.

It is not very hard to see that compactness and normality of $\mathcal{A}(M)$ is not hereditary to $\mathcal{A}(N)$, where $N \subset M$.

Theorem 5. *Let (M,d) be a nonempty metric space and N be a nonempty subset 1-local retract of M. Assume that $\mathcal{A}(M)$ is compact and normal. Then $\mathcal{A}(N)$ is compact and normal.*

Proof. The definition of 1-local retracts implies obviously that $\mathcal{A}(N)$ is compact. Let us prove that $\mathcal{A}(N)$ is normal. Choose $A \in \mathcal{A}(N)$ not reduced to one point. By Proposition 2, we have $\delta(cov(A)) = \delta(A)$ and by Lemma 4 we have $r(cov(A)) = r(A)$. Since $\mathcal{A}(M)$ is normal and $cov(A) \in \mathcal{A}(M)$, then $r(cov(A)) < \delta(cov(A))$. Hence we have $r(A) < \delta(A)$. The proof of Theorem 5 is therefore complete.

The following theorem is the main result of this work. The idea of the proof was inspired from [2].

Theorem 6. *Let (M,d) be a nonempty metric space for which $\mathcal{A}(M)$ is compact and normal. Let $(M_\beta)_{\beta\in\Gamma}$ be a decreasing family of 1-local retracts of M, where Γ is totally ordered. Then $\bigcap_{\beta\in\Gamma} M_\beta$ is not empty and is a 1-local retract of M.*

Proof. Consider the family

$$\mathcal{F} = \{A = \Pi_{\beta\in\Gamma} A_\beta \,;\, A_\beta \in \mathcal{A}(M_\beta) \text{ and } (A_\beta) \text{ is decreasing } \}.$$

$\mathcal{F}$ is not empty since $\Pi_{\beta\in\Gamma} M_\beta \in \mathcal{F}$. From Theorem 5, we know that $\mathcal{A}(M_\beta)$ is compact, for every $\beta \in \Gamma$. Therefore, $\mathcal{F}$ satisfies the assumptions of Zorn's lemma. Hence for every $D \in \mathcal{F}$, there exists a minimal element $A \in \mathcal{F}$ such that $A \subset D$. We claim that if $A = \Pi_{\beta\in\Gamma} A_\beta$ is minimal, then there exists $\beta_0 \in \Gamma$ such that $\delta(A_\beta) = 0$ for every $\beta \geq \beta_0$. Indeed, let $A = \Pi_{\beta\in\Gamma} A_\beta$ be a minimal element of $\mathcal{F}$. Fix $\beta \in \Gamma$. For every $D \subset M$, set

$$cov_\beta(D) = \bigcap_{x\in M_\beta} B(x, r_x(D)).$$

Consider $A' = \Pi_{\alpha\in\Gamma} A'_\alpha$ where

$$\begin{aligned} A'_\alpha &= cov_\beta(A_\beta) \cap A_\alpha \quad \text{if} \quad \alpha \leq \beta \\ A'_\alpha &= A_\alpha \qquad\qquad\qquad \text{if} \quad \alpha \geq \beta \end{aligned}$$

The family $(A'_{\alpha\geq\beta})$ is decreasing since $A \in \mathcal{F}$. Let $\alpha \leq \gamma \leq \beta$. Then $A'_\gamma \subset A'_\alpha$ since $A_\gamma \subset A_\alpha$ and $A_\beta = cov_\beta(A_\beta) \cap A_\beta$. Hence the family (A'_α) is decreasing. On the other hand if $\alpha \leq \beta$, then $cov_\beta(A_\beta) \cap A_\alpha \in \mathcal{A}(M_\alpha)$ since $M_\beta \subset M_\alpha$. Hence $A'_\alpha \in \mathcal{A}(M_\alpha)$. Therefore, we have $A' \in \mathcal{F}$. Since A is minimal, then $A = A'$ which implies

$$A_\alpha = cov_\beta(A_\beta) \cap A_\alpha, \text{ for every } \alpha \leq \beta\ .$$

Let $x \in M_\beta$ and $\alpha \leq \beta$. Since $A_\beta \subset A_\alpha$, then $r_x(A_\beta) \leq r_x(A_\alpha)$. Because $cov_\beta(A_\beta) = \bigcap\limits_{x'\in M_\beta} B(x', r_{x'}(A_\beta)$, then we have $cov_\beta(A_\beta) \subset B(x, r_x(A_\beta)$ which implies $r_x(cov_\beta(A_\beta) \leq r_x(A_\alpha)$. Since $A_\alpha \subset cov_\beta(A_\beta)$, then

$$r_x(A_\beta) \leq r_x(A_\alpha) \leq r_x(cov_\beta(A_\beta)) \leq r_x(A_\beta).$$

Therefore, we have $r_x(A_\alpha) = r_x(A_\beta)$ for every $x \in M_\beta$. Using the definition of r, we get

$$r(A_\alpha) \leq r(A_\beta).$$

Let $a \in A_\alpha$ and set $s = r_a(A_\alpha)$. Then $a \in cov_\beta(A_\beta)$ since $A_\alpha \subset cov_\beta(A_\beta)$. Hence $a \in \bigcap\limits_{x\in A_\beta} B(x, s) \bigcap cov_\beta(A_\beta)$. Since M_β is a 1-local retract, then

$$S_\beta = M_\beta \bigcap \bigcap_{x\in A_\beta} B(x, s) \bigcap cov_\beta(A_\beta) \neq \emptyset.$$

Since $A_\beta = M_\beta \bigcap cov_\beta(A_\beta)$, then we have

$$S_\beta = A_\beta \bigcap \bigcap_{x \in A_\beta} B(x,s).$$

Let $z \in S_\beta$, then $z \in \bigcap_{x \in A_\beta} B(x,s)$. Hence $r_z(A_\beta) \leq s$ which implies $r(A_\beta) \leq s = r_a(A_\alpha)$, for every $a \in A_\alpha$. Hence $r(A_\beta) \leq r(A_\alpha)$. Therefore we have

$$r(A_\beta) = r(A_\alpha), \text{ for every } \alpha \text{ and } \beta \text{ in } \Gamma.$$

Assume that $\delta(A_\beta) > 0$ for every $\beta \in \Gamma$. Set $A''_\beta = \mathcal{C}(A_\beta)$ for every $\beta \in \Gamma$. Since $r(A_\beta) = r(A_\alpha)$, for every α and β in Γ, then the family (A''_β) is decreasing. Indeed, let $\alpha \leq \beta$ and $x \in A''_\beta$. Then we have $r_x(A_\beta) = r(A_\beta)$. Since we proved that $r_z(A_\beta) = r_z(A_\alpha)$ for every $z \in M_\beta$, then

$$r_x(A_\alpha) = r_x(A_\beta) = r(A_\beta) = r(A_\alpha),$$

which implies that $x \in A''_\alpha$. Therefore, we have $A'' = \Pi_{\beta \in \Gamma} A''_\beta \in \mathcal{F}$. Since $A'' \subset A$ and A is minimal, we get $A = A''$. Therefore, we have $\mathcal{C}(A_\beta) = A_\beta$ for every $\beta \in \Gamma$. This contradicts the fact that $\mathcal{A}(M_\beta)$ is normal for every $\beta \in \Gamma$. Hence there exists $\beta_0 \in \Gamma$ such that

$$\delta(A_\beta) = 0, \text{ for every } \beta \geq \beta_0.$$

The proof of our claim is therefore complete. Then we have $A_\beta = \{a\}$, for every $\beta \geq \beta_0$. This clearly implies that

$$a \in \bigcap_{\beta \in \Gamma} M_\beta \neq \emptyset.$$

In order to complete the proof, we need to show that $S = \bigcap_{\beta \in \Gamma} M_\beta$ is a 1-local retract of M. Let $(B_i)_{i \in I}$ be a family of closed balls centered in S such that $\bigcap_{i \in I} B_i \neq \emptyset$. Set $D_\beta = \bigcap_{i \in I} B_i \bigcap M_\beta$ for $\beta \in \Gamma$. Since M_β is a 1-local retract of M, and the family (B_i) is centered in M_β, then D_β is not empty and $D_\beta \in \mathcal{A}(M_\beta)$. Therefore, $D = \Pi D_\beta \in \mathcal{F}$. Let $A = \Pi A_\beta \subset D$ be a minimal element of $\mathcal{F}$. The above proof shows that

$$\emptyset \neq \bigcap_{\beta \in \Gamma} A_\beta \subset \bigcap_{\beta \in \Gamma} D_\beta.$$

The proof of Theorem 6 is therefore complete.

Remark. From Theorem 6, one can deduce that any decreasing family $(M_\beta)_{\beta\in\Gamma}$ of 1-local retracts of M, where Γ is downward directed, has a nonempty intersection which is a 1-local retract of M.

4 Main fixed point theorem

As we mentioned in the introduction, it was unknown whether a bounded metric space with a compact and normal convexity structure enjoys the fixed point property for any commutative family of nonexpansive mappings.

Before we get to the main result of this work, let us note that under the assumptions of Theorem 1, the fixed point set of a nonexpansive mapping is a 1-local retract of M. Indeed, let $T : M \to M$ be a nonexpansive map. We know that $Fix(T)$ is not empty. Let $(B_i)_{ji\in I}$ be a family of closed balls centered in $Fix(T)$ with a nonempty intersection. Since T is nonexpansive, then $S = \bigcap_{i\in I} B_i$ is T-invariant, i.e. $T(S) \subset S$. The set S belongs to $\mathcal{A}(M)$, which implies that $\mathcal{A}(S)$ is compact and normal. Therefore by Theorem 1, T has a fixed point in S, i.e. $S \cap Fix(T) \neq \emptyset$. This completes the proof of our claim.

The next results discuss the existence of fixed points of commutative family of nonexpansive mappings and the structure of their common fixed point set.

Theorem 7. *Let (M, d) be a nonempty bounded metric space such that $\mathcal{A}(M)$ is compact and normal. Then any finite commuting family of nonexpansive mappings $T_1, T_2, .., T_n$, $T_i : M \to M$, has a common fixed point. Moreover if we denote by $Fix((T_i))$ the set of the common fixed points,i.e. $Fix((T_i)) = \{x \in M; T_i(x) = x \quad i = 1, .., n\}$, then $Fix((T_i))$ is a 1-local retract of M.*

Proof. First let us prove that $F = Fix((T_i))$ is not empty. Using Theorem 1, we know that $Fix(T_1)$ is not empty. Since $Fix(T_1)$ is a 1-local retract of M, then Theorem 5 implies that $\mathcal{A}(Fix(T_1))$ is compact and normal. On the other hand, we have $T_2(Fix(T_1)) \subset Fix(T_1)$ because T_1 and T_2 commute.

Hence T_2 has a fixed point in $Fix(T_1)$. If we restrict ourselves to $Fix(T_1, T_2)$, the common fixed point set of T_1 and T_2, then one can prove in an identical argument that T_3 has a fixed point in $Fix(T_1, T_2)$. Step by step, we can prove that the common fixed point set F of $T_1, .., T_n$ is not empty. The same argument used to prove that the fixed point set of a nonexpansive map is a 1-local retract, can be reproduced here to prove that F is a 1-local retract. The proof of Theorem 7 is therefore complete.

Now we are ready to prove the main result of this work.

Theorem 8. *Let (M, d) be a nonempty bounded metric space such that $\mathcal{A}(M)$ is compact and normal. Then any commuting family of nonexpansive mappings $(T_i)_{i\in I}$, $T_i : M \to M$, has a common fixed point. Moreover if we denote by $Fix((T_i))$ the set of the common fixed points, then $Fix((T_i))$ is a 1-local retract of M.*

Proof. Let $\Gamma = 2^I = \{\beta;\ \beta \subset I\}$. It is obvious that Γ is downward directed (the order on Γ is the set inclusion). Theorem 7 implies that for every $\beta \in \Gamma$, the set F_β of common fixed point set of the mappings $T_i,\ i \in \beta$, is a nonempty 1-local retract of M. Clearly the family $(F_\beta)_{\beta\in\Gamma}$ is decreasing. Using the remark following Theorem 6, we deduce that $\bigcap_{\beta\in\Gamma} F_\beta$ is nonempty and is a 1-local retract of M. The proof of Theorem 8 is complete.

Note that Kijima and Takahashi [8] proved a similar conclusion provided M is a compact metric space and $\mathcal{A}(M)$ is normal.

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